Matrix Algebra Cheat Sheet

Eigenvectors and Eigenvalues

If a matrix $\mathbf{A}$ transforms a non-zero vector $\mathbf{v}$ such that the output is a scalar multiple of $\mathbf{v}$,

$$\mathbf{Av} = \lambda\mathbf{v},$$

then $\mathbf{v}$ is called an **eigenvector** of matrix $\mathbf{A}$, with λ its corresponding **eigenvalue**. Eigenvectors are special to a matrix transformation, as their direction is unchanged during a linear transformation – they get stretched or squashed. The equation above is called the **eigenvalue equation**. Invariant lines of a transformation are parallel to eigenvectors. To find the eigenvectors and corresponding eigenvalues of a given matrix $\mathbf{A}$,

$$\mathbf{Av} = \lambda\mathbf{v} \;\Rightarrow\; \mathbf{Av} = \lambda\mathbf{Iv}$$

$$\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

where $\mathbf{I}$ is the identity matrix. For this to be true for a non-zero $\mathbf{v}$, $(\mathbf{A} - \lambda\mathbf{I})$ must be singular, i.e.

$$\det(A - \lambda I) = 0.$$

This is the **characteristic equation** of $\mathbf{A}$ and is a polynomial in λ. The roots are the eigenvalues of $\mathbf{A}$. The respective eigenvectors can then be found by plugging each root λ back into the equation.

Example: Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$.

Find and solve the characteristic equation to obtain the eigenvalues	$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix}$ $= (-\lambda)(-3-\lambda) - (-2)(1) = \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -1, -2$
Find the eigenvector for λ_1. Any multiple of this vector is also an eigenvector.	When $\lambda = -1$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$ Equating the top and bottom elements from either side $\begin{aligned} 0x + y &= -x \\ -2x - 3y &= -y \end{aligned} \Rightarrow \begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \end{aligned}$ Both equations give the same information and accept any solution if $y = -x \therefore \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an acceptable eigenvector.
Find the eigenvector for λ_2. Any multiple of this vector is also an eigenvector.	When $\lambda = -2$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$ Equating the top and bottom elements from either side $\begin{aligned} 0x + y &= -2x \\ -2x - 3y &= -2y \end{aligned} \Rightarrow \begin{aligned} 2x + y &= 0 \\ 2x + y &= 0 \end{aligned}$ Both equations give the same information and accept any solution if $y = -2x \therefore \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an acceptable eigenvector.

If $\mathbf{A}$ were a 3×3 matrix, the characteristic equation would be a cubic equation and there would be up to 3 eigenvalues and eigenvectors. A **normalised eigenvector** is an eigenvector of length 1. If $\mathbf{v}$ is an eigenvector, its normalised vector is given by $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$, where $|\mathbf{v}|$ is the length of $\mathbf{v}$.

Diagonalisation

A square matrix is **diagonal** if all elements that are not on its **leading diagonal** are zero. The leading diagonal is the line of elements from the **top left to bottom right** of a square matrix.

$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} -7 & 0 & 0 \\ 0 & e^{\pi i} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 9 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}$
Diagonal	Diagonal	Not Diagonal (The leading diagonal is not the only collection of non-zero elements)	Not Diagonal (The leading diagonal is not filled in)

Most square matrices can be transformed into a diagonal matrix. This process is called **diagonalisation**. The diagonal entries of this matrix are the eigenvalues of the original matrix.

Example: Diagonalise $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}$ into a diagonal matrix $\mathbf{D}$, such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where $\mathbf{P}$ is to be found.

Find and solve the characteristic equation to obtain the eigenvalues.	$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ 2 & -4 & 2-\lambda \end{vmatrix}$ $= (1-\lambda)\big((3-\lambda)(2-\lambda) - (0)(-4)\big) - (2)\big(0(2-\lambda) - (0)(2)\big) + (0)\big((0)(-4) - (2)(3-\lambda)\big)$ $= (1-\lambda)(\lambda^2 - 5\lambda + 6) = -(\lambda-1)(\lambda-2)(\lambda-3) \Rightarrow \lambda = 1, 2, 3$
Find the eigenvector for λ_1.	When $\lambda = 1$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\begin{aligned} x + 2y &= x \\ 3y &= y \\ 2x - 4y + 2z &= z \end{aligned} \Rightarrow \begin{aligned} y &= 0 \\ y &= 0 \\ z &= y - 2x \end{aligned}$ These equations accept any solution if $z = -2x \therefore \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$
Find the eigenvector for λ_2.	When $\lambda = 2$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\begin{aligned} x + 2y &= 2x \\ 3y &= 2y \\ 2x - 4y + 2z &= 2z \end{aligned} \Rightarrow \begin{aligned} x &= 2y \\ y &= 0 \\ x &= 2y \end{aligned}$ These equations accept any solution if $x = y = 0 \therefore \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
Find the eigenvector for λ_3.	When $\lambda = 3$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\begin{aligned} x + 2y &= 3x \\ 3y &= 3y \\ 2x - 4y + 2z &= 3z \end{aligned} \Rightarrow \begin{aligned} x &= y \\ y &= y \\ z &= 2x - 4y \end{aligned}$ These equations accept any solution if $x = y$, $z = 2x - 4y$. Taking $x = y = 1$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$
$\mathbf{P}$ is the matrix of eigenvectors. $\mathbf{D}$ is the diagonal matrix of eigenvalues, in the same order as their eigenvectors in $\mathbf{P}$.	$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

