Complex Numbers I Cheat Sheet

Complex Numbers and Complex Algebra

Complex numbers are a superset of the real numbers. Since being introduced to the modern number system, they have proved useful in many fields - including quantum mechanics and electronics. They serve as a useful mathematical tool to model complicated situations and behaviours.

Imaginary Numbers

One way we can easily find solutions for quadratic equations of the form;

$$ax^2 + bx + c = 0$$

(where $a, b, c \in \mathbb{R}$) is by applying the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, when the discriminant, $(b^2 - 4ac)$, is less than zero, the equation has no real solutions - as the square root of a negative number is being taken. Introducing imaginary numbers allows us to represent the complex solutions of the equation.

- In Cartesian form, complex numbers are represented as $z = a + bi$, where $a, b \in \mathbb{R}$. They consist of a real part, $Re(z) = a$, and an imaginary part, $Im(z) = b$.
- i is the letter used to denote the unit imaginary number and is defined by $i = \sqrt{-1}$.

Example 1: Express $\sqrt{-49}$ as an imaginary number in the form bi, where $b \in \mathbb{R}$.

Use the surd rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to rewrite $\sqrt{-49}$ so that $\sqrt{-1}$ is factored out.	$\sqrt{-49} = (\sqrt{49})(\sqrt{-1})$
Apply the definition of i to simplify further.	$\sqrt{-49} = (\sqrt{49})(\sqrt{-1}) = (\sqrt{49})i = 7i$

Complex Arithmetic

Complex addition is similar to vector addition. Similar to how we add the x and y components of a vector separately, we must add the real and imaginary parts separately.

Example 2: Simplify $z = (11 - 2i) - (4 + 2i)$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

Add the real parts together.	$11 - 4 = 7$
Add the imaginary parts together.	$-2i - 2i = -4i$
Combine the real and imaginary results.	$z = 7 - 4i$

Complex Multiplication

Similar to complex addition and subtraction, complex multiplication is identical to the usual real number multiplication except for a key difference; when two imaginary numbers are multiplied together, a real number is produced since $i^2 = (\sqrt{-1})(\sqrt{-1}) = -1$.

Example 3: Expand $(7 + 2i)(5 - i)$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

Expand the brackets.	$(7 + 2i)(5 - i) = 35 - 7i + 10i - 2i^2$
Apply the $i^2 = -1$ definition.	$35 - 7i + 10i - 2i^2 = 35 - 7i + 10i - 2(-1)$ $= 35 - 7i + 10i + 2$
Simplify the expression.	$35 - 7i + 10i + 2 = 37 + 3i$

For questions dealing with i to the power of an exponent, it is useful to memorise the results $i^3 = -i$ and $i^4 = 1$ in order to simplify the expression.

Example 4: Find i^{100} and i^{75}.

Use laws of indices of separate out i^4.	$i^{100} = i^{4(25)} = (1)^{25} = 1$
Use laws of indices to again factor out an i^4 term.	$i^{75} = i^{72+3} = (i)^{72} \times i^3 = i^{4(18)} \times -i$ $= 1 \times -i = -i$

Complex Conjugation

A given complex number has an associated complex conjugate. The complex conjugate of $z = a + bi$ is given by:

$$z^* = a - bi$$

Example 5: Using the definition of the complex conjugate, show that $zz^* = a^2 + b^2$.

First write down the definition of the complex conjugate z^*.	If $z = a + bi$, then $z^* = a - bi$
Multiply z and z^* together.	$zz^* = (a + bi)(a - bi)$ $= a^2 + b^2 i - b^2 i - b^2 i^2$ $= a^2 - b^2 i^2$ $= a^2 - b^2(-1)$ $= a^2 + b^2$

Example 6: Find the complex number z such that $z + 3z^* = 2 + 2i$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

Write down the general Cartesian forms of a complex number and its conjugate.	$z = a + bi, \quad z^* = a - bi$
Substitute these into the given equation.	$(a + bi) + 3(a - bi) = 2 + 2i$ $\Rightarrow a + bi + 3a - 3bi = 2 + 2i$ $\Rightarrow 4a - 2bi = 2 + 2i$
Equate real and imaginary parts on both sides.	$4a = 2 \Rightarrow a = \frac{2}{4} = \frac{1}{2}$ $-2b = 2 \Rightarrow b = \frac{2}{-2} = -1$
Write down z.	$z = \frac{1}{2} - i$

Complex Division

When dividing two complex numbers by each other, it is necessary to make use of the complex conjugate in order to realise the denominator of the fraction.

