Volumes of Revolution

A solid of revolution is obtained by rotating a curve $y = f(x)$ about some axis, with a full turn corresponding to rotation by an angle of 2π radians. Integration is used to find the volume of these solids, which are called volumes of revolution. Two cases are considered here: rotation about the x-axis and rotation about the y-axis.

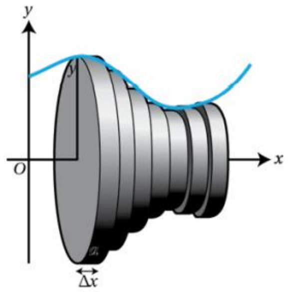
Rotation About the x-Axis

Consider a curve $y = f(x)$ between the points $x = a$ and $x = b$. Rotating this curve 2π radians around the x-axis will produce a solid of revolution whose volume, V, is given by the following formula.

$$V = \pi\int_a^b y^2 dx.$$

The derivation is as follows:

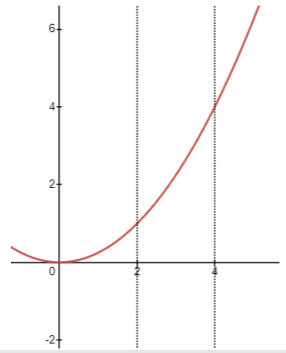
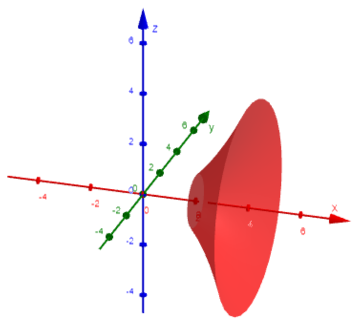
1. The solid of revolution is approximated as a collection of thin cylinders, each of width Δx. To find the volume of the entire solid, the volumes of all these cylinders are added together.
2. The y-coordinate of each cylinder is its radius, as these cylinders are symmetric about the x-axis. Using the formula for the volume of a cylinder, each of these cylinders has a volume of $\pi y^2 \Delta x$. These volumes are then summed over the whole cylinder, so from $x = a$ to $x = b$. π here is a constant, and so can be taken to the front.
$$V \approx \sum_{x=a}^{b} \pi y^2 \Delta x = \pi \sum_{x=a}^{b} y^2 \Delta x.$$
3. In the limit $\Delta x \to 0$, so as the cylinders get thinner, the approximation to the true volume of the solid becomes more accurate. The sum will become an integral, resulting in the required formula.
$$V = \lim_{\Delta x \to 0} \pi \sum_{x=a}^{b} y^2 \Delta x = \pi \int_a^b y^2 dx.$$



Ref: CUP AQA A Level Further Mathematics Book 1

If the curve is rotated by a fraction of a full turn, the volume of revolution will be given by the same formula but multiplied by this fraction. Note that performing successive full rotations will not increase the volume of the solid.

Example 1: Find the exact volume of the solid generated by rotating the curve $y = \frac{x^2}{4}$ between the points $x = 2$ and $x = 4$ by 2π radians around the x-axis.

On the left is the graph of $y = \frac{x^2}{4}$ in the x-y plane, with the lines $x = 2$ and $x = 4$ added. On the right is the corresponding solid of revolution, which is the solid we will be finding the volume of.	 
To find the volume, use the integral formula, first finding y^2, with the limits $x = 2$ and $x = 4$.	$y^2 = \left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}$ $V = \pi\int_2^4 \frac{x^4}{16} dx = \frac{\pi}{16}\left[\frac{x^5}{5}\right]_2^4 = \frac{\pi}{80}\left((4^5) - (2^5)\right) = \frac{62}{5}\pi.$ Therefore, the exact volume of this solid of revolution is $\frac{62}{5}\pi$.

Rotation About the y-Axis

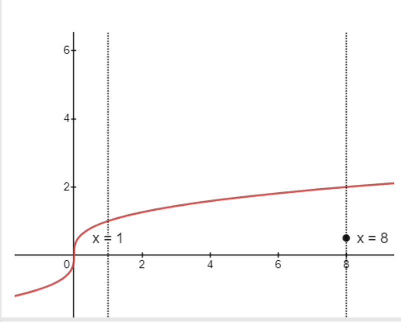
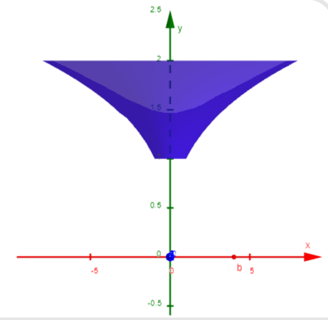
A similar formula is used when the curve is rotated about the y-axis, however this time the integral is evaluated with respect to y, and so the equation $y = f(x)$ will need to be rearranged to find x in terms of y. Additionally, if the portion of the curve that is rotated is given by x values, the corresponding y values will need to be found. These will be the limits of integration.

So, consider a curve $y = f(x)$ between the points $y = c$ and $y = d$. Rotating this curve 2π radians about the y-axis will produce a solid of revolution whose volume, V, is given by the following formula.

$$V = \pi\int_c^d x^2 dy.$$

The derivation of this formula is like that for rotation around the x-axis, with the orientation of the cylinders changing such that the widths of the cylinders are Δy, and the limit $\Delta y \to 0$ is considered.

Example 2: The curve $y = \sqrt[3]{x}$ is rotated 2π radians around the y-axis, between the points $x = 1$ and $x = 8$. Find the exact volume of the generated solid.

On the left is the graph of $y = \sqrt[3]{x}$ in the x-y plane, and on the right is the solid of revolution generated by rotating it 2π radians around the y-axis. This is the solid that we are calculating the volume of.	 
To use the above formula, the limits must be in terms of y. To find these limits, substitute the x values into $f(x)$ to find the values of y.	For $x = 1$, y takes the value $y = \sqrt[3]{1} = 1$ For $x = 8$, y takes the value $y = \sqrt[3]{8} = 2$ These are the lower and upper limits of the integral, respectively.
The integral is integrated with respect to y, and so it is necessary to find x in terms of y.	Rearranging $y = f(x)$ gives $y = \sqrt[3]{x} \Rightarrow x = y^3$
The volume of revolution is found using the integral formula.	With the upper limit as $y = 2$ and lower limit as $y = 1$, the integral evaluates as $V = \pi\int_1^2 (y^3)^2 dy = \int_1^2 y^6 dx = \pi\left[\frac{y^7}{7}\right]_1^2 = \frac{127\pi}{7}.$ So, the exact volume of this solid of revolution is $\frac{127\pi}{7}$.