Further Algebra and Functions IV Cheat Sheet

AQA A Level Further Maths: Core

Graphs of Rational Functions with Linear Terms

A rational function with linear terms has the form:

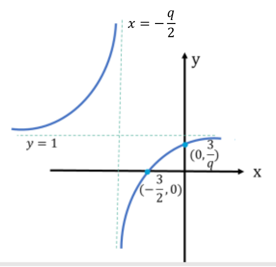
$$y = \frac{ax+b}{cx+d}$$

where $x \neq -\frac{d}{c}$. This is a transformation of the reciprocal graph $y = \frac{1}{x}$ and is a particular type of curve known as a hyperbola. The properties of the graph are useful to know to make sketching easier. The x-intercept occurs when $y = 0$, so $x = -\frac{b}{a}$. The y-intercept occurs when $x = 0$ and therefore $y = \frac{b}{d}$.

Finding Asymptotes

Rational function graphs have asymptotes. These are straight lines which the graph never touches but will approach as x or y tends towards infinity. The vertical asymptote occurs when the denominator is zero, so at $x = -\frac{d}{c}$. The horizontal asymptote occurs for large values of x, so the denominator can be approximated to cx and the numerator to ax. Therefore, the asymptote happens at $y = \frac{a}{c}$.

Example 1: Sketch the function $y = \frac{2x+3}{2x+q}$ where $q > 3$.

The intercepts are found by determining where $x = 0$ and $y = 0$.	When $x = 0$, $y = \frac{3}{q}$ When $y = 0$, $x = -\frac{3}{2}$
The asymptotes are found at $x = -\frac{d}{c}$. and $y = \frac{a}{c}$.	The asymptotes occur at $x = -\frac{q}{2}$ and $y = \frac{2}{2} = 1$
There is now enough information to sketch the graph with labelled points.	

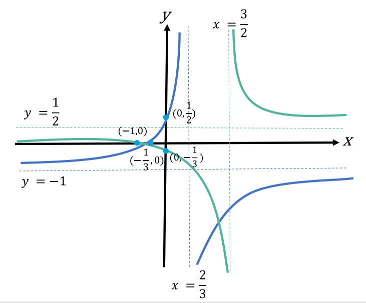
Finding Points of Intersection

Example 2: The function $f(x)$ is given by $f(x) = \frac{3x}{2x-1}$ and the line ℓ is given by $y = mx + 1$. Given that ℓ is a tangent to $f(x)$, calculate all possible values of m.

The points of intersection occur when the two equations are equal. It is possible to solve for a value of x from the quadratic equation.	$\frac{3x}{2x-1} = mx + 1$ $3x = 2mx^2 + 2x - mx - 1$ $0 = 2mx^2 - (m+1)x - 1$
The discriminant can then be used to determine the value of m. As the line ℓ forms a tangent, the discriminant equation to solve is $b^2 - 4ac = 0$. This equation can be solved in multiple ways. In this solution, the completing the square method has been used to solve the quadratic for m.	$(-(m+1))^2 - 4(2m)(-1) = 0$ $m^2 + 2m + 1 + 8m = 0$ $m^2 + 10m + 1 = 0$ $m = -\frac{10 \pm \sqrt{10^2 - 4(1)(1)}}{2}$ $m = -5 \pm 2\sqrt{6}$

Inequalities Involving Rational Functions with Linear Terms

Example 3: There are two functions: $f(x) = \frac{3x+1}{2-3x}$ and $g(x) = \frac{x+1}{2x-3}$. By sketching $g(x)$ and $f(x)$ on the same axes, determine the exact values where $f(x) > g(x)$.

The intercepts are found by determining where $x = 0$ and $y = 0$.	For $f(x)$, when $x = 0$, $y = \frac{1}{2}$ and $y = 0, x = -\frac{1}{3}$ For $g(x)$, when $x = 0$, $y = -\frac{1}{3}$ and $y = 0, x = -1$
The asymptotes are found at $x = -\frac{d}{c}$. and $y = \frac{a}{c}$.	For $f(x)$, the asymptotes occur at $x = \frac{2}{3}, y = -\frac{3}{3} = -1$ For g(x), the asymptotes occur at $x = \frac{3}{2}, y = \frac{1}{2}$
Using these details, the two graphs can be sketched out with the details that as the graphs approach the asymptotes they tend towards infinity. You need to ensure the asymptotes and axes intercepts are labelled.	
To solve the inequality, it is important to use the relationship of the inequalities and solve for x. This provides two solutions which form the limits of the solution. The other limits come from the asymptotes at $x = \frac{3}{2}$ and $x = \frac{2}{3}$ because the values of x must be less than these due to the nature of asymptotes.	$\frac{3x+1}{2-3x} > \frac{x+1}{2x-3}$ $6x^2 - 7x - 3 = -3x^2 - x + 2$ $9x^2 - 6x - 5 = 0$ $x = \frac{6 \pm \sqrt{6^2 - 4(9)(-5)}}{18}$ $x = \frac{1}{3} \pm \sqrt{\frac{2}{3}}$ Therefore, the solution is $\frac{1}{3} - \sqrt{\frac{2}{3}} < x < \frac{2}{3}$ and $\frac{1}{3} + \sqrt{\frac{2}{3}} < x < \frac{3}{2}$

