Reducible Differential Equations Cheat Sheet

Many real-life situations can be modelled using differential equations. As a result, it is very useful to be able to solve these equations. Numerous methods can be employed to solve differential equations of different forms.

First-order differential equations

Substitutions allow us to solve a first-order differential equation, either by separating the variables or using an integrating factor.

Example 1: Using the substitution $u = x + y$, transform the differential equation $\frac{dy}{dx} = \frac{x+y-1}{5-x-y}$ into the form $4dx = (5-u)du$, and solve with the initial condition that $y(0) = 2$.

First, differentiate the substitution. Note that $y' = \frac{dy}{dx}$.	$u = x + y, \Rightarrow u' = 1 + y'$ $\Rightarrow y' = u' - 1$
Substitute y' into the differential equation and rearrange.	$u' - 1 = \frac{u-1}{5-u} \Rightarrow u' = \frac{u-1}{5-u} + 1$ $\Rightarrow u' = \frac{4}{5-u}$ $\Rightarrow \frac{du}{dx} = \frac{4}{5-u} \Rightarrow (5-u)du = 4dx$
This can be solved using separation of variables.	$\int (5-u)du = \int 4dx$ $5u - \frac{u^2}{2} = 4x + c$
Substitute u back in.	$5(x+y) - \frac{(x+y)^2}{2} = 4x + c$ $\Rightarrow 10(x+y) - (x+y)^2 = 8x + 2c$ $\Rightarrow 10x + 10y - x^2 - 2xy - y^2 = 8x + c_1$ $\Rightarrow 2x + 10y - 2xy - x^2 - y^2 = c_1$
Use the given condition $y(0) = 2$ to find c_1.	$2(0) + 10(2) - 2(0)(2) - (0)^2 - (2)^2 = c_1$ $\Rightarrow c_1 = 16$
State the particular solution.	$2x + 10y - 2xy - x^2 - y^2 = 16$

As per the specification, the substitutions will be given, and be reducible to the forms of differential equations you will have seen previously.

Example 2: Use the substitution $z = \sqrt{x}$ to transform the differential equation $\frac{dx}{dt} + t^2x = t^2\sqrt{x}$ into one in terms of t and z. Using the transformed equation, find the general solution to the original equation.

Rearrange the substitution to make x the subject and find the derivative. Remember that z is a function of x and t.	$z = x^{\frac{1}{2}} \Rightarrow x = z^2 \Rightarrow \frac{dx}{dt} = 2z\frac{dz}{dt}$
Substitute $\frac{dx}{dt}$ into the original equation.	$2z\frac{dz}{dt} + t^2z^2 = t^2z$
Divide both sides by $2z$.	$\frac{dz}{dt} + \frac{t^2}{2}z = \frac{t^2}{2}$
Notice that the transformed equation can be solved using an integrating factor.	$I = e^{\int \frac{t^2}{2}dt}$ $I = e^{\frac{t^3}{6}}$
Multiply both sides of the equation by the integrating factor and solve.	$e^{\frac{t^3}{6}}\frac{dz}{dt} + e^{\frac{t^3}{6}}\frac{t}{2}z = e^{\frac{t^3}{6}}\frac{t^2}{2}$ $\frac{d}{dt}\left(e^{\frac{t^3}{6}}z\right) = e^{\frac{t^3}{6}}\frac{t^2}{2}$ Integrating both sides with respect to t we get, $e^{\frac{t^3}{6}}z = e^{\frac{t^3}{6}} + c$
Rearrange to find z.	$z = 1 + \frac{c}{e^{\frac{t^3}{6}}}$
Reverse the original substitution and write down the general solution.	$z = 1 + \frac{c}{e^{\frac{t^3}{6}}} \Rightarrow \sqrt{x} = 1 + \frac{c}{e^{\frac{t^3}{6}}}$ $\Rightarrow x = (1 + \frac{c}{e^{\frac{t^3}{6}}})^2$

Second order differential equations

Substitutions can also reduce second order differential equations into those of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$. Again, in an exam you will be given the substitution necessary.

