Complex Numbers III Cheat Sheet

AQA A Level Further Maths: Core

Loci

A locus is defined as the set of all points that satisfy a given constraint. We can use loci to represent regions on an Argand diagram that correspond to given constraints such as inequalities. In this cheat sheet we examine loci located in the complex plane through circles, perpendicular bisectors, and half-lines.

Distance Between Points

Modifying the definition for the modulus introduced in "Complex Numbers I" allows us to find the distance between any two points in the complex plane. The distance between two general points, $w = x_1 + iy_1$ and $z = x_2 + iy_2$, on an Argand diagram is given by the equation:

$$|z - w| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Circles

The locus of all complex numbers located a given distance r from a point a on an Argand diagram is given by:

$$|z - a| = r$$

The above equation represents a circle on an Argand diagram. This becomes apparent when we substitute in the Cartesian form $z = x + iy$ and let a be a general point (x_1, y_1):

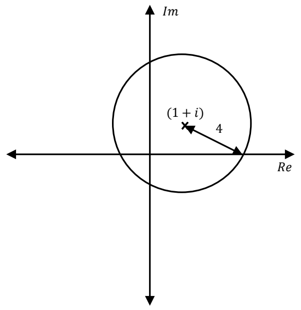
$$|x + iy - (x_1 + iy_1)| = r$$

$$\Rightarrow \sqrt{(x - x_1)^2 + (y - y_1)^2} = r$$

$$\Rightarrow (x - x_1)^2 + (y - y_1)^2 = r^2,$$

which is the standard Cartesian equation of a circle.

Example 1: Draw the locus of $|z - 1 - i| = 4$ on an Argand diagram.

First, we rewrite $\lvert z - 1 - i\rvert$ so that we can determine the centre of the circle on an Argand diagram.	$\lvert z - 1 - i\rvert = \lvert z - (1 + i)\rvert$ Centre $= (1,1)$ Radius $= 4$
Draw a circle centred at $(1,1)$ with radius 4 on an Argand diagram.	

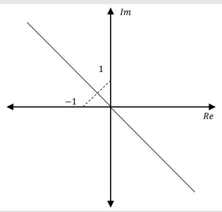
Perpendicular Bisectors

The locus of complex numbers that satisfy equations of the form:

$$|z - z_1| = |z - z_2|,$$

where z_1 and z_2 are two points in the complex plane, lie on the perpendicular bisector between the two points.

Example 2: Sketch the locus of points that satisfy $|z - i| = |z + 1|$.

The perpendicular bisector is the line directly between $(0,1)$ and $(-1,0)$. In words, the equation represents "all the complex numbers z that are the same distance away from $(0,1)$ as they are from $(-1,0)$".	

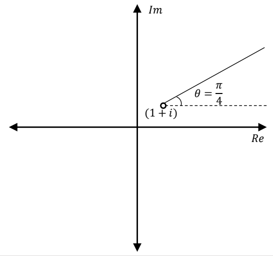
Half-Lines

Loci can also be constructed using arguments. For example, half-lines (lines that originate at a single point and go on infinitely in only one direction) are represented by:

$$\arg(z - z_1) = \theta,$$

where z_1 is the starting point of the half-line and θ is the angle (in radians) made with the positive real axis.

Example 3: Sketch the locus of points that satisfy $\arg(z - 1 - i) = \frac{\pi}{4}$.

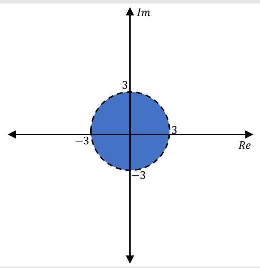
Rewrite $\arg(z - 1 - i) = \frac{\pi}{4}$ in the form $\arg(z - z_1) = \theta$.	$\arg(z - 1 - i) = \arg(z - (1 + i)) = \frac{\pi}{4}$
Draw a half line beginning at $(1,1)$ that makes an angle $\frac{\pi}{4}$ with the positive real axis (shown as dashed baseline). The point $(1,1)$ is not included in the locus, so it is represented by an empty circle.	

Sketching Regions

When representing inequalities using loci, we must:

- Use a dashed line for the less than $<$ and greater than $>$ symbols; all the points on the line are not included
- Use a solid line for the less than or equal to $\leq$ and greater than or equal to $\geq$ symbols; all the points on the line are included

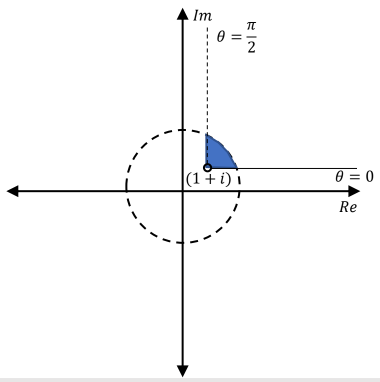
Example 4: On an Argand diagram, represent the inequality $|z| < 3$.

