Differential Equations III Cheat Sheet (A Level Only)

Solving Second Order Non-Homogeneous Differential Equations with Constant Coefficients Using the Complementary Function and Particular Integral

A second order non-homogeneous differential equation with constant coefficients is a differential equation of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x),$$

where a and b are constants. Notice a coefficient of $\frac{d^2y}{dx^2}$ can be handled by dividing the whole equation by this coefficient.

Solving second order non-homogeneous differential equations with constant coefficients is more difficult than their associated homogeneous versions (where $f(x) = 0$). However, since their forms are so similar, the general solution of the associated homogenous differential equation, known as the complementary function, is present in the general solution of the inhomogeneous differential equation. The other part of the general solution is the particular integral, which is any solution to the non-homogeneous differential equation and depends on $f(x)$. So overall the general solution to the differential equation is:

$$y = y_{CF} + y_{PI}$$

Where y_{CF} is the complementary function and y_{PI} is the particular integral. The method for finding the complementary function is covered in the Differential Equations II Cheat Sheet. Finding the particular integral requires guessing a trial function. This choice of trial function depends on the form of $f(x)$. For this course, there are three forms $f(x)$ could take. The table below shows the possible forms of $f(x)$ and the functions which should be trialled, assuming that the complementary function does not also contain a function of the same form. If the complementary function does contain a function of the same form as the trial function then, like with the general solution for doubled roots of the auxiliary equation, multiply the trial function by x.

$f(x)$	Trial Function
Polynomial	Polynomial of the same order
Ae^{bx}	Pe^{bx}
$a\cos bx$ or $a\sin bx$	$p\sin bx + q\cos bx$

Example 1: Solve $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 6y = 4x + 3$ where at $x = 0, y = \frac{20}{9}$ and $\frac{dy}{dx} = \frac{2}{3}$.

Find the roots of the auxiliary equation to the associated homogeneous differential equation $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 6y = 0$.	$0 = \lambda^2 + 7\lambda + 6 = (\lambda + 6)(\lambda + 1) \Rightarrow \lambda_1 = -6,\ \lambda_2 = -1$
Write complementary function y_{CF}.	$y_{CF} = Ae^{-x} + Be^{-6x}$
Use a polynomial trial function of order 1.	$y_{PI} = px + q$
Find first and second derivatives of y_{PI}.	$\frac{dy}{dx} = p, \quad \frac{d^2y}{dx^2} = 0$
Substitute derivatives into the differential equation.	$(0) + 7(p) + 6(px + q) \equiv 4x + 3$
Equate coefficients.	x^1: $6p = 4 \Rightarrow p = \frac{2}{3}$ x^0: $7p + 6q = 3 \Rightarrow q = \frac{1}{6}\left(3 - \frac{14}{3}\right) = -\frac{5}{18}$
Write particular integral y_{PI}.	$y_{PI} = \frac{2}{3}x - \frac{5}{18}$
Sum complementary function and particular integral to obtain the general solution.	$y = y_{CF} + y_{PI} = Ae^{-x} + Be^{-6x} + \frac{2}{3}x - \frac{5}{18}$
Find $\frac{dy}{dx}$ to substitute initial conditions into.	$\frac{dy}{dx} = -Ae^{-x} - 6Be^{-6x} + \frac{2}{3} \Rightarrow \frac{2}{3} = -Ae^{-(0)} - 6Be^{-6(0)} + \frac{2}{3}$ $\Rightarrow A = -6B$
Substitute initial conditions into the general solution.	$y = Ae^{-x} + Be^{-6x} + \frac{2}{3}x - \frac{5}{18} \Rightarrow \frac{20}{9} = Ae^{-(0)} + Be^{-6(0)} + \frac{2}{3}(0) - \frac{5}{18}$ $\Rightarrow \frac{5}{2} - B = A$
Solve equations simultaneously to obtain A and B.	$-6B = \frac{5}{2} - B \Rightarrow B = -\frac{1}{2} \Rightarrow A = 3$
Write the particular solution.	$y = 3e^{-x} - \frac{1}{2}e^{-6x} + \frac{2}{3}x - \frac{5}{18}$

Example 2: Consider the differential equation $x\frac{d^2y}{dx^2} + (-10x + 2)\frac{dy}{dx} + (25x - 10)y = 3e^{4x}$ where $x \neq 0$.

a) Use the substitution $u = xy$ to rewrite the differential equation as $\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = 3e^{4x}$ where a and b are constants to be found.

b) Hence solve $x\frac{d^2y}{dx^2} + (-10x + 2)\frac{dy}{dx} + (25x - 10)y = 3e^{4x}$.

