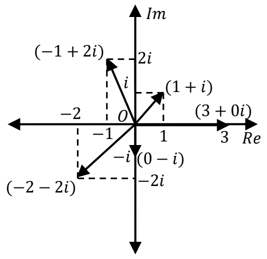
Complex Numbers II Cheat Sheet

Geometric Representation of Complex Numbers

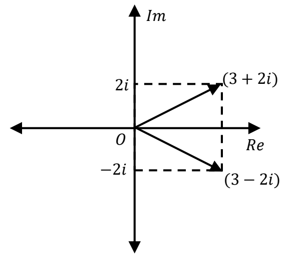
Argand Diagrams

All real numbers exist on a straight, infinite number line; all complex numbers exist beyond the real number line in the **complex plane**. To geometrically represent complex numbers, we use an **Argand diagram** which represents this complex plane.

An Argand diagram consists of two axes: the **horizontal real axis** and the **vertical imaginary axis**. Plotting complex numbers on an Argand diagram is identical to plotting Cartesian coordinates on a Cartesian diagram. When plotting a given complex number $z = a + bi$, we treat the real part as the x-coordinate and the imaginary part as the y-coordinate. So, the coordinate (a, b) represents the complex number $z = a + bi$. A few examples have been plotted on the right.



Example 1: Plot the complex number $z = 3 + 2i$ and its conjugate on an Argand diagram.

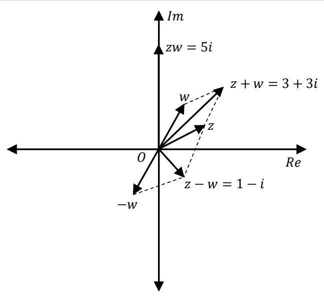
First write down the conjugate of z.	If $z = 3 + 2i$, then $z^* = 3 - 2i$
On an Argand diagram, the coordinates of z are $(3, 2)$ and z^* are $(3, -2)$. Thus, both z and z^* are 3 units to the right on the real axis. z is 2 units up on the real axis, whilst z^* is 2 units down.	

Note that on an Argand diagram, the complex conjugate of a complex number is always a reflection in the real axis.

Additive and Multiplicative Operations on an Argand Diagram

As mentioned in "Complex Numbers I", complex numbers add and multiply like vectors. This also applies to their addition, subtraction and multiplication on Argand diagrams.

Example 2: Given that $z = 2 + i$ and $w = 1 + 2i$, show $z + w$, $z - w$ and zw on an Argand diagram.

First calculate $z + w$, $z - w$ and zw.	$z + w = (2+i) + (1+2i) = 3 + 3i$ $z - w = (2+i) - (1+2i) = 1 - i$ $zw = (2+i)(1+2i)$ $= 2 + 4i + i + 2i^2$ $= 2 + 5i + 2(-1)$ $= 5i$
Plot the points $(3,3)$, $(1,-1)$ and $(0,5)$ corresponding to $z + w = 3 + 3i$, $z - w = 1 - i$ and $zw = 5i$ respectively on an Argand diagram. z, w and $-w$ have been included in the diagram with dashed lines forming a parallelogram with $z + w$ and $z - w$ to display the similarity to vector addition.	

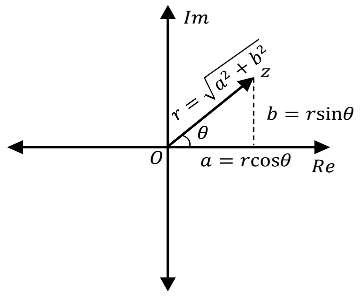
Modulus-Argument Form

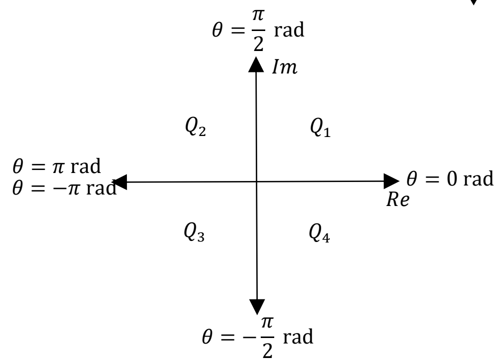
$z = a + bi$ is the **Cartesian form** of a complex number. A more geometrically useful way of representing complex numbers is known as the **modulus-argument** form.