Notice that the less than symbol is used, so we must use a dashed line. In words, the inequality represents "all the complex numbers z that are less than 3 units away from the origin". Hence, we must shade inside the circle.	

Intersecting Regions

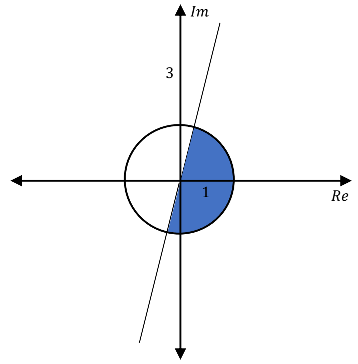
Example 5: On an Argand diagram, shade the region which satisfies:

$$\{z \in \mathbb{C}: |z| < 2\} \cap \{z \in \mathbb{C}: 0 \leq \arg(z - 1 - i) < \frac{\pi}{2}\}$$

$\lvert z\rvert < 2$ represents a circle that is centred at the origin and has radius 2. The less than symbol is used so we use a dashed line to represent the circumference. $0 \leq \arg(z - 1 - i) < \frac{\pi}{2}$ represents a wedge which is essentially two half-lines. The wedge begins at $(1,1)$. Whilst the half-line for $0 \leq \arg(z - 1 - i)$ will be a solid line, the half-line for $\arg(z - 1 - i) < \frac{\pi}{2}$ will be a dashed line. We shade the region inside the circle (since we want all the complex numbers less than 2 units away from the origin) that is *also* in the range 0 to $\frac{\pi}{2}$ starting from $(1 + i)$ because the $\cap$ (and) intersection symbol is used.	

Example 6: On an Argand diagram, shade the region which satisfies:

$$\{z \in \mathbb{C}: |z - 1| \leq |z - 3i|\} \cap \{z \in \mathbb{C}: |z| \leq 2\}$$

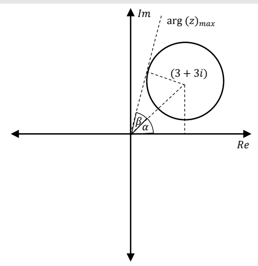
We draw the same circle as in example 5; however, we use a solid line since the $\leq$ symbol is used. In words, the inequality $\lvert z - 1\rvert \leq \lvert z - 3i\rvert$ represents "all complex numbers z where the distance from z to 1 is less than the distance from z to $3i$". A solid line is used for the perpendicular bisector because of the $\leq$ symbol. To satisfy both loci (due to the presence of the "and" $\cap$ symbol), we shade inside the circle and under the perpendicular bisector.	

Further Loci Problems

Example 7: Given that $|z| = 4$ and $|z + i| = |z - 1|$, find the complex number z.

Substitute $z = x + iy$ into $\lvert z\rvert = 4$.	$\lvert x + iy\rvert = 4 \Rightarrow x^2 + y^2 = 16$
Substitute $z = x + iy$ into $\lvert z + i\rvert = \lvert z - 1\rvert$.	$\lvert x + iy + i\rvert = \lvert x + iy - 1\rvert$ $\Rightarrow x^2 + (y + 1)^2 = (x - 1)^2 + y^2$ $\Rightarrow x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$ $\Rightarrow 2y = -2x \Rightarrow y = -x$
Substitute $y = -x$ into $x^2 + y^2 = 16$.	$x^2 + x^2 = 16 \Rightarrow 2x^2 = 16 \Rightarrow x^2 = 8$ $\Rightarrow x = \pm 2\sqrt{2} \Rightarrow y = \mp 2\sqrt{2}$ $z = 2\sqrt{2} - 2i\sqrt{2}, \quad z = -2\sqrt{2} + 2i\sqrt{2}$

Example 8: Find the maximum value of $\arg(z)$ given that $|z - (3 + 3i)| = 2$.

First, we draw $\lvert z - (3 + 3i)\rvert = 2$ on an Argand diagram. Next, we construct triangles onto the diagram to help find the maximum argument.	
We can find α using the rightmost dashed right-angled triangle.	$\alpha = \arctan\left(\frac{3}{3}\right) = \frac{\pi}{4}$ rad
To find β, first find $\lvert 3 + 3i\rvert$. Use the fact that the radius of the circle is 2 to solve for β using the leftmost dashed right-angled triangle.	$\lvert 3 + 3i\rvert + \sqrt{3^2 + 3^2} = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$ $\beta = \arcsin\left(\frac{2}{3\sqrt{2}}\right) = 0.490\ldots$ rad
Write down $\arg(z)_{\max}$.	$\arg(z)_{\max} = \alpha + \beta = 1.28$ rad (3. sig. figs)

A similar method to the one above can be used to find the minimum argument of a circle.