Recurrence Relations Cheat Sheet

Edexcel FP2

Recurrence relations are used to reduce and model problems using iterative processes. Populations, interest levels, drug levels in the bloodstream and many more scenarios can be modelled using recurrence relations, as the previous 'level' forms the basis for the next one.

Forming recurrence relations

Recurrence relations describe each term of a sequence in terms of the previous term or terms. For example, if there are 250 bacteria in a petri dish, and the population increases by 25% every 5 hours, with 50 bacteria taken out for a sample after each set of 5 hours, then we can model how big the population will get, for example after 15 or 20 hours:

- $u_{m+1} = 1.25\,u_m - 50$, with $u_0 = 250$

This is a first-order relation, as u_n is given as a function of only n and u_{n-1}

Each recurrence relation will have an initial condition, be it the initial population or a starting amount of money in the bank, often dented by u_0.

Example 1: A man takes out a loan for £10,000. At the end of every month, the interest rate of 0.75% is added, and he makes a payment of £250. Construct a recurrence relation and calculate how much money is owed after 4 months

Construct the recurrence relation, using the initial condition and information given in the question	$u_{m+1} = 1.0075\,u_m - 250, u_0 = 10{,}000$
Calculate u_1.	$u_1 = 1.0075(10000) - 250$ $u_1 = 9825$
Calculate u_2 by substituting the value calculated for u_1 into the relation	$u_2 = 1.0075(9825) - 250$ $u_2 = 9648.69$
Continue substituting the values until the value for u_4 is calculated	$u_3 = 1.0075(9648.69) - 250$ $u_3 = 9471.05$ $u_4 = 1.0075(9471.05) - 250$ $u_4 = 9292.08$

Solving first-order recurrence relations

To solve a first-order recurrence relation, you have to find a closed form for the terms in the sequence in the form $u_n = f(n)$. A first-order linear relation can be written in the form $u_n = au_{n-1} + g(n)$, where a is a real constant. If $g(n) = 0$, the equation is called homogeneous. Some recurrence relations can be solved using back substitution.

Example 2: Solve the recurrence relation $a_n = 4a_{n-1}$, with $a_0 = 2$

Notice that $a_{n-1} = 4a_{n-2}$ and substitute into the equation given	$a_n = 4(4(a_{n-2}))$ $a_n = 4^2 a_{n-2}$
Continue substituting, eg $a_{n-2} = 4a_{n-3}$	$a_n = 4^2 \times 4 \times a_{n-3}$
Notice the pattern forming	$a_n = 4^b a_{n-b}$
Continue the back-substitution until you can substitute a_0 in $(b = n)$	$a_n = 4^n a_0 = 4^n(2)$

Normally, you will be asked to prove the closed form by induction, but in an exam you will not need to do this for homogeneous recurrence relations of the form we have seen before. You can think of having general and particular solutions to recurrence relations, much like solving differential equations. The general solution to the recurrence relation $u_n = au_{n-1}$ will be in the form $u_n = ca^n$, where c is and arbitrary constant and the initial condition(s) will give the value of c. Some non-homogeneous relations can also be solved using back-substitution.

Example 3: Find a solution to the recurrence relation $u_n = u_{n-1} + n^2$, with $u_0 = 1$

Use iteration to back-substitute	$u_n = u_{n-1} + n^2$ $u_n = (u_{n-2} + (n-1)^2) + n^2$ $\vdots \quad \vdots$ $u_n = u_0 + \sum_{r=1}^{n} r^2$
Substitute in the formula $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ and the value for u_0	$u_n = 1 + \frac{n(n+1)(2n+1)}{6}$

This can be extended to a general form:

- The solution to the first-order non-homogeneous linear recurrence relation $u_n = u_{n-1} + g(n)$ is given by $u_n = u_0 + \sum_{r=1}^{n} g(r)$.

Example 4: Solve the recurrence relations $u_n = u_{n-1} + 4n^3$, with $u_0 = 5$, $r_n = r_{n-1} + 2^n + 1$, with $r_1 = 4$

Use the formula given	$u_n = u_0 + \sum_{r=1}^{n} g(r)$ $u_n = 5 + 4\sum_{r=1}^{n} r^3$
Use the standard result for $\sum_{r=1}^{n} r^3$.	$u_n = 5 + 4\left(\frac{n^2(n+1)^2}{4}\right)$ $u_n = 5 + n^2(n+1)^2$
Find an expression for r_0	$r_n = r_{n-1} + 2^n + 1$ $r_1 = r_0 + 2 + 1$ $\Rightarrow r_0 = r_1 - 3$ $r_0 = 4 - 3 = 1$
Use the formula given	$r_n = r_0 + \sum_{a=1}^{n} 2^a + \sum_{a=1}^{n} 1$
Use the standard result for $\sum_{a=1}^{n} 1$ and the fact that $\sum_{a=1}^{n} 2^n$ is a geometric series with initial term 2 and common ratio 2	$r_n = 1 + \frac{2(1-2^n)}{1-2} + n$

