Differential Equations II Cheat Sheet (A Level Only)

Solving Homogenous 2nd Order Differential Equations with Constant Coefficients Using the Auxiliary Equation

A homogeneous 2nd order differential equation with constant coefficients is a differential equation of the form:

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

Where a and b are constants. Notice a co-efficient of $\frac{d^2y}{dx^2}$ can be handled by dividing the whole equation by it.

These equations can be solved by first guessing a solution of the form $y=e^{\lambda x}$, where λ is a constant to be determined. Then find the first and second derivatives and substitute into the differential equation:

$$\frac{dy}{dx}=\lambda e^{\lambda x},\quad \frac{d^2y}{dx^2}=\lambda^2e^{\lambda x}\ \Rightarrow\ \lambda^2e^{\lambda x}+a\lambda e^{\lambda x}+be^{\lambda x}=0$$

Then divide both sides by $e^{\lambda x}$ to obtain the auxiliary equation shown below. This can always be done as $e^{\lambda x}\neq 0$ for all $\lambda, x\in\mathbb{C}$.

$$\lambda^2+a\lambda+b=0$$

As with any quadratic, there are three cases for the types of roots which can be classified using the discriminant:

a) $\Delta=a^2-4b>0$	Distinct real roots: $\lambda_1,\ \lambda_2$
b) $\Delta=a^2-4b=0$	Repeated roots: λ
c) $\Delta=a^2-4b<0$	Complex roots: $\lambda_1=\alpha+\beta i,\ \lambda_2=\alpha-\beta i$

Distinct Real Roots

In an equation with distinct real roots $\lambda_1,\ \lambda_2$ the general solution takes the form $y=Ae^{\lambda_1x}+Be^{\lambda_2x}$ where A and B are arbitrary constants.

Example 1: Consider the differential equation $3\frac{d^2y}{dx^2}+6\frac{dy}{dx}-24y=0$

a) Find the auxiliary equation.

b) Hence state and verify the general solution.

a) Write differential equation into correct form by dividing through by 3.	$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-8y=0$
Write auxiliary equation using the method demonstrated above. Notice the co-efficient of n^{th} order derivative is the same as the n^{th} power of λ.	$\lambda^2+2\lambda-8=0$
b) Factorise the auxiliary equation to find the roots.	$(\lambda-2)(\lambda+4)=0\ \Rightarrow\ \lambda_1=2,\ \lambda_2=-4$
Write general solution.	$y=Ae^{2x}+Be^{-4x}$
Verify by finding 1st and 2nd derivatives and substituting into the original differential equation.	$\frac{dy}{dx}=2Ae^{2x}-4Be^{-4x},\quad \frac{d^2y}{dx^2}=4Ae^{2x}+16Be^{-4x}$ $3(4Ae^{2x}+16Be^{-4x})+6(2Ae^{2x}-4Be^{-4x})-24(Ae^{2x}+Be^{-4x})\equiv 0$

Repeated Roots

In an equation with repeated roots λ, the general solution takes the form $y=(A+Bx)e^{\lambda x}$, where A and B are arbitrary constants. This is because in the case of a repeated root, the other solution can be obtained by multiplying through by x as justified below.

Proof 1: For the differential equation $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+a^2y=0$, where $\lambda=a$ is a repeated root of the auxiliary equation for a constant a, $y=xe^{\lambda x}$ is a solution.

First notice that the auxiliary equation of all homogeneous 2nd order differential equations with repeated roots $\lambda=a$ can be written in the same form.	Auxiliary equation: $0=\lambda^2-2a\lambda+a^2=(\lambda-a)^2$
Find 1st and 2nd derivatives of $y=xe^{\lambda x}$.	$\frac{dy}{dx}=e^{\lambda x}+\lambda xe^{\lambda x}=e^{\lambda x}(1+\lambda x),\quad \frac{d^2y}{dx^2}=\lambda e^{\lambda x}(1+\lambda x)+\lambda e^{\lambda x}=\lambda e^{\lambda x}(2+\lambda x)$
Verify by substituting into the original differential equation. Recall $\lambda=a$ and $(\lambda-a)^2=0$.	$\lambda e^{\lambda x}(2+\lambda x)-2a\left(e^{\lambda x}(1+\lambda x)\right)+a^2\left(xe^{\lambda x}\right)=e^{\lambda x}\left(2(\lambda-a)+x(\lambda-a)^2\right)\equiv 0$

Complex Roots

Notice that the complex roots are a conjugate pair as they are the solutions of a quadratic with real coefficients. For an equation with complex roots $\lambda_1=\alpha+\beta i,\ \lambda_2=\alpha-\beta i$, although the form of the general solution for distinct real roots is valid, Euler's formula can be used to rewrite the general solution as $y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$.

Proof 2: Given that $y=Ce^{\lambda_1x}+De^{\lambda_2x}$ and $\lambda_1=\alpha+\beta i,\ \lambda_2=\alpha-\beta i$, then $y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$ where A, B, C and D are constants.

Substitute λ_1 and λ_2 into $y=Ce^{\lambda_1x}+De^{\lambda_2x}$ and factorise out $e^{\alpha x}$.	$y=Ce^{(\alpha+\beta i)x}+De^{(\alpha-\beta i)x}=e^{\alpha x}(Ce^{\beta ix}+De^{-\beta ix})$
Use Euler's formula to rewrite $e^{\beta ix}$ and $e^{-\beta ix}$. Recall $\sin-\beta x=-\sin\beta x$.	$y=e^{\alpha x}(C(\cos\beta x+i\sin\beta x)+D(\cos\beta x-i\sin\beta x))$
Collect terms and rename co-efficients. $A=C+D$ and $B=(C-D)i$.	$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

Example 2: Solve the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+\mu y=0$ when

a) $\mu=1$
b) $\mu=5$
c) $\mu=0$

a) Write auxiliary equation.	$0=\lambda^2+2\lambda+1=(\lambda+1)^2\ \Rightarrow\ \lambda=-1$
Substitute into the general solution for repeated roots.	$y=(A+Bx)e^{-x}$
b) Write auxiliary equation.	$0=\lambda^2+2\lambda+5=\left(\lambda-(-1+2i)\right)\left(\lambda-(-1-2i)\right)\ \Rightarrow\ \lambda=-1\pm 2i$
Substitute into the general solution for complex roots.	$y=e^{-x}(A\cos 2x+B\sin 2x)$
c) Write auxiliary equation.	$0=\lambda^2+2\lambda=\lambda(\lambda+2)\Rightarrow\ \lambda_1=0,\lambda_1=-2$
Substitute into the general solution for distinct real roots.	$y=A+Be^{-2x}$

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