a) Apply the product rule to $u = xy$ to find $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$.	$\frac{du}{dx} = y + x\frac{dy}{dx}, \quad \frac{d^2u}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + x\frac{d^2y}{dx^2}$
Rearrange to make $\frac{dy}{dx}$ and $x\frac{d^2y}{dx^2}$ the subject of the equation respectively.	$\frac{dy}{dx} = \frac{1}{x}\left(\frac{du}{dx} - y\right), \quad x\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} - 2\frac{dy}{dx} = \frac{d^2u}{dx^2} - 2\left(\frac{1}{x}\left(\frac{du}{dx} - y\right)\right)$
Substitute into the differential equation and simplify.	$\frac{d^2u}{dx^2} - 2\left(\frac{1}{x}\left(\frac{du}{dx} - y\right)\right) + (-10x + 2)\left(\frac{1}{x}\left(\frac{du}{dx} - y\right)\right) + (25x - 10)y = 3e^{4x}$ $\Rightarrow \frac{d^2u}{dx^2} + \left(\frac{-2}{x} - 10 + \frac{2}{x}\right)\frac{du}{dx} + \left(\frac{2}{x} + 10 - \frac{2}{x} - 10\right)y + 25u = 3e^{4x}$ $\Rightarrow \frac{d^2u}{dx^2} - 10\frac{du}{dx} + 25u = 3e^{4x} \Rightarrow a = -10, b = 25$
b) Find the roots of the auxiliary equation to the associated homogeneous differential equation $\frac{d^2u}{dx^2} - 10\frac{du}{dx} + 25u = 0$.	$0 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 \Rightarrow \lambda = 5$
Write complementary function u_{CF}.	$u_{CF} = (A + Bx)e^{5x}$
Use an exponential trial function.	$u_{PI} = Pe^{4x}$
Find first and second derivatives of y_{PI}.	$\frac{du}{dx} = 4Pe^{4x}, \quad \frac{d^2u}{dx^2} = 16Pe^{4x}$
Substitute derivatives into the differential equation.	$16Pe^{4x} - 10(4Pe^{4x}) + 25(Pe^{4x}) \equiv 3Pe^{4x}$
Equate coefficients.	e^{4x}: $16P - 40P + 25P = 3P \Rightarrow P = \frac{1}{3}$
Write out u_{PI}.	$u_{PI} = \frac{1}{3}e^{4x}$
Sum complementary function and particular integral to obtain the general solution. Rewrite in terms of y.	$u = u_{CF} + u_{PI} = (A + Bx)e^{5x} + \frac{1}{3}e^{4x}$ $\Rightarrow y = \frac{1}{x}\left((A + Bx)e^{5x} + \frac{1}{3}e^{4x}\right)$

Example 3: Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos x + e^{2x}$

Find the roots of the auxiliary equation to the associated homogeneous differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.	$0 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$
Write complementary function y_{CF}.	$y_{CF} = Ae^{x} + Be^{2x}$
Use a trial function for both trigonometric functions and exponential noticing that y_{CF} contains e^{2x} so multiply that through by x.	$y_{PI} = p\cos x + q\sin x + rxe^{2x}$
Find first and second derivatives of y_{PI}.	$\frac{dy}{dx} = -p\sin x + q\cos x + re^{2x}(1 + 2x)$, $\frac{d^2y}{dx^2} = -p\cos x - q\sin x + 2re^{2x}(2 + 2x)$
Substitute derivatives into the differential equation.	$\left(-p\cos x - q\sin x + 2re^{2x}(2 + 2x)\right)$ $- 3\left(-p\sin x + q\cos x + re^{2x}(1 + 2x)\right)$ $+ 2(p\cos x + q\sin x + rxe^{2x}) \equiv \cos x + e^{2x}$
Equate coefficients. Notice that guessing xe^{2x} meant that another equation was obtained and not just $0 = 0$, which is what would have been obtained if just e^{2x} was guessed.	$\cos x$: $-p - 3q + 2p = 1 \Rightarrow p = 1 + 3q$ $\sin x$: $-q + 3p + 2q = 0 \Rightarrow 3p + q = 0$ e^{2x}: $4r - 3r = 1 \Rightarrow r = 1$ xe^{2x}: $4r - 6r + 2r = 0 \Rightarrow 0 = 0$
Solve simultaneous equations to find p and q.	$3(1 + 3q) + q = 0 \Rightarrow q = -\frac{3}{10} \Rightarrow p = \frac{1}{10}$
Write out y_{PI}.	$y_{PI} = \frac{1}{10}\cos x - \frac{3}{10}\sin x + xe^{2x}$
Sum complementary function and particular integral to obtain the general solution.	$y = y_{CF} + y_{PI} = Ae^{x} + Be^{2x} + \frac{1}{10}\cos x - \frac{3}{10}\sin x + xe^{2x}$