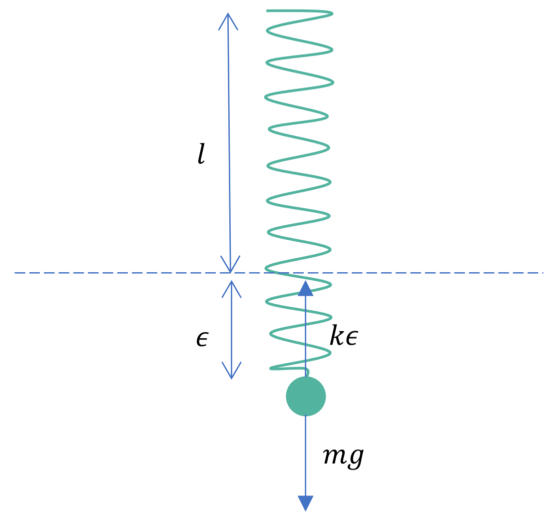
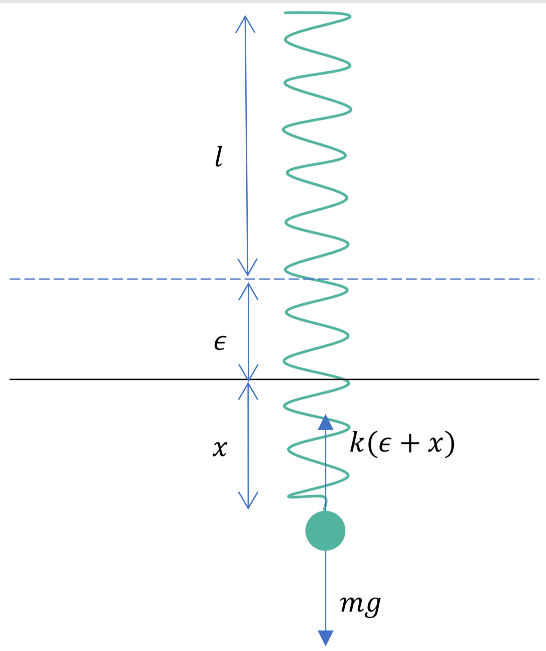
Using Hooke's Law to Formulate Differential Equations for Simple Harmonic Motion

The process of observing a system and modelling it using a differential equation requires some knowledge of the forces acting in the system. In springs, when they are compressed or stretched, they exert a restoring force to bring the length of the spring back to its natural length (the natural length is the length of the spring when it is not acted on by any tension or compression forces). This force can be described by Hooke's Law which states $F = kx$, the restorative force F exerted on a spring is proportional to the extension x acting in the opposite direction to the extension. Notice this also describes compression which can be thought of as negative extension. The constant k is known as to as the spring constant or stiffness. Using Hooke's Law in combination with Newton's 2nd Law ($F = ma$), differential equations for simple harmonic motion in a spring can be formulated.

Example 1: A light spring is attached to a fixed point and a 2kg mass is attached vertically below to the other end of the spring. The spring has a spring constant of $k = 80\ \text{Nm}^{-1}$ and natural length $l = 0.6\text{m}$.

a) Find the equilibrium length of the spring.

b) If at $t = 0$ the spring is then pulled down a further 0.2m and released from rest show that the spring undergoes simple harmonic motion and find the particular solution

a) Draw a diagram of the situation at equilibrium.	
Form an equation and substitute in values to the find equilibrium length $l + \epsilon$.	$k\epsilon = mg \Rightarrow \epsilon = \frac{mg}{k} = 0.25\text{m} \Rightarrow l + \epsilon = 0.85\text{m}$
b) Update the diagram. The black line is the equilibrium position the particle oscillates about. Notice that the extension is from the blue dashed line and here x denotes the displacement from the equilibrium and not the extension.	
Substitute Hooke's Law into $F = ma$ to obtain a differential equation.	$mg - k(x + \epsilon) = m\ddot{x} \Rightarrow 0 = \ddot{x} + \omega^2 x = 0, \quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{80}{2}} = 2\sqrt{10}$
Substitute into the standard solution for simple harmonic motion starting at the amplitude.	$x = 0.2\cos\omega t, \quad \omega = 2\sqrt{10}$

Coupled 1st Order Differential Equations

There are many systems which involve two quantities (dependent variables which are each governed by a 1st order differential equation where the derivatives are with respect to an independent variable – often time). For example, a predator-prey relationship over a time period where the number of predators affects the number of prey and vice versa. These systems can be solved by forming a 2nd order differential equation by differentiating the equation of one of the dependent variables and eliminating the other dependent variable by rearranging the two initial equations. This method is demonstrated in the simple example below.

