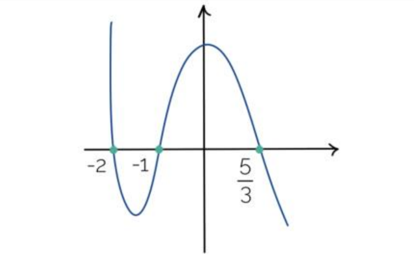
Further Algebra and Functions III Cheat Sheet

Inequalities Involving Polynomial Equations

This section will be largely familiar and looks at sketching graphs and solving inequalities for polynomial equations. Cubic and quartic inequalities are solved in precisely the same way as quadratic inequalities.

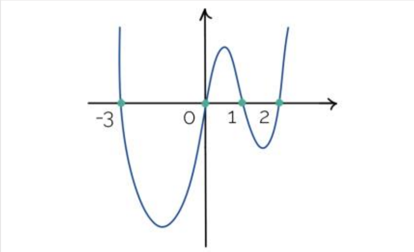
Cubic Inequalities

Example 1: Solve $(5-3x)(x+2)(x+1) \geq 0$.

Sketch the graph of: $y=(5-3x)(x+2)(x+1)$ by plotting the roots of the equation and recognising that a cubic with a negative coefficient of x^3 will start in the upper-left quadrant.	
Write the regions of the graph that are touching or above the y-axis.	$x \leq -2$ or $-1 \leq x \leq \frac{5}{3}$

Quartic Inequalities

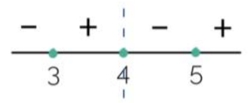
Example 2: Find the values of x for which $x^4+6x<7x^2$.

Rearrange the equation to have the RHS be 0.	$x^4-7x^2+6x<0$
Factorise out the x. Notice that $f(1)=0$, where $f(x)=x^3-7x+6$, so $(x-1)$ must be a factor of $f(x)$ by the factor theorem. Use this to fully factorise the LHS.	$x(x^3-7x+6)<0$ $x(x-1)(x^2+x-6)<0$ $x(x-1)(x-2)(x+3)<0$
Sketch the graph of: $y=x(x-1)(x-2)(x+3)$ by plotting the roots of the equation and recognising that a quartic with a positive coefficient of x^4 will start in the upper-left quadrant.	
Write the regions of the graph that are below the y-axis.	$-3<x<0$ or $1<x<2$.

Solving Inequalities Algebraically

Inequalities should not be algebraically solved by cross-multiplying, as with an equation. This is because if any denominators involve x, they could be positive or negative, which would cause the inequality sign to flip. Instead, move all terms onto one side by subtraction or addition and simplify the expression from there. An alternative (but often longer) method is to multiply both sides by the square of the denominator.

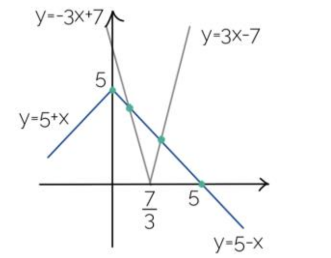
Example 3: Solve the inequality $\frac{3x-11}{x-4} \geq x-1$.

Rearrange the inequality to have everything on the LHS.	$\frac{3x-11}{x-4}+(1-x) \geq 0$
Rewrite the LHS as one simplified fraction.	$\frac{3x-11}{x-4}+\frac{(1-x)(x-4)}{x-4} \geq 0$ $\frac{3x-11+(1-x)(x-4)}{x-4} \geq 0$ $\frac{3x-11+(-x^2+5x-4)}{x-4} \geq 0$ $\frac{3x-11-x^2+5x-4}{x-4} \geq 0$ $\frac{-x^2+8x-15}{x-4} \geq 0$ $\frac{x^2-8x+15}{x-4} \leq 0$
Factorise the numerator.	$\frac{(x-3)(x-5)}{x-4} \leq 0$
Mark the critical points on a number line. Plug values in between the critical points into $\frac{(x-3)(x-5)}{x-4}$ and mark whether the output is positive or negative.	
The solutions will be the regions where the output is negative.	$x \leq 3$ or $4 \leq x \leq 5$

Inequalities Involving the Modulus of Functions (A-Level Only)

The modulus (or absolute value) function is represented by two vertical lines. For instance, the modulus of $y=f(x)$ is written as $y=|f(x)|$. This results in any y values that were previously below the x-axis being reflected in the x-axis. The part of the graph that has been reflected will hence have equation $y=-f(x)$. Questions involving moduli are best solved by means of a sketch.

Example 4: Solve $|3x-7|<5-|x|$.