There is a relationship between any $\mathbf{A}$, $\mathbf{P}$ and $\mathbf{D}$ in general:

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}.$$

If $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, its entries are a reflection along the leading diagonal. Such a matrix is called **symmetric**, and its **normalised eigenvectors are always orthogonal to each other**. A matrix made of orthogonal vectors is called an **orthogonal matrix**. Orthogonal matrices have the property that **their transpose is their inverse** and vice versa. In this case, **orthogonal diagonalisation** is used.

If $\mathbf{A}$ is a symmetric matrix, $\mathbf{D} = \mathbf{P}^{T}\mathbf{AP}$ since if $\mathbf{A}$ is symmetric, $\mathbf{P}$ is orthogonal. Hence, $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$.

Example: Diagonalise $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ by finding a matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

Find and solve the characteristic equation to obtain the eigenvalues.	$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix}$ $= (3-\lambda)^2 - 2^2 = -\lambda^2 - 6\lambda + 5 = 0 = (\lambda - 1)(5 - \lambda) = 0 \Rightarrow \lambda = 1, 5$
Since $\mathbf{A}$ is symmetric, use orthogonal diagonalisation. Find the normalised eigenvector for λ_1.	When $\lambda = 1$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ Equating elements from either side, $\begin{aligned} 3x + 2y &= x \\ 2x + 3y &= y \end{aligned} \Rightarrow \begin{aligned} x + y &= 0 \\ x + y &= 0 \end{aligned}$ These equations accept any solution if $y = -x$, $\therefore \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\therefore \hat{\mathbf{v}} = \frac{1}{\sqrt{1^2 + (-1)^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$
Find the normalised eigenvector for λ_2.	When $\lambda = 5$, $\mathbf{Av} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$ Equating elements from either side, $\begin{aligned} 3x + 2y &= 5x \\ 2x + 3y &= 5y \end{aligned} \Rightarrow \begin{aligned} -x + y &= 0 \\ x - y &= 0 \end{aligned}$ These equations accept any solution if $y = x$, $\therefore \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\therefore \hat{\mathbf{v}} = \frac{1}{\sqrt{1^2 + 1^2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
$\mathbf{P}$ is the matrix of normalised eigenvectors, and $\mathbf{D}$ is the diagonal matrix of eigenvalues, as before.	$\mathbf{P} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

Diagonal matrices are easier to perform long calculations with. For any diagonal $k \times k$ matrix $\mathbf{D}$,

$$\text{If } \mathbf{D} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}, \quad \mathbf{D}^n = \begin{pmatrix} a_1^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k^n \end{pmatrix}.$$

Cayley-Hamilton Theorem

This theorem states that every square matrix $\mathbf{A}$ is a solution to its own characteristic equation. Plugging a matrix into a polynomial feels unnatural, but it works by considering $\mathbf{I}$ as 1.

Example: Verify the Cayley-Hamilton Theorem on $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ and use it to find its inverse.

Find the characteristic equation of $\mathbf{A}$.	$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0 = \lambda(\lambda + 3) + 2 = \lambda^2 + 3\lambda + 2 = 0$
Use the Cayley-Hamilton Theorem.	$\mathbf{A}^2 + 3\mathbf{A} + 2\mathbf{I} = \mathbf{0} \xrightarrow{\textit{Multiply by } \mathbf{A}^{-1}} \mathbf{A} + 3\mathbf{I} + 2\mathbf{A}^{-1} = \mathbf{0}$
Rearrange for $\mathbf{A}^{-1}$.	$\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} - \frac{3}{2}\mathbf{I} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3/2 & -1/2 \\ 1 & 0 \end{pmatrix}$