The t-formulae Cheat Sheet

This topic builds upon previous knowledge of trigonometric identities. The t-formulae are a group of formulae that enable trigonometric identities to be expressed in a different way. They are very useful in solving some trigonometric equations and evaluating certain integrals involving trigonometric functions by converting them to algebraic solutions, which also allows for easier modelling- a real life application of this is looking at blood pressure.

The t-formulae

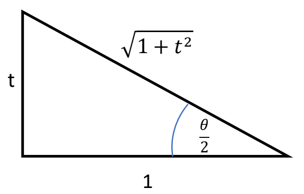
The t-formulae allow trigonometric identities to be expressed in terms of a single variable, t. Unlike in parametric equations, t is defined as $t = \tan\frac{\theta}{2}$, therefore we can write expressions involving $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of it:

When $t = \tan\frac{\theta}{2}$:

- $\sin\theta = \frac{2t}{1+t^2}$
- $\cos\theta = \frac{1-t^2}{1+t^2}$
- $\tan\theta = \frac{2t}{1-t^2}$

These formulae are **not** given in the formula book, you must learn them and their derivations.

<u>Example 1:</u> Derive the t-formulae

The easiest way to derive the t-formulae is to draw a diagram of a right-angled triangle with acute angle $\frac{\theta}{2}$. By Pythagoras' theorem the hypotenuse is $\sqrt{1+t^2}$.	
Apply the SOH CAH TOA ratios to the triangle	$\tan\frac{\theta}{2} = t$ $\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$ $\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$
However, we want to find $\sin\theta$, $\cos\theta$ and $\tan\theta$, so we need to use the double angle formulae: $\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}}$ $= \mathbf{\frac{2t}{1+t^2}}$ $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \mathbf{\frac{1-t^2}{1+t^2}}$ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} = \mathbf{\frac{2t}{1-t^2}}$

Although we assume that $\frac{\theta}{2}$ is acute, the formulae do hold in general. This can be seen by algebraic proofs:

- $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2}$
- $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos^2\frac{\theta}{2}(1 - \tan^2\frac{\theta}{2}) = \frac{1-\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}$
- And $\tan\theta$ can be proved exactly as before, using $\frac{\sin\theta}{\cos\theta}$ from the above forms in t.

<u>Example 2:</u> Given that $\tan\frac{\theta}{2} = \frac{5}{8}$, find the exact value of $\sec\theta$.

Substitute the value of $\tan\frac{\theta}{2}$ into the t-formula for cos	$\cos\theta = \frac{1-t^2}{1+t^2}$ $\cos\theta = \frac{1-(\frac{5}{8})^2}{1+(\frac{5}{8})^2} = \frac{39}{89}$
Use the identity that $\sec\theta = \frac{1}{\cos\theta}$	$\sec\theta = \frac{89}{39}$

Exam questions may also require you to use other trigonometric identities and give you a range that θ could be in, requiring you to select the appropriate answers- make sure you are confident with this!

Applying the t-formulae to trigonometric identities

The t-formulae can also be used to prove trigonometric identities

<u>Example 3:</u> Using the t-formulae, prove $\sin^2\theta + \cos^2\theta = 1$

Recall the t-formulae for $\sin\theta$ and $\cos\theta$.	$\sin\theta = \frac{2t}{1+t^2}, \cos\theta = \frac{1-t^2}{1+t^2}$
Substitute the formulae into the equation	$(\frac{2t}{1+t^2})^2 + (\frac{1-t^2}{1+t^2})^2 = 1$
Simplify	$\frac{4t^2}{(1+t^2)^2} + \frac{1+t^4-2t^2}{(1+t^2)^2} = 1$ $\frac{t^4+4t^2+1}{(1+t^2)^2} = 1$ $\frac{(1+t^2)^2}{(1+t^2)^2} = 1$

Solving trigonometric equations

The t-formulae can be used to change equations from being in terms of trigonometric functions of θ into algebraic equations in terms of t, which can then be solved algebraically. The substitution then has to be 'undone', finding the values of θ that correspond to the values of t. The substitution is only defined when $\tan\frac{\theta}{2}$ is defined- solutions of the form $\theta = (2n+1)\pi, n \in \mathbb{Z}$ can't be found using this method.

<u>Example 4:</u> Using the t-formulae, solve '$2\cot\theta - \sec\theta = 0$' for θ in the range $0 \leq \theta \leq 2\pi$.

