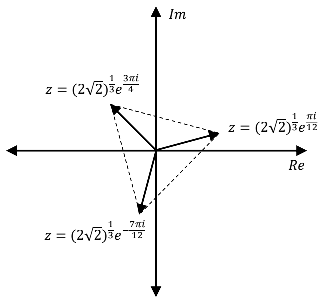
Complex Numbers V Cheat Sheet

Finding nth Roots of Complex Numbers

Any complex number of the form $z^n = a$ (where z and a are complex numbers and $n \in \mathbb{Z}^+$) has n distinct n^{th} roots. These roots can be found by utilising De Moivre's theorem.

Example 1: Solve the equation $z^3 = 2 + 2i$ and plot the solutions on an Argand diagram. Give your answers in modulus-argument form.

Firstly, write $2 + 2i$ in exponential form.	$\lvert 2+2i\rvert = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ $\arg(2+2i) = \arctan\left(\frac{2}{2}\right) = \frac{\pi}{4}$ $2 + 2i = 2\sqrt{2}e^{\frac{\pi}{4}i}$
To represent all the possible solutions for z^3, add $2\pi k i$ (where $k \in \mathbb{Z}$) to the argument.	$z^3 = 2\sqrt{2}e^{\frac{\pi}{4}i + 2\pi k i} = 2\sqrt{2}e^{\pi i\left(2k+\frac{1}{4}\right)} = 2\sqrt{2}e^{\pi i\left(\frac{8k+1}{4}\right)}$
Next, take the cube root of the exponential form. Knowing that there will be three distinct cube roots, substitute $k = 0$, $k = 1$, and $k = 2$ separately to find all the solutions. We want arguments in the range $-\pi < \theta \leq \pi$, so subtract 2π from $\frac{17\pi}{12}$.	$z = (2\sqrt{2})^{\frac{1}{3}}e^{\pi i\left(\frac{8k+1}{12}\right)}$ $k = 0$, $z = (2\sqrt{2})^{\frac{1}{3}}e^{\frac{\pi i}{12}}$ $k = 1$, $z = (2\sqrt{2})^{\frac{1}{3}}e^{\frac{9\pi i}{12}} = (2\sqrt{2})^{\frac{1}{3}}e^{\frac{3\pi i}{4}}$ $k = 2$, $z = (2\sqrt{2})^{\frac{1}{3}}e^{\frac{17\pi i}{12}} = (2\sqrt{2})^{\frac{1}{3}}e^{-\frac{7\pi i}{12}}$ Thus, $z_1 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$, $z_2 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$, $z_3 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$ $= (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{7\pi}{12}\right) - i\sin\left(\frac{7\pi}{12}\right)\right)$
Plot the solutions on an Argand diagram.	

Notice that the three roots form a regular equilateral triangle. The general form of this result holds true for all complex roots:

The n distinct solutions of $z^n = a$ form the vertices of a regular n-gon.

The arguments corresponding to each vertex are separated by an angle of $\frac{2\pi}{n}$.

Some questions may require you to draw the vertices of an n-sided polygon that is not centred at the origin. The equations for these will be of the form:

$$(z - c)^n = a,$$

where z and a are complex numbers, $n \in \mathbb{Z}^+$, and c is the centre of the polygon. In this scenario, solve as above for $z - c$ then add c to these solutions (for $z - c$) in order to find the solutions for z.

Example 2: Find the roots of the equation $(z - i)^3 = 2 + 2i$, giving your answers correct to 3 s.f.

Identify that the solutions for z will form a triangle centred at $(0,1)$. This is a translation of the triangle in Example 1 by $+1$ in the imaginary axis.	From Example 1, we know the solutions for z are, $z_1 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$, $z_2 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$ $z_3 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{7\pi}{12}\right) - i\sin\left(\frac{7\pi}{12}\right)\right)$
Add i to each of the solutions for z with a calculator.	$z_1 + i = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right) + i$ $= 1.37 + 1.37i$, $z_2 + i = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) + i$ $= -1.00 + 2.00i$, $z_3 + i = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{7\pi}{12}\right) - i\sin\left(\frac{7\pi}{12}\right)\right) + i$ $= -0.366 - 0.366i$

nth Roots of Unity

The solutions to $z^n = 1$ are known as the n^{th} roots of unity and are usually given in the form:

$$z = e^{\frac{2\pi k i}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

Where $k = 0,1,2,\dots,(n-1)$. We can also express an n^{th} root of unity as:

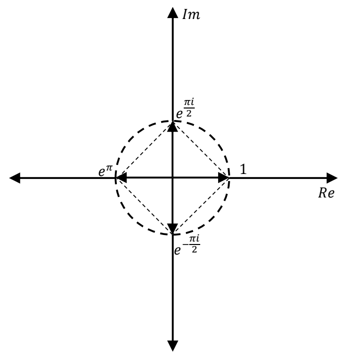
$$\omega = \left(e^{\frac{2\pi i}{n}}\right)^k$$

It follows that the n^{th} roots of unity are $1, \omega, \omega^2, \omega^3, \dots, \omega^{n-1}$, for $k = 0,1,2,\dots,(n-1)$.

