Further Algebra and Functions I Cheat Sheet

AQA A Level Further Maths: Core

Relationship Between the Roots and Coefficients of a Polynomial

The roots of a polynomial are the points at which the curve crosses the x-axis. In the 15th century, French mathematician François Viète discovered a connection between the sums and products of the roots of a polynomial and its coefficients.

Quadratic Equations

For a quadratic equation of the form $ax^2 + bx + c = 0$ with roots α, β, where $a \neq 0$:

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

Cubic Equations

For a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ, where $a \neq 0$:

- $\alpha + \beta + \gamma = -\frac{b}{a}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
- $\alpha\beta\gamma = -\frac{d}{a}$

Quartic Equations

For a quartic equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α, β, γ, δ, where $a \neq 0$:

- $\sum\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$
- $\sum\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
- $\sum\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
- $\sum\alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{e}{a}$

Example 1: α, β, γ and δ are the roots of the quartic equation $4x^4 + 7x^3 + 8x^2 - x - 12 = 0$. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$.

Rewrite $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ using the expressions that relate the roots of the polynomial to its coefficients.	$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\sum\alpha\beta\gamma}{\sum\alpha\beta\gamma\delta}$
Calculate $\sum\alpha\beta\gamma$ and $\sum\alpha\beta\gamma\delta$. Use the expression from the first line of working to find the required result.	$\sum\alpha\beta\gamma = -\frac{(-1)}{4} = \frac{1}{4}$ $\sum\alpha\beta\gamma\delta = \frac{-12}{4} = -3$ $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\frac{1}{4}}{-3} = -\frac{1}{12}$

Linear Transformation of the Roots of a Polynomial

It is possible to form a new polynomial equation whose roots are a linear transformation of a given polynomial equation.

There are two common methods to tackle these problems, but as the AQA A-Level Further Mathematics course just involves linear transformations of cubics and quartics, the substitution method is generally most suitable. This method is used in the example below.

Example 2: $3z^3 + z^2 - 4z + 1 = 0$ has roots α, β, and γ. Find a new polynomial with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$.

Define w to be one of the roots of the original polynomial. The first root has been chosen here.	Let $w = 2\alpha + 1$ $\therefore \alpha = \frac{w-1}{2}$
$f(\alpha) = 0$ by the factor theorem. Hence, substitute α into the original polynomial to derive a new polynomial in terms of w.	$3\left(\frac{w-1}{2}\right)^3 + \left(\frac{w-1}{2}\right)^2 - 4\left(\frac{w-1}{2}\right) + 1 = 0$
Expand each bracket and leave the new polynomial in simplified form.	$3\left(\frac{w^3 - 3w^2 + 3w - 1}{8}\right) + \left(\frac{w^2 - 2w + 1}{4}\right) - 4\left(\frac{w-1}{2}\right) + 1 = 0$ $3(w^3 - 3w^2 + 3w - 1) + 2(w^2 - 2w + 1) - 16(w - 1) + 8 = 0$ $3w^3 - 9w^2 + 9w - 3 + 2w^2 - 4w + 2 - 16w + 16 + 8 = 0$ $3w^3 - 7w^2 - 11w + 23 = 0$

Series as a Summation

A series is the sum of the terms in a given sequence. In general, $S_n = u_1 + u_2 + u_3 + \cdots + u_n$, where S_n denotes the sum of the first n terms of the sequence, and u_n denotes the n^{th} term of the sequence. Using sigma notation, S_n can be more efficiently written as: $\sum_{r=1}^{n} u_r$. In words, this means to sum u_r from 1 to n.

This section will look at problems involving summing series from the following standard results:

- $\sum_{r=1}^{n} 1 = n$
- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
- $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

These are given in the data booklet.

Sums of Integers

Example 3: Given that $f(r) = ar + b$ and $\sum_{r=1}^{n} f(r) = 3n^2 + 7n$, find the constants a and b.

Substitute $f(r) = ar + b$ into the summation.	$\sum_{r=1}^{n} ar + b = 3n^2 + 7n$
Apply the distributive properties that $\sum(u_r + v_r) = \sum u_r + \sum v_r$ and $\sum cu_r = c\sum u_r$ to the LHS.	$a\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1 = 3n^2 + 7n$
Rewrite the summations using the standard results for natural numbers.	$a\left[\frac{1}{2}n(n+1)\right] + b(n) = 3n^2 + 7n$
Expand and simplify the LHS.	$\frac{a}{2}n^2 + \frac{a}{2}n + bn = 3n^2 + 7n$
Group the like terms and compare coefficients to identify a and b. It can be useful to check answers using the sum function on a calculator.	$\frac{a}{2}n^2 + \left(\frac{a}{2} + b\right)n = 3n^2 + 7n$ Comparing coefficients: $\frac{a}{2} = 3 \Rightarrow a = 6$ $\frac{a}{2} + b = 7 \Rightarrow 3 + b = 7 \Rightarrow b = 4$

Sums of Squares and Cubes

Example 4: Find $\sum_{r=n+1}^{2n} r^2$.

