Improper Integrals (A Level Only)

Integrals With Undefined Points

When integrating a function over an interval, it is possible that it will be undefined at one or more points within it. This point could be at either end of the interval, or somewhere in the middle. For example, the function $f(x) = \frac{1}{x}$ is undefined at the point $x = 0$ - and so the integral $\int_0^3 \frac{1}{x}dx$ is improper.

To tackle these types of questions the point at which the function is undefined, say at $x = k$, is replaced by a variable, say b, and the limit $b \to k$ is taken. For an integral $\int_a^c f(x)dx$, if the point at which $f(x)$ is undefined is at the beginning of the interval, when $x = a$, the following limit is calculated:

$$\int_a^c f(x)dx = \lim_{b\to a}\int_b^c f(x)dx.$$

Similarly, if $f(x)$ is undefined at the upper limit, the limit $b \to c$ is taken:

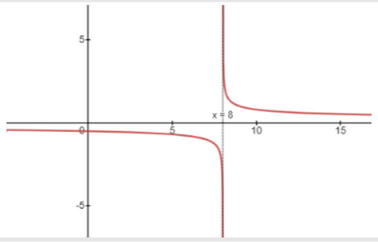
$$\int_a^c f(x)dx = \lim_{b\to c}\int_a^b f(x)dx.$$

When the integrand is undefined at $x = k$ between the integration limits, the following result is used:

$$\int_a^c f(x)dx = \lim_{b\to k}\int_a^b f(x)dx + \lim_{b\to k}\int_b^c f(x)dx$$

The integral is split into two, and a limit taken for each. The first takes the limit $b \to k$ 'from below' and the second 'from above'.

Example 1: Evaluate the integral $\int_0^{10} \frac{1}{\sqrt[3]{x-8}}dx$.

This is the graph of $y = \frac{1}{\sqrt[3]{x-8}}$ along with the line $x = 8$. The point where the function is undefined can be seen here as the point at which the graph 'jumps' from being below the x-axis to above it.	
This integral is improper as it is undefined at the point $x = 8$, leading to division by zero. Therefore, it is split into two integrals at $x = 8$, with the limit $b \to 8$ taken for both.	$\int_0^{10} \frac{1}{\sqrt[3]{x-8}}dx = \lim_{b\to8}\int_0^b \frac{1}{\sqrt[3]{x-8}}dx + \lim_{b\to8}\int_b^{10}\frac{1}{\sqrt[3]{x-8}}dx$
Evaluate each integral separately. It can be useful to rewrite the integrands with indices. Leave the answers in terms of b for now, ready for the limit to be taken in the next step.	$\int_0^b \frac{1}{\sqrt[3]{x-8}}dx = \int_0^b (x-8)^{-\frac{1}{3}}dx = \left[\frac{3}{2}(x-8)^{\frac{2}{3}}\right]_0^b$ $= \frac{3}{2}\left((b-8)^{\frac{2}{3}} - (-8)^{\frac{2}{3}}\right)$ $\int_b^{10} \frac{1}{\sqrt[3]{x-8}}dx = \int_b^{10}(x-8)^{-\frac{1}{3}}dx = \left[\frac{3}{2}(x-8)^{\frac{2}{3}}\right]_b^{10}$ $= \frac{3}{2}\left((2)^{\frac{2}{3}} - (b-8)^{\frac{2}{3}}\right)$
Evaluate each limit and sum them together to arrive at the final answer, leaving it in the simplest possible form.	$\lim_{b\to8}\int_0^b \frac{1}{\sqrt[3]{x-8}}dx = \lim_{b\to8}\frac{3}{2}\left((b-8)^{\frac{2}{3}} - (-8)^{\frac{2}{3}}\right) = -\frac{3}{2}(-8)^{\frac{2}{3}} = -6$ $\lim_{b\to8}\int_b^{10}\frac{1}{\sqrt[3]{x-8}}dx = \lim_{b\to8}\frac{3}{2}\left((2)^{\frac{2}{3}} - (b-8)^{\frac{2}{3}}\right) = \frac{3}{2}(2)^{\frac{2}{3}} = \frac{3}{\sqrt[3]{2}}$ $\therefore$ $\int_0^{10}\frac{1}{\sqrt[3]{x-8}}dx = \lim_{b\to8}\int_0^b \frac{1}{\sqrt[3]{x-8}}dx + \lim_{b\to8}\int_b^{10}\frac{1}{\sqrt[3]{x-8}}dx = \frac{3}{\sqrt[3]{2}} - 6$

