Further Algebra and Functions II Cheat Sheet

Maclaurin Series of a Function and the General Term (A-Level Only)

The Maclaurin series is a method of approximating a function by expressing the function as an infinite series polynomial.

Let this function $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

Differentiating yields:

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$

$f''(x) = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + 5 \times 4a_5x^3 + \cdots$

$f^3(x) = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + 5 \times 4 \times 3a_5x^2 + 6 \times 5 \times 4a_6x^3 + \cdots$

$f^4(x) = 4 \times 3 \times 2a_4 + 5 \times 4 \times 3 \times 2a_5x + 6 \times 5 \times 4 \times 3a_6x^2 + 7 \times 6 \times 5 \times 4a_7x^3 + \cdots$

$\vdots$

Evaluating $f^n(x)$ at $x = 0$:

$f(0) = a_0$

$f'(0) = a_1$

$f''(0) = 2a_2$

$f^3(0) = 3 \times 2a_3$

$f^4(0) = 4 \times 3 \times 2a_4$

$\vdots$

It becomes clear that $f^n(0) = n!\,a_n$. Hence, $a_n = \frac{f^n(0)}{n!}$.

$f(x)$ can thus be rewritten as $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^3(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n$. This is the general formula for the Maclaurin series of a function $f(x)$ up to the term in x^n. The **general term** is $\frac{f^n(0)}{n!}x^n$.

Note that the assumption that $f(x)$ can be written as an infinite series is only true if the expansion converges. Furthermore, a Maclaurin series is only valid if $f^n(0)$ is defined for all integer values of $n \geq 0$. For instance, it is not possible to write a Maclaurin series for $ln(x)$, as $f'(0) = \frac{1}{0}$ is undefined. However, it is possible to write a Maclaurin series for $ln(1+x)$, as $f^n(0)$ is defined for all $n \in \mathbb{Z}^+$.

The **range of validity** is the values for which a Maclaurin expansion will converge. For questions where no range of validity is given, it is fair to assume the expression is valid for all $x \in \mathbb{R}$.

Example 1: Find the first three terms and the general term of the Maclaurin series for $f(x) = (2x+1)^{-1}$.

Calculate the first few derivatives of $f(x)$ using the chain rule. The more terms calculated, the easier it will be to spot the general term.	$f(x) = (2x+1)^{-1}$ $f'(x) = (2)(-1)(2x+1)^{-2} = -2(2x+1)^{-2}$ $f''(x) = (-2)(2)(-2)(2x+1)^{-3} = 8(2x+1)^{-3}$ $f^3(x) = (8)(2)(-3)(2x+1)^{-4} = -48(2x+1)^{-4}$ $f^4(x) = (-48)(2)(-4)(2x+1)^{-5} = 384(2x+1)^{-5}$
Evaluate each derivative at $x = 0$.	$f(0) = (2 \times 0 + 1)^{-1} = 1$ $f'(0) = -2(2 \times 0 + 1)^{-2} = -2$ $f''(0) = 8(2 \times 0 + 1)^{-3} = 8$ $f^3(0) = -48(2 \times 0 + 1)^{-4} = -48$ $f^4(0) = 384(2 \times 0 + 1)^{-5} = 384$
Use the general formula for the Maclaurin series to write the first few terms of $f(x)$.	$f(x) = 1 + (-2)x + \frac{8}{2!}x^2 + \frac{-48}{3!}x^3 + \frac{384}{4!}x^4 + \cdots$ $= 1 - 2x + 4x^2 - 8x^3 + 16x^4 + \cdots$
Identify that the coefficients of x^n form a geometric sequence with first term $a = 1$ and common ratio $r = -2$. Use the n^{th} term formula for a geometric sequence ($u_n = ar^{n-1}$) to write the general term for the Maclaurin series.	First three terms: $1, -2x, 4x^2$ General term: $1(-2)^{n-1}x^n = (-2)^{n-1}x^n$

Maclaurin Series of Standard Functions

The table below shows the standard functions given in the data booklet, alongside their ranges of validity. These can be used to find the Maclaurin series for expressions that typically involve products of functions or otherwise tedious differentiation.

Maclaurin Series	Range of Validity
$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots$	All x
$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{r+1}\frac{x^r}{r} + \cdots$	$-1 < x \leq 1$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots$	All x
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots$	All x
$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$	$\lvert x \rvert < 1, r \in \mathbb{R}$

Example 2: Show that the Maclaurin series for $(1+x)e^x$, as far as the term in x^3 is $1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3$.

Rewrite the expression using the Maclaurin series for e^x.	$(1+x)e^x = (1+x)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$
Expand the brackets. Note that it is not necessary to include the $\frac{x^4}{3!}$ term and beyond, as the question only asks for the expansion up to the term in x^3.	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + x + x^2 + \frac{x^3}{2!}$
Collect like terms and simplify to acquire the necessary result.	$= 1 + 2x + \left(\frac{1}{2!} + 1\right)x^2 + \left(\frac{1}{3!} + \frac{1}{2!}\right)x^3$ $\therefore (1+x)e^x = 1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3$

Example 3: Find the first three non-zero terms in the series expansion of $ln\left(\frac{1-5x}{\sqrt{1+4x}}\right)$, and state the values of x for which the expansion is valid.