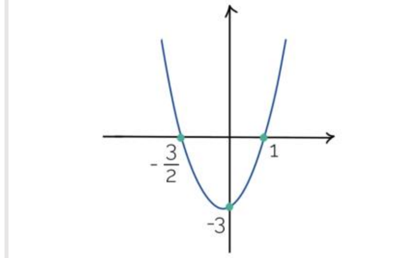
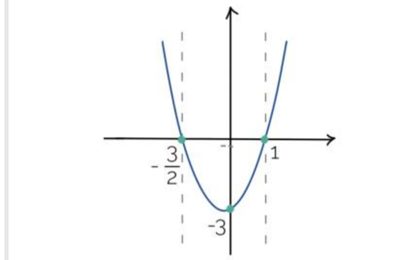
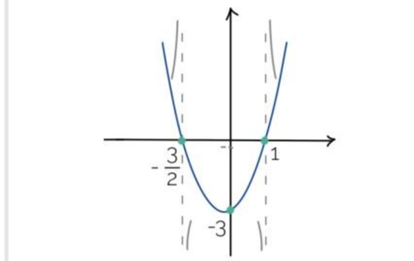
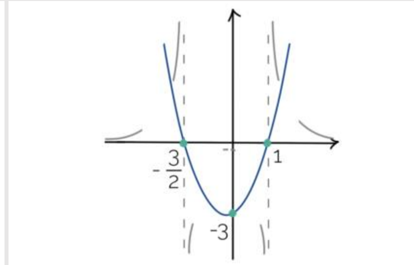
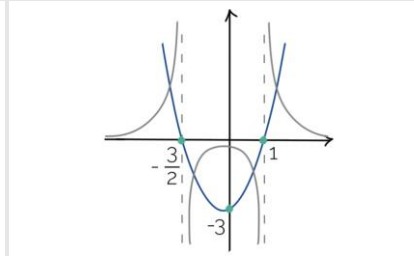
Sketch graphs of $y=|3x-7|$ and $y=5-|x|$ on one set of axes, labelling each half of a graph with their respective equations. Note that the graph of $y=5-|x|$ is sketched by first drawing $y=x$, then reflecting all the negative y-values in the x-axis to yield $y=|x|$. After, reflect in the x-axis to get $y=-|x|$, before finally translating by $\begin{bmatrix}0\\5\end{bmatrix}$ for $y=5-|x|$.	
Notice that the points of intersection are found when $-3x+7=5-x$ and $3x-7=5-x$. Solve each equation separately.	$-3x+7=5-x$ $\Rightarrow 2x=2$ $\Rightarrow x=1$ $3x-7=5-x$ $\Rightarrow 4x=12$ $\Rightarrow x=3$
Identify the region where the graph of $y=|3x-7|$ is below the graph of $y=5-|x|$.	$1<x<3$

Graphs of Reciprocal and Modulus Functions (A-Level Only)

Given a function $f(x)$, it is possible to sketch its reciprocal $\frac{1}{f(x)}$ purely from the graph of $f(x)$ alone. Some key points are needed to sketch reciprocal graphs. These points are summarised in the following table.

$f(x)$	$\frac{1}{f(x)}$
$f(a)=0$	Vertical asymptote at $x=a$
Vertical asymptote at $x=a$	$\frac{1}{f(a)}=0$
Horizontal asymptote at $y=a$	Horizontal asymptote at $y=\frac{1}{a}$
$f(x)>0$	$\frac{1}{f(x)}>0$
$f(x)<0$	$\frac{1}{f(x)}<0$
$f(x)\to 0$	$\frac{1}{f(x)}\to\infty$ or $\frac{1}{f(x)}\to-\infty$
$f(x)\to+\infty$	$\frac{1}{f(x)}\to 0$ (from above)
$f(x)\to-\infty$	$\frac{1}{f(x)}\to 0$ (from below)
Local maximum at $(a,f(a))$	Local minimum at $\left(a,\frac{1}{f(a)}\right)$
Local minimum at $(a,f(a))$	Local maximum at $\left(a,\frac{1}{f(a)}\right)$
a is the y-intercept	$\frac{1}{a}$ is the y-intercept

Example 5: If $f(x)=2x^2+x-3$, $g(x)=|x|$, sketch **a)** the graph of $y=\frac{1}{f(x)}$, **b)** the graph of $y=gf(x)$.

Factorise $f(x)$ to find its roots. Find the minimum of $f(x)$ by writing $f(x)$ in completed-square form.	$f(x)=(2x+3)(x-1)$ $f(x)=2\left(x+\frac{1}{2}\right)^2-\frac{25}{8}$ Minimum at $\left(-\frac{1}{2},-\frac{25}{8}\right)$
Sketch $f(x)$.	
a) Draw vertical asymptotes where $f(x)=0$. Also, sketch on the y-intercept $\left(\frac{1}{-3}\right)$ and local maximum $\left(-\frac{1}{4},-\frac{8}{25}\right)$ of $\frac{1}{f(x)}$.	
Near the roots of $f(x)$, $\frac{1}{f(x)}$ will be very large, tending to infinity at the vertical asymptotes. Note that $\frac{1}{f(x)}$ will be positive where $f(x)>0$ and negative where $f(x)<0$. Add this to the sketch.	
$f(x)\to+\infty$ in both directions, so $\frac{1}{f(x)}\to 0$ in both directions from above.	
Join up the lines with smooth curves.	
b) Identify that $f(x)$ is negative for values: $-\frac{3}{2}<x<1$. Reflect $f(x)$ for these values in the x-axis to yield the graph of $y=gf(x)=|f(x)|$.