Integrals With an Infinite Range

Sometimes one, or both, of the limits of integration will extend to infinity, with the upper limit approaching $+\infty$ and the lower $-\infty$. In certain cases, the integral will still be a finite number. If so, the integral is said to converge. If not, it is said to diverge.

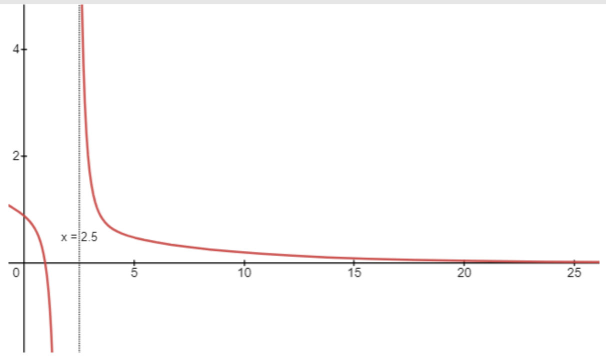
As in the previous case, the infinite limit is replaced by a variable, say b, and then the limit $b \to \pm\infty$ is considered. To evaluate the improper integral $\int_a^\infty f(x)dx$, the following limit is evaluated:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx = \lim_{b\to\infty}\{I(b) - I(a)\}$$

where $I(a)$ and $I(b)$ are the integral evaluated at $x = a$ and $x = b$ respectively. If the lower limit is $-\infty$, a similar approach is used, where we consider the limit as $a \to -\infty$. If both limits have an infinite range, the following is evaluated:

$$\int_{-\infty}^\infty f(x)dx = \lim_{b\to-\infty}\int_b^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx.$$

Example 2: Evaluate the integral $\int_{\frac{5}{2}}^\infty e^{-\frac{x}{6}} - \frac{1}{(2-x)^3}dx$.

The graph of $y = e^{-\frac{x}{6}} - \frac{1}{(2-x)^3}$ As x becomes larger, the graph flattens quickly, indicating that the area under the curve, and so the integral, will converge to a finite number. This function is undefined at $x = 2$, hence the steep increase as x approaches it.	
This integral is improper since its upper limit extends to infinity, and so this limit is replaced by the variable b. Since the lower limit is $\frac{5}{2} > 2$, the point where the function is undefined is avoided.	$\int_{\frac{5}{2}}^\infty e^{-\frac{x}{6}} - \frac{1}{(2-x)^3}dx = \lim_{b\to\infty}\int_{\frac{5}{2}}^b e^{-\frac{x}{6}} - \frac{1}{(2-x)^3}dx$ $= \lim_{b\to\infty}\int_{\frac{5}{2}}^b e^{-\frac{x}{6}}dx - \lim_{b\to\infty}\int_{\frac{5}{2}}^b \frac{1}{(2-x)^3}dx$
Evaluate each integral separately. The limits of each of these results are taken and summed to give the final answer. This answer can be found by remembering that $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty} x^{-k} = 0$.	$\int_{\frac{5}{2}}^b e^{-\frac{x}{6}}dx = -6\left[e^{-\frac{x}{6}}\right]_{\frac{5}{2}}^b = -6\left(e^{-\frac{b}{6}} - e^{-\frac{5}{12}}\right)$ $\int_{\frac{5}{2}}^b \frac{1}{(2-x)^3}dx = \frac{1}{2}\left[\frac{1}{(2-x)^2}\right]_{\frac{5}{2}}^b = \frac{1}{2}\left(\frac{1}{(2-b)^2} - 4\right)$ $\therefore \int_{\frac{5}{2}}^\infty e^{-\frac{x}{6}} - \frac{1}{(2-x)^3}dx = \lim_{b\to\infty}\left(-6\left(e^{-\frac{b}{6}} - e^{-\frac{5}{12}}\right) - \frac{1}{2}\left(\frac{1}{(2-b)^2} - 4\right)\right)$ $= 2\left(3e^{-\frac{5}{12}} + 1\right)$