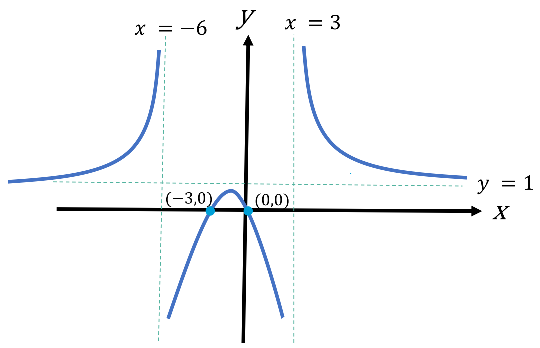
Graphs of Rational Functions with Quadratic Terms

Some rational functions can contain quadratic expressions and have the form:

$$y = \frac{ax^2+bx+c}{dx^2+ex+f}.$$

For these graphs, the y-intercept occurs when $y = \frac{c}{f}$ and the x-intercept occurs when $ax^2 + bx + c = 0$. It is also possible to investigate the y-values for the function by letting $y = k$. This forms a quadratic equation for which you can use the quadratic discriminant to determine if there is a solution for the equation. The vertical asymptotes occur for the solutions to $dx^2 + ex + f = 0$. The horizontal asymptote occurs for large values for x, so when $y = \frac{a}{d}$.

Example 4: Sketch $y = \frac{x^2+3x}{x^2+3x-18}$ including the asymptotes and axes intercepts.

When sketching the graph, it is important to remember to include features of the graph including the asymptotes and intercepts. The vertical asymptotes occur at solutions to $dx^2 + ex + f = 0$ and the horizontal when $y = \frac{a}{d}$.	The axes intercepts occurs at (0,0) and (−3,0). The asymptotes occur at $x = -6, x = 3$ and $y = -1$. 

Example 5: Find the stationary points for $y = \frac{x^2 - 4x + 4}{x^2 - 9}$.

Let $y = k$ and rearrange to find a quadratic expression.	Let $y = k$: $k = \frac{x^2 - 4x + 4}{x^2 - 9}$ $kx^2 - 9k = x^2 - 4x + 4$ $(k-1)x^2 + 4x - (9k+4) = 0$
Using the discriminant ($\Delta = b^2 - 4ac$), find the range of values for k and determine where the stationary points are.	$\Delta = 4^2 + 4(k-1)(9k+4)$ $= 36k^2 - 20k$ $= 2k(18k - 10)$ There are solutions when $k < 0$ and $k > \frac{5}{9}$.
The solutions suggest that there are no y-values between 0 and $\frac{5}{9}$. This suggests when $y = 0$ there must be a maximum and when $y = \frac{5}{9}$ there must be a minimum. The values for x at these coordinates can be found by substituting them back into the equation.	A maximum occurs when $y = 0$ and $x = 2$. A minimum occurs when $y = \frac{5}{9}$ and $x = \frac{9}{2}$.

Finding Oblique Asymptotes (A-Level Only)

If in the quadratic rational function form $d = 0$, then the rule that the horizontal asymptote occurs at $y = \frac{a}{d}$ is no longer applicable. Instead, the rational function needs to be simplified into a polynomial and a rational function. In terms of the asymptote, this means that for large values of x, the graph does not tend towards a constant and the asymptote is a non-horizontal line. This type of line is known as an oblique asymptote.

Example 6: A curve C is given by $f(x) = \frac{2x^2-5x-1}{x-3}$.

- **a)** Find the oblique asymptote equation for C.
- **b)** By finding a condition on for the number of solutions to find the coordinates of any stationary points for C.
- **c)** Sketch a graph of $f(x)$ including any stationary points and asymptotes.

a) The quadratic rational function can be rewritten in the form $Ax + B + \frac{C}{x-3}$ using polynomial long division or by comparing the coefficients of the two forms. The oblique asymptote equation occurs at the leftover rational function $\frac{2}{x-3}$. This is true because when the values for x are very large, this term is very small.	$f(x) = \frac{2x^2 - 5x - 1}{x - 3} = Ax + B + \frac{C}{x-3}$ $\frac{2x^2 - 5x - 1}{x-3} = \frac{2x^2 - 5x - 3 + 2}{x - 3}$ $= \frac{(2x+1)(x-3) + 2}{x-3}$ $= 2x + 1 + \frac{2}{x-3}$ Therefore $A = 2, B = 1$ and $C = 2$. The oblique asymptote is at $y = 2x + 1$.
b) The stationary points occur where $f'(x) = 0$. From this, it is possible to determine the x-coordinates of the stationary points and substitute those values back into $f(x)$ to find the corresponding y-coordinate.	$f'(x) = \frac{2(x^2 - 6x + 8)}{(x-3)^2} = 0$ $x^2 - 6x + 8 = (x-4)(x-2) = 0$ When $x = 2$, $f(2) = 3$ and when $x = 4$, $f(4) = 11$ The stationary points occur at $(2,3)$ and $(4,11)$.
c) Using the details found in part **a)** and **b)**, sketch the graph. Remember the oblique asymptote is a diagonal line.