Example 2: Given the pair of differential equations below, find the general solution for x and y in terms of t.

$$\frac{\text{d}x}{\text{d}t} = 2x - y \quad (1), \qquad \frac{\text{d}y}{\text{d}t} = 3y - 2x \quad (2)$$

Differentiate the first equation to get a 2nd order differential equation.	$\frac{\text{d}^2x}{\text{d}t^2} = 2\frac{\text{d}x}{\text{d}t} - \frac{\text{d}y}{\text{d}t}$
Substitute (2) into the 2nd order differential equation.	$\frac{\text{d}^2x}{\text{d}t^2} = 2\frac{\text{d}x}{\text{d}t} - (3y - 2x)$
Rearrange (1) to make y the subject and substitute into the 2nd order differential equation.	$\frac{\text{d}^2x}{\text{d}t^2} = 2\frac{\text{d}x}{\text{d}t} - \left(3\left(2x - \frac{\text{d}x}{\text{d}t}\right) - 2x\right) \Rightarrow \frac{\text{d}^2x}{\text{d}t^2} - 5\frac{\text{d}x}{\text{d}t} + 4x = 0$
Solve 2nd order equation using auxiliary equation.	$\lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda = 1, 4 \Rightarrow x = Ae^t + \text{B}e^{4t}$
Differentiate solution of x and substitute into (1) to obtain the solution for y.	$\frac{\text{d}x}{\text{d}t} = Ae^t + 4\text{B}e^{4t} \Rightarrow y = 2(Ae^t + \text{B}e^{4t}) - (Ae^t + 4\text{B}e^{4t}) = Ae^t - 2\text{B}e^{4t}$

With some initial conditions a particular solution can be found and analysed to gain insight into the situation being modelled.

Example 3: A scientist wishes to study the population dynamics of lions (L) and zebras (Z). From previous research the scientist estimates that lions have a birth rate of $0.05Z + 0.5$ and a death rate of $0.1L$, and zebras have a birth rate of $0.1Z + 2$ and a death rate of $0.25L$.

a) Use these birth and death rates to write two differential equations in terms of $\dot{L}, \dot{Z}, Z$ and L.

b) Given that initially there are 28 lions and 52 zebras, solve the two differential equations from (a)

c) Find the minimum and maximum number of lions the model predicts throughout the study

a) Use Change in population = Birth rate – Death rate to form differential equations.	$\dot{L} = 0.05Z + 0.5 - 0.1L\ (*), \qquad \dot{Z} = 0.1Z + 2 - 0.25L\ (**)$
b) Differentiate (*) to form a 2nd order differential equation and substitute (**) into this 2nd order differential equation.	$\ddot{\text{L}} = 0.05(0.1\text{Z} + 2 - 0.25\text{L}) - 0.1\dot{\text{L}}$
Rearrange (*) to make Z the subject and substitute into the 2nd order differential equation.	$\ddot{L} = 0.05(0.1(20(\dot{L} - 0.5 + 0.1L)) + 2 - 0.25L) - 0.1\dot{L}$
Simplify 2nd order differential equation and put into standard form to solve.	$\ddot{L} + \omega^2 L = 0.05, \qquad \omega = 0.05$
Solve the 2nd order differential equation by guessing a particular integral of a constant and substituting in.	$L_{PI} = p \Rightarrow \dot{L}_{PI} = 0, \ddot{L}_{PI} = 0 \Rightarrow \omega^2 p = 0.05 \Rightarrow \text{p} = 20$
Combine L_{PI} with $L_{CF} = A\cos\omega t + B\sin\omega t$, the solution of the simple harmonic motion equation, to obtain the general solution for L.	$L = L_{PI} + L_{CF} = A\cos\omega t + B\sin\omega t + 20$
Differentiate the general solution for L to substitute into (*) to find the general solution for Z. Observe simplification due to $20\omega = 1$.	$\dot{L} = -A\omega\sin\omega t + B\omega\cos\omega t$ $\Rightarrow \text{Z} = 20\big((-A\omega\sin\omega t + B\omega\cos\omega t) - 0.5 + 0.1(A\cos\omega t + B\sin\omega t + 20)\big)$ $\Rightarrow \text{Z} = (20\text{B}\omega + 2\text{A})\cos\omega t + (-20A\omega + 2B)\sin\omega t + 30$ $= (B + 2A)\cos\omega t + (-A + 2B)\sin\omega t + 30$
Substitute initial conditions into the general solution for L to fix A.	$28 = A\cos 0 + B\sin 0 + 20 \Rightarrow \text{A} = 8$
Substitute initial conditions and $A = 8$ to fix Z.	$52 = \big(B + 2(8)\big)\cos 0 + (-8 + 2B)\sin\omega t + 30 \Rightarrow 22 = \text{B} + 16 \Rightarrow \text{B} = 6$
Write the particular solutions for L and Z.	$L = 8\cos\omega t + 6\sin\omega t + 20, \qquad Z = 22\cos\omega t + 4\sin\omega t + 30$
c) Rewrite $A\cos\omega t + B\sin\omega t$ as $R\cos(\omega t - \alpha)$ using trigonometry.	$R\cos(\omega t - \alpha) \equiv (R\cos\alpha)\cos\omega t + (R\sin\alpha)\sin\omega t \equiv 8\cos\omega t + 6\sin\omega t$ $\Rightarrow R\cos\alpha = 8, \qquad R\sin\alpha = 6$ $(R\cos\alpha)^2 + (R\sin\alpha)^2 = 8^2 + 6^2 \Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 100 \Rightarrow \text{R} = 10$ $\frac{R\sin\alpha}{R\cos\alpha} = \frac{6}{8} \Rightarrow \tan\alpha = \frac{3}{4} \Rightarrow \alpha = \tan^{-1}\frac{3}{4} = 0.644\ (3s.f.)$ $\Rightarrow 8\cos\omega t + 6\sin\omega t \equiv 10\cos(\omega t - 0.644)$
Use amplitude of the wave to find L_{max} and L_{min}.	$L = 10\cos(\omega t - 0.644) + 20 \Rightarrow L_{min} = 10,\ L_{max} = 30$