When we add all the n^{th} roots of unity together, we form a geometric series that sums to 0 (as $\omega^n = 1$):

$$1 + \omega + \omega^2 + \dots + \omega^{n-1} = \frac{(1 - \omega^n)}{1 - \omega} = 0$$

Example 3: Find the fourth roots of unity and draw the polygon they produce on an Argand diagram. Hence, show that $2\cos\left(\frac{\pi}{2}\right) + \cos(\pi) = -1$.

To find the fourth roots of unity, we solve $z^4 = 1$	$z^4 = 1 \Rightarrow z = e^{\frac{2\pi k i}{4}} = e^{\frac{\pi k i}{2}}$ For $k = 0,1,2,3$: $z = 1, e^{\frac{\pi i}{2}}, e^{\pi i}, e^{\frac{3\pi i}{2}} \left(= e^{-\frac{\pi i}{2}}\right)$
Draw the roots on an Argand diagram.	
We need to rewrite the roots of unity in modulus-argument form and use the fact that the roots of unity sum to 0 to prove that: $2\cos\left(\frac{\pi}{2}\right) + \cos(\pi) = -1$	$0 = 1 + e^{\frac{\pi i}{2}} + e^{\pi i} + e^{-\frac{\pi i}{2}}$ $0 = 1 + \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) + \cos(\pi) + i\sin(\pi) + \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$ $0 = 1 + \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) + \cos(\pi) + 0 + \cos\left(\frac{\pi}{2}\right) - i\sin\left(-\frac{\pi}{2}\right)$ $0 = 1 + 2\cos\left(\frac{\pi}{2}\right) + \cos(\pi)$ $-1 = 2\cos\left(\frac{\pi}{2}\right) + \cos(\pi)$

Note that the four roots of unity in Example 3 lie on a circle of modulus 1. This is true for all n^{th} roots of unity.

Solving Geometric Problems

If z_1 is a root of the equation $z^n = a$, whose roots of unity are $1, \omega, \omega^2, \omega^3, \dots, \omega^{n-1}$, the roots of $z^n = a$ are:

$$z_1, z_1\omega, z_1\omega^2, z_1\omega^3, \dots, z_1\omega^{n-1}$$

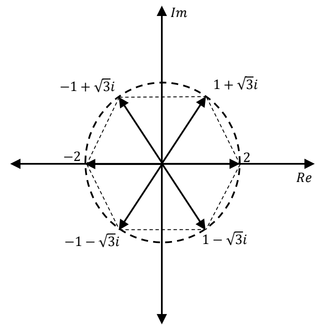
Geometrically, this means that it is possible to find all the vertices of any n-gon centred at the origin by knowing one vertex and continually rotating that point about the origin by $\frac{2\pi}{n}$.

Example 4: A regular hexagon on an Argand diagram has its centre at the origin and one of its vertices at $(1, -\sqrt{3})$.

a) Find the coordinates of the other five vertices.

b) Represent the roots on an Argand diagram.

c) Calculate the area of the hexagon, leaving your answer in terms of π.

a) We rewrite the given vertex in exponential form. Note that as z_1 is in the fourth quadrant, its argument is $-\frac{\pi}{3}$ and not $\frac{\pi}{3}$.	$z_1 = 1 - \sqrt{3}i$ $\lvert z_1\rvert = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ $\arg(z_1) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$ $z_1 = 2e^{-\frac{\pi i}{3}}$
Write an expression for the 6th root of unity with $k = 1$.	$\omega = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}$
We multiply each vertex by ω until all five vertices are found.	$z_2 = z_1\omega = 2e^{\frac{\pi i}{3}} \times e^{\frac{\pi i}{3}} = 2e^{\frac{2\pi i}{3}}$ $z_3 = z_1\omega^2 = 2e^{\frac{\pi i}{3}} \times e^{\frac{2\pi i}{3}} = 2e^{\pi i}$ $z_4 = z_1\omega^3 = 2e^{\frac{\pi i}{3}} \times e^{\pi i} = 2e^{\frac{4\pi i}{3}}$ $z_5 = z_1\omega^4 = 2e^{\frac{\pi i}{3}} \times e^{\frac{4\pi i}{3}} = 2e^{\frac{5\pi i}{3}}$ $z_6 = z_1\omega^5 = 2e^{\frac{\pi i}{3}} \times e^{\frac{5\pi i}{3}} = 2e^{2\pi i}$
Write each vertex as a coordinate by first converting into modulus-argument form.	$z_2 = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -1 + \sqrt{3}i \therefore (-1, \sqrt{3})$ $z_3 = 2(\cos(\pi) + i\sin(\pi)) = -2 \therefore (-2, 0)$ $z_4 = 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -1 - \sqrt{3}i \therefore (-1, -\sqrt{3})$ $z_5 = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = 1 - \sqrt{3}i \therefore (1, -\sqrt{3})$ $z_6 = 2(\cos(2\pi) + i\sin(2\pi)) = 2 \therefore (2, 0)$
b) Plot the roots on an Argand diagram.	
c) Notice that the hexagon is made up of six identical triangles, each with legs of length 2 and vertex angle $\frac{\pi}{3}$, therefore the other two angles must also be $\frac{\pi}{3}$, so the triangle is equilateral.	The formula $A = \frac{1}{2}r^2\sin\theta$ can be used for the area of a single triangle. $A = \frac{6}{2} \times 4\sin\frac{\pi}{3} = 6\sqrt{3}$