Recall the required t-formulae	Using the substitution $t = \tan\frac{\theta}{2}$ $\cos\theta = \frac{1-t^2}{1+t^2} \Rightarrow \sec\theta = \frac{1+t^2}{1-t^2}$ $\tan\theta = \frac{2t}{1-t^2} \Longrightarrow \cot\theta = \frac{1-t^2}{2t}$
Substitute in the t-formula and simplify	$2\cot\theta - \mathrm{cosec}\,\theta = 0$ $2\left(\frac{1-t^2}{2t}\right) - \frac{1+t^2}{1-t^2} = 0$ $\frac{1-t^2}{t} - \frac{1+t^2}{1-t^2} = 0$ $\frac{(1-t^2)(1-t^2)}{t(1-t^2)} - \frac{t(1+t^2)}{t(1-t^2)} = 0$ $\frac{t^4-t^3-2t^2-t+1}{t(1-t^2)} = 0$
Solve the quartic equation. For equations where it is a fraction=0, it is enough to solve the equation where the numerator is 0.	$t^4 - t^3 - 2t^2 - t + 1 = 0$ $t = 2.081, t = 0.4805$
Find the values of θ in the range required by substituting in the values we found for t.	$2.081 = \tan\frac{\theta}{2}$ $1.123 = \frac{\theta}{2}$ $2.246 = \theta$ As $\tan\frac{x}{2}$ has a period of 2π (not π !), another solution could be found by adding 2π to the solution already found, however this is outside of the range required by the question $\theta = 2.246$ $0.4805 = \tan\frac{\theta}{2}$ $0.448 = \frac{\theta}{2}$ $0.896 = \theta$ So the two solutions are $\theta = 2.246, \theta = 0.896$

Modelling with trigonometry

Trigonometric functions are incredibly useful in modelling, especially models that are periodic. More advanced situations can be modelled by adding and subtracting multiples of trigonometric functions, which can be simplified with the t-formulae.

<u>Example 5:</u> The displacement of a particle moving in a straight line, s m, at time x seconds is given by $s = \frac{5}{2} - 3\cos x + 4\sin 2x$, $0 \leq x \leq 2\pi$. Find all values of x for which the particle is stationary.

Differentiate the displacement with respect to time to find the velocity	$v = \frac{ds}{dx} = 3\sin x + 8\cos 2x$
Substitute in the t-formulae and the double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$ We have deliberately taken time as x so we don't confuse it with our t-formulae	$v = 3\left(\frac{2t}{1+t^2}\right) + 8(1 - 2(\frac{2t}{1+t^2})^2)$
Simplify the equation	$v = \frac{6t}{1+t^2} + 8 - 16(\frac{4t^2}{(1+t^2)^2})$ $v = \frac{6t}{1+t^2} + 8 - \frac{64t^2}{(1+t^2)^2}$ $v = \frac{6t(1+t^2) + 8(1+t^2)^2 - 64t^2}{(1+t^2)^2}$ $v = \frac{6t + 6t^3 + 8 + 16t^2 + 8t^4 - 64t^2}{(1+t^2)^2}$ $v = \frac{8t^4 + 6t^3 - 48t^2 + 6t + 8}{(1+t^2)^2}$
To find the values of t where the particle is stationary, we need to set $v = 0$. Solve the equation (polynomial solver or plot graph on graphics calculator)	$0 = 8t^4 + 6t^3 - 48t^2 + 6t + 8$ $t = 1.9708, t = 0.5074, t = -0.3471,$ $t = -2.881$
Substitute t back into $t = \tan\frac{\theta}{2}$	$1.9708 = \tan\frac{\theta}{2}$ $\frac{\theta}{2} = 1.101$ $\theta = 2.2024$ $0.5074 = \tan\frac{\theta}{2} \Longrightarrow \theta = 0.9391$ $-0.3471 = \tan\frac{\theta}{2} \Longrightarrow \theta = -0.6682$ $-2.881 = \tan\frac{\theta}{2} \Longrightarrow \theta = -2.4734$
Find the values of θ within the range required	$\theta = -0.6682 + 2\pi = 5.615$ $\theta = -2.4734 + 2\pi = 3.809$
	$\theta = 0.9391, \theta = 2.2024, \theta = 3.809,$ $\theta = 5.615$