# Further Calculus III Cheat Sheet (A-Level Only)



## Integration Using Partial Fractions

### Integration using Partial Fractions with Linear Factors

Partial fraction decomposition is used to simplify integrals involving the product of two (or more) binomials in the denominator of the integrand. These decompositions are more easily integrated, and most often will integrate to become logarithms. The following table gives the partial fraction decomposition for the three main types of expressions:

| Type of Expression | Partial Fraction Decomposition |
|---|---|
| $\frac{px+q}{(ax+b)(cx+d)}$ | $\frac{A}{ax+b}+\frac{B}{cx+d}$ |
| $\frac{px^2+qx+r}{(ax+b)(cx+d)(ex+f)}$ | $\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{ex+f}$ |
| $\frac{px^2+qx+r}{(ax+b)^2(cx+d)}$ | $\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{cx+d}$ |

The following result is often used within these types of questions:

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c,$$

where $c$ is the constant of integration. This result can often be arrived at by multiplying the integrand by a constant, for example:

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\ln|x^2+1| + c$$

Example 1: Evaluate the indefinite integral

$$\int \frac{32x^2-46x-45}{(6x+1)(2x-1)(x+7)}dx$$

Write the integrand as its partial fraction decomposition in terms of unknown constants.

$$\frac{32x^2-46x-45}{(6x+1)(2x-1)(x+7)} = \frac{A}{6x+1}+\frac{B}{2x-1}+\frac{C}{x+7}$$

Multiply the equation by the product of the three binomials. Then, find one unknown constant by choosing the value of $x$ such that the binomials before the two other constants vanish. Then, solve for the desired constant. Repeat this for the other unknown. Find the final one by equating the coefficients of $x^2$ on each side of the equation. Finally, rewrite the integrand as its partial fraction decomposition, ready to be integrated.

$$32x^2-46x-45 = A(2x-1)(x+7)+B(6x+1)(x+7)+C(6x+1)(2x-1)$$

Set $x=-7$ to cancel out terms with a $(x+7)$ factor to find $C$,

$$32(-7)^2-46(-7)-45 = C(6(-7)+1)(2(-7)-1) \Rightarrow 1845 = 615C \Rightarrow c = \frac{1845}{615} = 3$$

Set $x=\frac{1}{2}$ to cancel out terms with a $(2x-1)$ factor to find $B$,

$$32\left(\tfrac{1}{2}\right)^2-46\left(\tfrac{1}{2}\right)-45 = B\left(6\left(\tfrac{1}{2}\right)+1\right)\left(\tfrac{1}{2}+7\right) \Rightarrow -60 = 30B \Rightarrow B = -\tfrac{60}{30} = -2$$

Equate coefficients of $x^2$ on the left and right-hand sides to find the last unknown $A$,

$$32 = 2A+6B+12C \Rightarrow 32 = 2A-12+36 \Rightarrow 32 = 2A+24 \Rightarrow 8 = 2A$$
$$\Rightarrow A = 4$$
$$\therefore \frac{32x^2-46x-45}{(6x+1)(2x-1)(x+7)} = \frac{4}{6x+1}-\frac{2}{2x-1}+\frac{3}{x+7}$$

Split the integral into three using the partial fraction decomposition. Use $\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$ to find the integrals. Use the laws of logarithms to simplify the final answer. Remember to finally add on the constant of integration.

$$\int \frac{32x^2-46x-45}{(6x+1)(2x-1)(x+7)}dx = \int\frac{4}{6x+1}dx - \int\frac{2}{2x-1}dx + \int\frac{3}{x+7}dx$$
$$\int\frac{4}{6x+1}dx = \frac{4}{6}\int\frac{6}{6x+1}dx, \quad \int\frac{3}{x+7}dx = 3\int\frac{1}{x+7}dx$$
$$\therefore \int \frac{32x^2-46x-45}{(6x+1)(2x-1)(x+7)}dx = \frac{4}{6}\int\frac{6}{6x+1}dx - \int\frac{2}{2x-1}dx + 3\int\frac{1}{x+7}dx$$
$$= \frac{4}{6}\ln|6x+1| - \ln|2x-1| + 3\ln|x+7| + c = \ln\left|\frac{|6x+1|^{\frac{2}{3}}|x+7|^3}{|2x-1|}\right| + c$$