Extremal Limits of Polynomial-Exponential and Polynomial-Logarithm Functions (A Level Only)

When evaluating improper integrals, the following two results are often used:

$$\lim_{x\to\infty} x^k e^{-x} = 0 \qquad \lim_{x\to0} x^k \ln(x) = 0.$$

These two relations are examples of an extremal limit of a polynomial-exponential function and of a polynomial-logarithm function respectively. These limits are often used to evaluate the limits used in improper integrals.

Example 3: Evaluate $\int_0^\infty 4xe^{-6x}\,dx$.

This integral is improper as the upper limit is infinite. So, the infinite limit is replaced by b and the limit $b \to \infty$ is taken.	$\int_0^\infty 4xe^{-6x}dx = \lim_{b\to\infty}\int_0^b 4xe^{-6x}dx$.
Evaluate the integral using integration by parts. Differentiating the $4x$ and integrating the e^{-6x} will lead to a simpler integral.	$u = 4x,\ \frac{dv}{dx} = e^{-6x} \Rightarrow \frac{du}{dx} = 4,\ v = \frac{-e^{-6x}}{6}$. $\int_0^b 4xe^{-6x}dx = \left[-\frac{4xe^{-6x}}{6}\right]_0^b + \frac{4}{6}\int_0^b e^{-6x}dx$ $= \left[-\frac{4xe^{-6x}}{6} - \frac{4}{36}e^{-6x}\right]_0^b$ $= -\frac{4be^{-6b}}{6} - \frac{4}{36}e^{-6b} + \frac{4}{36}$
Finally, take the limit of the result of the integral to get to the final answer. This includes the use of the limit: $\lim_{x\to\infty} x^k e^{-x} = 0$ with $k = 1$.	$\lim_{b\to\infty}\int_0^b 4xe^{-6x}dx = \lim_{b\to\infty}\left(-\frac{4}{6}be^{-6b} - \frac{4}{36}e^{-6b} + \frac{4}{36}\right)$ $= 0 + 0 + \frac{4}{36} = \frac{4}{36} = \frac{1}{9}$ $\therefore \int_0^\infty 4xe^{-6x}dx = \frac{1}{9}$

Example 4: Evaluate $\int_0^4 x^2\ln(x)\,dx$.

Since the point at which the function is undefined is at the lower limit of integration, the variable b replaces it and the limit $b \to 0$ is taken.	$\int_0^4 x^2\ln(x)\,dx = \lim_{b\to0}\int_b^4 x^2\ln(x)\,dx$.
Perform integration by parts to evaluate this integral. Differentiating $\ln(x)$ here rather than x^2 will lead to a simple integral.	$u = \ln(x),\ \frac{dv}{dx} = x^2 \Rightarrow \frac{du}{dx} = \frac{1}{x},\ v = \frac{x^3}{3}$ $\int_b^4 x^2\ln(x)\,dx = \left[\frac{1}{3}x^3\ln(x)\right]_b^4 - \int_b^4 \frac{1}{3}x^2dx$ $= \left[\frac{1}{3}x^3\ln(x) - \frac{x^3}{9}\right]_b^4 = \frac{64}{3}\ln(4) - \frac{64}{9} - \frac{b^3}{3}\ln(b) + \frac{b^3}{9}$
Take the limit $b \to 0$. The result $\lim_{x\to0} x^k\ln(x) = 0$ is used here, with $k = 3$.	$\lim_{b\to0}\int_b^4 x^2\ln(x)\,dx = \lim_{b\to0}\left(\frac{64}{3}\ln(4) - \frac{64}{9} - \frac{b^3}{3}\ln(b) + \frac{b^3}{9}\right)$ $= \frac{64}{3}\ln(4) - \frac{64}{9}$ $\therefore \int_0^4 x^2\ln(x)\,dx = \frac{64}{3}\ln(4) - \frac{64}{9}$