Example 3: Find the general solution of the differential equation $x^2\frac{d^2y}{dx^2} + 12x\frac{dy}{dx} + 10y = 0$ using the substitution $x = e^u$, which can be solved by considering the auxiliary equation. (Remember that u is a function of x)

To make the substitution, we need to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{du}$. We can differentiate $\frac{d}{du}\left(e^u\frac{dy}{dx}\right)$ by using the product rule and chain rule together. Use $\frac{dx}{du} = e^u = x$ to simplify $e^u\frac{d^2y}{dx^2}\frac{dx}{du}$.	$x = e^u \Rightarrow \frac{dx}{du} = e^u = x$ Using the chain rule, $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u\frac{dy}{dx} = x\frac{dy}{dx}$ Using the product rule and chain rule, $\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(e^u\frac{dy}{dx}\right)$ $= e^u\frac{dy}{dx} + e^u\frac{d^2y}{dx^2}\frac{dx}{du}$ $= \frac{dy}{du} + x^2\frac{d^2y}{dx^2}$ $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$
Substitute the expressions found for $x^2\frac{d^2y}{dx^2}$ and $x\frac{dy}{dx}$ into the original differential equation and simplify.	$x^2\frac{d^2y}{dx^2} + 12x\frac{dy}{dx} + 10y = 0$ $\Rightarrow \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + 12\frac{dy}{du} + 10y = 0$ $\Rightarrow \frac{d^2y}{du^2} + 11\frac{dy}{du} + 10y = 0$
Write down and solve the auxiliary equation.	$m^2 + 11m + 10 = 0$ $m_1 = -1, \quad m_2 = -10$
Write down the general solution in terms of u.	$y = Ae^{m_1u} + Be^{m_2u}$ $y = Ae^{-u} + Be^{-10u}$ Where A and B are arbitrary constants
Reverse the substitution.	$x = e^u \Rightarrow u = \ln x$ $y = Ae^{-\ln x} + Be^{-10\ln x}$ $y = \frac{A}{x} + \frac{B}{x^{10}}$

Modelling with differential equations

Questions may be asked in different ways, for example you may be asked to model real-life situations such as the velocity of a particle, displacement of a particle, or movement of a spring. You may have to state conditions for the system or explain what happens to the model over time.

Example 4: The velocity, v, of a particle at time t seconds is modelled using the differential equation,

$$2t\frac{dv}{dt} + 2v = -3t^2v^3$$

a) Use the substitution $z = vt$ to transform the differential equation into one in terms of z and t.

b) Given that when $t = 1, v = 1$, find the general solution.

c) Find the velocity at $t = 3$s.

a) Find expressions for $\frac{dz}{dt}$ and v.	$z = vt \Rightarrow \frac{dz}{dt} = t\frac{dv}{dt} + v$ $\Rightarrow \frac{dv}{dt} = \frac{\frac{dz}{dt} - v}{t}$ $v = \frac{z}{t}$
Substitute into the original differential equation.	$2t\left(\frac{\frac{dz}{dt} - v}{t}\right) + 2v = -3t^2\left(\frac{z}{t}\right)^3$ $\Rightarrow 2\left(\frac{dz}{dt}\right) = -3t^2\frac{z^3}{t^3} \Rightarrow 2\left(\frac{dz}{dt}\right) = -\frac{3z^3}{t}$
b) Separate the variables and integrate.	$2\left(\frac{dz}{dt}\right) = -\frac{3z^3}{t} \Rightarrow \frac{dz}{dt} = -\frac{3z^3}{2t}$ $\Rightarrow -\frac{1}{z^3}dz = \frac{3}{2t}dt$ $\int -\frac{1}{z^3}\,dz = \int \frac{3}{2t}\,dt$ $\Rightarrow \frac{1}{2z^2} = \frac{3}{2}\ln t + c$
Rearrange for z.	$2z^2 = \frac{2}{3\ln t + 2c}$ $\Rightarrow z = \sqrt{\frac{1}{3\ln t + 2c}}$
Reverse the substitution.	$vt = \sqrt{\frac{1}{3\ln t + 2c}} \Rightarrow vt = \sqrt{\frac{1}{3\ln t + c_1}}$ Thus, $v = \frac{1}{t}\sqrt{\frac{1}{3\ln t + c_1}}$
Substitute in the conditions $t = 1, v = 1$ to find c_1.	$1 = \sqrt{\frac{1}{c_1}}$ $c_1 = 1$
Thus, state the particular solution.	$v = \frac{1}{t}\sqrt{\frac{1}{3\ln t + 1}}$
c) Substitute in $t = 3$ to find v.	$v = \frac{1}{3}\sqrt{\frac{1}{3\ln 3 + 1}}$ $v = 0.161(3.\text{sig. figs})$