As the recurrence relations get more complicated, for example of the from $u_n = au_{n-1} + g(n)$, where $a \neq 1$, then the back substitution also becomes increasingly complicated. It can be useful to solve the non-homogeneous recurrence relation first, in this case $u_n = au_{n-1}$. This solution is called the complementary function, and then a particular solution will need to be added to the recurrence relation, which can take a variety of forms:

Form of $g(n)$	Form of particular solution
p with $a \neq 1$	λ
$pn + q$ with $a \neq 1$	$\lambda n + \mu$
kp^n with $p \neq a$	λk^n
ka^n	$\lambda n a^n$
p with $a = 1$	λn
$pn + q$ with $a = 1$	$\lambda n^2 + \mu n$

The particular solution will satisfy the recurrence relation but may not satisfy the initial conditions, which you will need to check.

Example 5: Solve the recurrence relation $u_n = 4u_{n-1} + 3n + 2$, with $u_1 = 2$

Find the general solution to the homogeneous recurrence relation	Homogeneous relation: $u_n = 4u_{n-1}$ From the previous work, the general solution to this, and the complementary function overall is $u_n = c(4^n)$
Using the forms in the table above, find the particular solution. $g(n)$ is of the form $pn + q$, so the particular solution is of the form $\lambda n + \mu$.	$u_n = 4u_{n-1} + 3n + 2$ The particular solution is of the form $u_n = \lambda n + \mu$, and so $u_{n-1} = \lambda(n-1) + \mu$ $\lambda n + \mu = 4(\lambda(n-1) + \mu) + 3n + 2$ $\lambda n + \mu = 4\lambda n - 4\lambda + 4\mu + 3n + 2$ $0 = 3\lambda n - 4\lambda + 3\mu + 3n + 2$
Group together the coefficients of n and the constant coefficients, and equate to the left-hand side	$0 = (3\lambda + 3)n - 4\lambda + 3\mu + 2$ $\Rightarrow 3\lambda + 3 = 0$ $\Rightarrow -4\lambda + 3\mu + 2 = 0$ $\lambda = -1, \mu = -2$
State the particular solution and the general solution (complementary function + particular solution)	The particular solution to this recurrence relation is $u_n = -n - 2$ The general solution is $u_n = c(4^n) - n - 2$
Substitute in the initial conditions to find c and the solutions.	Since $u_1 = 2$ $2 = c(4^1) - 1 - 2$ $5 = 4c$ $c = \frac{5}{4}$ $u_n = \frac{5}{4}(4^n) - n - 2$

Solving second-order recurrence relations

A second-order linear recurrence relation can be written in the from $u_n = au_{n-1} + bu_{n-2} + g(n)$, with $a, b \in \mathbb{R}$. Again, if $g(n) = 0$, then the equation is homogeneous.

- If $u_n = F(n)$ and $u_n = G(n)$ are particular solutions to a linear recurrence relation, then $u_n = aF(n) + bG(n)$, where a and b are constants, is also a solution.

The general solution to a second-order homogeneous linear recurrence relation, $u_n = au_{n-1} + bu_{n-2}$ is found by considering the auxiliary, or characteristic equation $r^2 - ar - b = 0$.

Three cases need to be considered:

1. The auxiliary equation has distinct, real roots denoted α and β. The general solution has the form $u_n = A\alpha^n + B\beta^n$, where A and B are arbitrary constants
2. The auxiliary equation has a repeated root α. The general solution will have the form $u_n = (A + Bn)\alpha^n$, where A and B are arbitrary constants
3. The auxiliary equation has complex roots α and β. The general solution will have the form $u_n = r^n(A\cos n\theta + B\sin n\theta)$, or $u_n = A\alpha^n + B\beta^n$, with A and B arbitrary constants.