Write the expression as a subtraction involving two summations, both with a lower limit of $r = 1$.	$\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$
Rewrite the summations using the standard result for the sum of squares.	$\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1)$
Factorise and simplify the RHS.	$\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)[2(4n+1) - (n+1)]$ $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)[8n + 2 - n - 1]$ $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$

Method of Differences

Take a series of the general form $\sum_{r=1}^{n} f(r+1) - f(r)$. Writing the summation term-by-term yields:

$$\big(f(2) - f(1)\big) + \big(f(3) - f(2)\big) + \cdots + \big(f(n) - f(n-1)\big) + \big(f(n+1) - f(n)\big).$$

It quickly becomes apparent that most terms will cancel each other out, leaving $\sum_{r=1}^{n} f(r+1) - f(r) = f(n+1) - f(1)$. This is the basis behind the *method of differences*.

Method of Differences for General Numeric and Algebraic Series

Example 5: $\sum_{r=1}^{n} 2r = \sum_{r=1}^{n}[r(r+1) - (r-1)r]$. Use this result and the method of differences to prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Use the result given to derive an expression for $\sum_{r=1}^{n} r$ by dividing both sides by 2.	$\sum_{r=1}^{n} 2r = \sum_{r=1}^{n}[r(r+1) - (r-1)r]$ $\Rightarrow \sum_{r=1}^{n} r = \frac{1}{2}\sum_{r=1}^{n}[r(r+1) - (r-1)r]$
Calculate the first few terms and the last few terms until a pattern is spotted. This is done by substituting $r = 1, r = 2$, and $r = n - 1, r = n - 1$ separately. Avoid simplifying the pairs of terms to spot the pattern more easily.	Using the method of differences: $\sum_{r=1}^{n} r = \frac{1}{2}[((1 \times 2) - (0 \times 1)) + ((2 \times 3) - (1 \times 2)) + \cdots$ $+ ((n-1)n - (n-2)(n-1)) + (n(n+1) - (n-1)n)$
As the (1×2) terms cancel out from the first and second pairs of terms, it becomes clear that the left term of a pair of terms will cancel out with the right term of the next pair of terms. From this, it is possible to reduce the summation as shown. Physically crossing out terms will be useful.	Using the method of differences: $= \frac{1}{2}[((1 \times 2) - (0 \times 1)) + ((2 \times 3) - (1 \times 2)) + \cdots$ $+ ((n-1)n - (n-2)(n-1)) + (n(n+1) - (n-1)n)]$ $= \frac{1}{2}[0 + n(n+1)]$ $= \frac{1}{2}n(n+1)$ $\therefore \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, as required.

Method of Differences Involving Partial Fractions (A-Level Only)

Example 6: **a)** Express $\frac{2}{k(k+2)}$ in partial fractions. **b)** Hence find $\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$.

a.) Split the fraction into partial fractions by the standard method.	$\frac{2}{k(k+2)} \equiv \frac{A}{k} + \frac{B}{k+2}$ $2 \equiv A(k+2) + B(k)$ Let $k = 0$: $2 = 2A \Rightarrow A = 1$ Let $k = -2$ $2 = -2B \Rightarrow B = -1$ $\frac{2}{k(k+2)} \equiv \frac{1}{k} - \frac{1}{k+2}$
b.) Use the partial fractions to sum the series, not immediately simplifying the pairs of terms. Identify that the $\frac{1}{3}$ terms cancel out from the first and third pairs of terms. This means the right term of a pair of terms will always cancel out with the left term of the pair of terms that appears two pairs of terms later.	$\sum_{k=1}^{n} \frac{2}{k(k+2)} = \sum_{k=1}^{n}\left[\frac{1}{k} - \frac{1}{k+2}\right]$ Using the method of differences: $= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots$ $+ \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)$ $= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$
Note that as n tends to infinity, $\frac{1}{n+1}$ and $\frac{1}{n+2}$ both become progressively smaller, tending to 0. Hence, only the constant term will remain.	As $n \to \infty$, $\sum_{k=1}^{n} \frac{2}{k(k+2)} \to \infty$. $\therefore \sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \frac{3}{2}$