### Integrals with Quadratic Expressions in the Denominator

The most complicated case of a partial fraction decomposition will involve having a quadratic expression in the denominator. In these cases, the following decomposition is used:

$$\frac{px^2+qx+r}{(ax+b)(cx^2+d)} = \frac{A}{ax+b}+\frac{Bx+C}{(cx^2+d)}$$

Example 2: Find:

$$\int \frac{9x^2-x+94}{x^3-x^2+16x-16.}dx.$$

It is hard to tell just by looking at the integrand what type of partial fraction decomposition will be needed, as the denominator has not been factorised. By substituting in $x=1$, the polynomial vanishes, and so by the factor theorem, it is divisible by $(x-1)$. Comparing coefficients of $x^3, x$ and the constant terms will then generate equations to find the other term in the denominator. After the factorisation, the integrand can be written as a partial fraction decomposition with unknown constants.

Let $f(x) = x^3-x^2+16x-16$.

$$f(1) = (1)^3-(1)^2+16-16 = 0 \therefore (x-1) \mid x^3-x^2+16x-16$$
$$\therefore (x-1)(Ax^2+Bx+C) = x^3-x^2+16x-16$$

Compare coefficients of $x^3$,

$$1 = 1\cdot A \Rightarrow A = 1$$

Compare the constant terms,

$$-16 = (-1)\cdot C \Rightarrow C = 16$$

Compare the coefficients of $x$,

$$16 = -B+C \Rightarrow 16 = -B+16 \Rightarrow B = 0$$
$$\therefore (x-1)(x^2+16) = x^3-x^2+16x-16$$
$$\therefore \frac{9x^2-x+94}{x^3-x^2+16x+16} = \frac{9x^2-x+94}{(x-1)(x^2+16)} = \frac{S}{x-1}+\frac{Px+Q}{x^2+16}$$

Multiply the partial fraction decomposition by the product of the two binomials. The first unknown coefficient, $S$, can be found by equating the term $(x-1)$ to 0, by setting $x=1$. The others, $P$ and $Q$, are most easily found by comparing the coefficients $x^2$ and the constant terms.

$$9x^2-x+94 = S(x^2+16)+(Px+Q)(x-1)$$
$$x = 1 \Rightarrow 9(1)^2-1+9 = S((1)^2+16) \Rightarrow 102 = 17S \Rightarrow S = \frac{102}{17} = 6$$

Comparing coefficients of $x^2$,

$$9 = S+P \Rightarrow 9 = 6+P \Rightarrow P = 3$$

Comparing the constant terms,

$$94 = 16S-Q \Rightarrow 94 = 96-Q \Rightarrow Q = 2$$
$$\therefore \frac{9x^2-x+94}{(x-1)(x^2+16)} = \frac{6}{x-1}+\frac{3x+2}{x^2+16}$$

Split the integral into two to begin with. The second integral is then split into two again, with the first found using the result $\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$. The second is found using the result $\int\frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$, with $a=4$. A constant of integration is introduced when finding each integral, but for the final result they can be absorbed into one constant. The final expression can be simplified using laws of logarithms.

$$\int\frac{9x^2-x+94}{(x-1)(x^2+16)}dx = \int\frac{6}{x-1}dx + \int\frac{3x+2}{x^2+16}dx$$
$$\int\frac{6}{x-1}dx = 6\int\frac{1}{x-1}dx = 6\ln|x-1| + c_1$$
$$\int\frac{3x+2}{x^2+16}dx = \int\frac{3x}{x^2+16}+\frac{2}{x^2+16}dx = \frac{3}{2}\int\frac{2x}{x^2+16}dx + 2\int\frac{1}{x^2+16}dx$$
$$= \frac{3}{2}\ln|x^2+16| + \frac{2}{4}\arctan\left(\frac{x}{4}\right) + c_2$$
$$\therefore \int\frac{9x^2-x+94}{(x-1)(x^2+16)}dx = 6\ln|x-1| + \frac{3}{2}\ln|x^2+16| + \frac{1}{2}\arctan\left(\frac{x}{4}\right) + c$$
$$= \frac{1}{2}\left(12\ln|x-1| + 3\ln|x^2+16| + \arctan\left(\frac{x}{4}\right)\right) + c$$
$$= \frac{1}{2}\left(\ln|(x-1)^{12}(x^2+16)^3| + \arctan\left(\frac{x}{4}\right)\right) + c$$