Example 6: Find a general solution to the recurrence relation $u_n = -2u_{n-1} - 4u_{n-2}, n \geq 2$. Given that $u_0 = 15$ and $u_1 = 7$, find a particular solution

Construct and solve the auxiliary equation	$r^2 + 2r + 4 = 0$ $\Rightarrow r = -1 \pm \sqrt{3}i$
The auxiliary equation has complex roots so the general solution will have the form $u_n = r^n(A\cos n\theta + B\sin n\theta)$	Putting the roots in exponential form, we obtain $-1 + \sqrt{3}i = 2e^{\frac{2\pi}{3}i}$ and $-1 - \sqrt{3}i = 2e^{-\frac{2\pi}{3}i}$, so $\theta = \frac{2\pi}{3}$, $r = 2$ and the general solution is of the form $u_n = 2^n(A\cos n\frac{2\pi}{3} + B\sin n\frac{2\pi}{3})$
Use the initial conditions and the general solution to write two simultaneous equations in A and B.	$n = 0$: $\quad 15 = A\cos 0 + B\sin 0$ $15 = A$ $n = 1$: $7 = 2(A\cos\frac{2\pi}{3} + B\sin\frac{2\pi}{3})$ $7 = 2(-A\frac{1}{2} + B\frac{\sqrt{3}}{2})$ $7 = -15 + B\sqrt{3}$ $B = \frac{22\sqrt{3}}{3}$
State the solution	$u_n = 2^n(15\cos n\frac{2\pi}{3} + \frac{22\sqrt{3}}{3}\sin n\frac{2\pi}{3})$

As with first-order recurrence relations, non-homogeneous linear second-order recurrence relations can be solved by considering the complementary function and finding a suitable particular solution.

To solve the recurrence relation $u_n = au_{n-1} + bu_{n-2} + g(n)$:

- Solve the associated homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2}$. This is the complementary function
- Choose the relevant general form for a particular solution and substitute into the original equation and find the values of coefficients
- The general solution is $u_n = \text{CF} + \text{PS}$.
- Use the initial conditions to find the values of the arbitrary constants

The relevant forms for the particular solution can be found in the table below, with α and β being the roots of the auxiliary equation:

Form of $g(n)$	**Form of particular solution**
p with $\alpha, \beta \neq 1$	λ
$pn + q$, with $\alpha, \beta \neq 1$	$\lambda n + \mu$
kp^n with $p \neq \alpha, \beta$	λp^n
p with $\alpha = 1, \beta \neq 1$	λn
$pn + q$ with $\alpha = 1, \beta \neq 1$	$\lambda n^2 + \mu n$
p with $\alpha = \beta = 1$	λn^2
$pn + q$ with $\alpha = \beta = 1$	$\lambda n^3 + \mu n^2$
$k\alpha^n$ with $\alpha \neq \beta$	$\lambda n\alpha^n$
$k\alpha^n$ with $\alpha = \beta$	$\lambda n^2\alpha^n$

The particular solution can't have any terms in common with the complementary function, much like with differential equations, and the bottom 6 rows show ways to avoid this.

Example 7: Solve the recurrence relation $u_n = u_{n-1} + 2u_{n-2} + 3(4^n)$

Solve the associated homogeneous recurrence relation $u_n = u_{n-1} + 2u_{n-2}$ by finding the auxiliary equation and the relevant complementary function	$r^2 - r - 2 = 0$ $\Rightarrow r = 2, r = -1$ So the complementary function is $u_n = A(-1)^n + B(2)^n$
Using the table, find the appropriate form of the particular solution ($g(n)$ is of the form kp^n with $p \neq \alpha, \beta$). Substitute into the original recurrence relation.	Try the particular solution $u_n = \lambda(4^n)$ $u_n = u_{n-1} + 2u_{n-2} + 3(4^n)$ $\lambda(4^n) = \lambda 4^{n-1} + 2\lambda 4^{n-2} + 3(4^n)$
Note that $4^{n-1} = 4^n \times 4^{-1}$. We can use this to put everything in terms of 4^n and equate coefficients.	$\lambda(4^n) = \frac{\lambda}{4}4^n + \frac{2\lambda}{4^2}4^n + 3(4^n)$
Equate the coefficients	$\lambda = \frac{\lambda}{4} + \frac{2\lambda}{16} + 3$ $\frac{5}{8}\lambda = 3$ $\Rightarrow \lambda = \frac{24}{5}$
State the particular solution and state the general solution	So a particular solution is $\frac{24}{5}(4^n)$ and the general solution is: $u_n = A(-1)^n + B(2)^n + \frac{24}{5}(4^n)$

Proving closed forms

Closed forms can be proven to be true by using mathematical induction.

For first-order recurrence relations:

1. Show that the basis step is true
2. Use this to assume that the closed form is true for $n = k$
3. Use induction, the basis step and the assumption to prove that the closed form is true for the required values.
4. Write a conclusion to show what you have proved

For second-order recurrence relations:

1. Show that the closed form is true for two consecutive values of n as the basis step
2. In the assumption step, assume that the closed form is true for both $n = k$ and $n = k - 1$
3. In the inductive steps, use your assumptions and induction to show that the closed form is true for $n = k + 1$
4. Remember to write a conclusion to show what you have proved