Example 7: Given $z = 4 + 3i$ and $w = 2 + 2i$, find $\frac{z}{w}$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

Write down the complex conjugate of w.	If $w = 2 + 2i$, then $w^* = 2 - 2i$
Multiply both the numerator and denominator of the fraction b w^*.	$\frac{z}{w} == \frac{4 + 3i}{2 + 2i} \times \frac{2 - 2i}{2 - 2i} = \frac{(4 + 3i)(2 - 2i)}{(2 + 2i)(2 - 2i)}$
Simplify.	$\frac{(4 + 3i)(2 - 2i)}{(2 + 2i)(2 - 2i)} = \frac{8 - 8i + 6i - 6i^2}{4 - 4i + 4i - 4i^2}$ $= \frac{8 - 2i - 6(-1)}{4 - 4(-1)} = \frac{14 - 2i}{8}$ $= \frac{7}{4} - \frac{1}{4}i$

Solving Equations with Complex Roots

Solving Quadratic Equations with Complex Roots

Complex roots of a quadratic equation always arise in conjugate pairs. If z is one root of a quadratic equation, then z^* is the other root.

Example 8: Solve $z^2 + 3z + 3 = 0$ for z. Give your answer in the form $a \pm bi$, where $a, b \in \mathbb{R}$.

Solving using the quadratic formula.	$a = 1, b = 3, c = 3$ $z = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 12}}{2}$ $= -\frac{3}{2} \pm \frac{\sqrt{-3}}{2} \Rightarrow z = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$
Or solve by completing the square.	$z^2 + 3z + 3 = 0 \Rightarrow \left(z + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 3 = 0$ $\Rightarrow \left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + 3 = 0 \Rightarrow \left(z + \frac{3}{2}\right)^2 + \frac{3}{4} = 0$ $\Rightarrow \left(z + \frac{3}{2}\right)^2 = -\frac{3}{4} \Rightarrow \left(z + \frac{3}{2}\right) = \pm\sqrt{-\frac{3}{4}}$ $\Rightarrow z = -\frac{3}{2} \pm \frac{\sqrt{-3}}{\sqrt{4}} \Rightarrow z = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$

If α and β are roots of a quadratic equation, we can write the equation as

$$(z - \alpha)(z - \beta) = 0$$

or

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0$$

Example 9: Given that $z = 3 + i$ is a root of the quadratic equation $z^2 + az + b = 0$ (where $a, b \in \mathbb{R}$), find the values a and b .

Write down both roots of the equation.	$\alpha = z = 3 + i, \quad \beta = z^* = 3 - i$
Substitute α and β into $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.	$\alpha + \beta = (3 + i) + (3 - i) = 6$ $\alpha\beta = zz^* = (3)^2 + (1)^2 = 10$ $z^2 - 6z + 10 = 0$ $a = -6, b = 10$

Solving Cubic and Quartic Equations with Complex Roots

A cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ (where $a, b, c, d \in \mathbb{R}$) will have either a pair of complex roots and a real root, or three real roots.

A quartic equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ (where $a, b, c, d, e \in \mathbb{R}$) will have either two pairs of complex roots, a pair of complex roots and two real roots, or four real roots.

Exam questions will typically give a root of the equation and so to find the remaining roots we have to use the factor theorem and then solve the remaining quadratic.

Example 10: Given that 7 is a root of the cubic equation $z^3 - 11z^2 + 41z - 91 = 0$, find the other two roots. Give your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

Use the factor theorem.	Since $z = 7$ is a solution, $(z - 7)$ must be a factor
Factor out $(z - 7)$ from the cubic using long division.	$(z - 7)(z^2 - 4z + 13)$
Use the quadratic equation to solve the quadratic for the other two roots.	$a = 1, b = -4, c = 13$ $z = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2}$ $= 2 \pm \frac{\sqrt{36}}{2}i = 2 \pm 3i$ $z = 7, \quad z = 2 + 3i, \quad z = 2 - 3i$

Example 11: Given that $1 + i$ is a root of the quartic equation $z^4 - 16z^3 + 128z^2 - 224z + 196 = 0$, find the other three roots.

Write down the other root of the quartic by using knowledge of complex conjugates.	If $1 + i$ is a root, then $1 - i$ is also a root
Use $z^2 - (\alpha + \beta)z + \alpha\beta$ to obtain a quadratic.	$\alpha = 1 + i, \quad \beta = 1 - i$ $\alpha + \beta = (1 + i) + (1 - i) = 2$ $\alpha\beta = (1 + i)(1 - i) = 1 - i^2 = 1 + 1 = 2$ $z^2 - 2z + 2$
Factor out the quadratic from the quartic using long division.	$z^4 - 16z^3 + 128z^2 - 224z + 196$ $= (z^2 - 2z + 2)(z^2 - 14z + 98)$
Use the quadratic equation to solve the remaining quadratic for the other 2 roots.	$a = 1, b = -14, c = 98$ $z = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(98)}}{2}$ $= \frac{14 \pm \sqrt{-196}}{2} = 7 \pm \frac{\sqrt{196}}{2}i = 7 \pm 7i$ $z = 1 \pm i, \quad z = 7 \pm 7i$