The **modulus** of a complex number z is the distance from the origin to z. It is denoted using $|z|$ or r and is given by,

$$|z| = r = \sqrt{a^2 + b^2}$$

The **argument** of a complex number z is the anticlockwise angle (**in radians**) between the positive real axis and the line joining z to the origin. It is denoted using $\arg(z)$ or θ and is conventionally given in the principle range $-\pi \leq \theta \leq \pi$.





To find the argument, we use $\tan\theta = \frac{|b|}{|a|}$ and then modify the result depending upon which quadrant the complex number falls in.

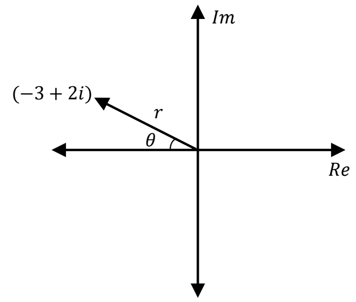
- If θ falls in Q_1 $(0 \leq \theta \leq \frac{\pi}{2})$, then $\arg(z) = \arctan\left(\frac{|b|}{|a|}\right)$
- If θ falls in Q_2 $(\frac{\pi}{2} < \theta \leq \pi)$, then $\arg(z) = \pi - \arctan\left(\frac{|b|}{|a|}\right)$
- If θ falls in Q_3 $(-\pi \leq \theta < -\frac{\pi}{2})$, then $\arg(z) = -\left(\pi - \arctan\left(\frac{|b|}{|a|}\right)\right)$
- If θ falls in Q_4 $(-\frac{\pi}{2} < \theta \leq 0)$, then $\arg(z) = -\arctan\left(\frac{|b|}{|a|}\right)$

The modulus and argument are combined to produce the modulus-argument form:

$$z = r(\cos\theta + i\sin\theta)$$

We can use the above equation to convert between the modulus-argument form and Cartesian form.

Example 3: Find the modulus-argument form of the complex number $z = -3 + 2i$.

First draw z on an Argand diagram.	
Next, we find r and θ.	$r = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (2)^2} = \sqrt{9+4} = \sqrt{13}$ Since z is in Q_2, we need to modify the arctan result. $\arg(z) = \theta = \arctan\left(\frac{\lvert 2\rvert}{\lvert -3\rvert}\right) = 0.588$ rad Use $\pi - \arctan\left(\frac{b}{a}\right)$ to get the angle from the positive real axis to z, $\pi - 0.588$ rad $= 2.55$ rad
Express in modulus-argument form.	$z = -3 + 2i = \sqrt{13}(\cos(2.55) + i\sin(2.55))$

Example 4: Convert $z = 4\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$ to Cartesian form.

Expand the brackets.	$a = 4\cos\left(\frac{3\pi}{4}\right) = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$ $b = 4\sin\left(\frac{3\pi}{4}\right) = 4i\left(\frac{\sqrt{2}}{2}\right) = (2\sqrt{2})$
Express in Cartesian form.	$z = -2\sqrt{2} + (2\sqrt{2})i$

Multiplying and Dividing in Modulus-Argument Form

Given two complex numbers z and w, the following moduli and argument results can be used:

$$|zw| = |z||w|$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$\arg(zw) = \arg(z) + \arg(w)$$

$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

Example 5: Given that $z = \sqrt{2}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$ and $w = 4\left(\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right)$, find zw and $\frac{z}{w}$. Give your answers in modulus-argument form.

First, we must rewrite w.	$\cos(x) = \cos(-x), -\sin(x) = \sin(-x)$ $\Rightarrow w = 4\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$
To find zw, multiply the moduli of z and w and add the arguments. Combine the results into the modulus-argument form.	$\lvert zw\rvert = \lvert z\rvert\lvert w\rvert = (\sqrt{2})(4) = 4\sqrt{2}$ $\arg(zw) = \arg(z) + \arg(w) = \frac{3\pi}{2} - \frac{\pi}{2} = \pi$ $zw = 4\sqrt{2}(\cos(\pi) + i\sin(\pi))$
To find $\frac{z}{w}$, divide the moduli of z and w and subtract the arguments. Combine the results into the modulus-argument form.	$\left\lvert\frac{z}{w}\right\rvert = \frac{\lvert z\rvert}{\lvert w\rvert} = \frac{\sqrt{2}}{4}$ $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) = \frac{3\pi}{2} - \left(-\frac{\pi}{2}\right)$ $= 2\pi$ $\frac{z}{w} = \frac{\sqrt{2}}{4}(\cos(2\pi) + i\sin(2\pi))$