Rewrite the expression using the log rules: $ln\left(\frac{a}{b}\right) = lna - lnb$ and $lna^b = blna$.	$ln\left(\frac{1-5x}{\sqrt{1+4x}}\right) = ln(1-5x) - ln(\sqrt{1+4x})$ $= ln(1-5x) - \frac{1}{2}ln(1+4x)$
Substitute $x = 5x$ and $x = 4x$ respectively into the Maclaurin series for $ln(1+x)$.	$= \left(-5x - \frac{(-5x)^2}{2} + \cdots\right) - \frac{1}{2}\left(4x - \frac{(4x)^2}{2} + \cdots\right)$
Expand and simplify. Then collect like terms.	$= \left(-5x - \frac{25}{2}x^2 - \frac{125}{3}x^3 - \cdots\right) - \frac{1}{2}\left(4x - \frac{16}{2}x^2 + \frac{64}{3}x^3 - \cdots\right)$ $= \left(-5x - \frac{25}{2}x^2 - \frac{125}{3}x^3 - \cdots\right) - 2x + 4x^2 - \frac{32}{3}x^3 + \cdots$ $= -7x - \frac{17}{2}x^2 - \frac{157}{3}x^3 - \cdots$
Find the range of validity for $ln(1-5x)$ and $ln(1+4x)$ by letting $x = -5x$ and $x = 4x$ respectively in the range of validity for $ln(1+x)$. Note that dividing by -5 reverses the inequalities.	$ln(1-5x)$ is valid for $-1 < -5x \leq 1$ $\Rightarrow -\frac{1}{5} \leq x < \frac{1}{5}$ $ln(1+4x)$ is valid for $-1 < 4x \leq 1$ $\Rightarrow -\frac{1}{4} < x \leq \frac{1}{4}$
Choose the inequality that satisfies both intervals.	$\therefore$ The expansion is valid for $-\frac{1}{5} \leq x < \frac{1}{5}$.

Evaluation of Limits (A-Level Only)

The idea of limits was first introduced in the topic of *differentiation from first principles*. Just as it is possible to find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, it is possible to find $\lim_{x \to c} f(x)$. When $f(c)$ can be properly evaluated, it is valid to find $\lim_{x \to c} f(x)$ by a direct substitution of $x = c$. In cases where $f(c)$ leads to an indeterminant form of $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, other methods will have to be employed. This section will explore two methods of evaluating such limits: using Maclaurin series and L'Hôpital's rule.

Using Maclaurin Series

Maclaurin series can be used when wanting to find the limit of a function that involves composite standard functions.

Example 4: Using Maclaurin series, find $\lim_{x \to 0} \frac{e^x - 1}{ln(2x+1)}$.

Use the Maclaurin series for e^x: $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots$ in place of e^x. Similarly, use the Maclaurin series for $ln(1+2x)$: $2x - \frac{(2x)^2}{2} + \cdots + (-1)^{r+1}\frac{(2x)^r}{r} + \cdots$ in place of $ln(2x+1)$. Note that the Maclaurin series for $ln(1+2x)$ is found by letting $x = 2x$ in the Maclaurin series for $ln(1+x)$.	$\lim_{x \to 0} \frac{e^x - 1}{ln(2x+1)} = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \cdots\right) - 1}{2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \cdots}$
Simplify the expression.	$= \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \cdots}{2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \cdots}$ $= \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \cdots}{2x - 2x^2 + \frac{8x^3}{3} - \cdots}$ $= \lim_{x \to 0} \frac{6x + 3x^2 + \ldots}{12x - 12x^2 + 16x^3 - \cdots}$ $= \lim_{x \to 0} \frac{6 + 3x^2 + \ldots}{12 - 12x + 16x^2 - \cdots}$
Find the required limit by substituting $x = 0$ into the fraction.	$= \frac{6 + 3(0)^2 + \cdots}{12 - 12(0) + 16(0)^2 - \cdots}$ $= \frac{1}{2}$

Using L'Hôpital's Rule

L'Hôpital's rule provides a way to find the limit of a function of the form $\frac{f(x)}{g(x)}$, when $\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{f(x)}{g(x)} = \pm\frac{\infty}{\infty}$. L'Hôpital's rule states that:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

and may only be used when $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ can be properly evaluated at c.

Example 5: Using L'Hôpital's rule, find $\lim_{x \to 0} \frac{e^x - 1}{ln(2x+1)}$.

Rewrite the limit using L'Hôpital's rule, where $f(x) = e^x - 1$ and $g(x) = ln(2x+1)$.	$\lim_{x \to 0} \frac{e^x - 1}{ln(2x+1)} = \lim_{x \to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(ln(2x+1))}$
Calculate the derivatives, using the chain rule to find $\frac{d}{dx}(ln(2x+1))$. Simplify the expression.	$= \lim_{x \to 0} \frac{e^x}{\frac{2}{2x+1}} = \lim_{x \to 0} \frac{e^x(2x+1)}{2}$
Find the required limit by substituting $x = 0$ into the fraction.	$= \frac{e^0(2 \times 0 + 1)}{2} = \frac{1}{2}$