Inequalities Cheat Sheet

Edexcel Further Pure 1

In this chapter, you will learn to solve inequalities with algebraic fractions or modulus, by using either the algebraic or graphical approach.

Algebraic Methods

For inequalities which involve an algebraic fraction, for example,

$$\frac{x}{x+3} < x$$

We cannot simply multiply x by $(x+3)$ and cancel out x, as this will only solve for one of the critical values.

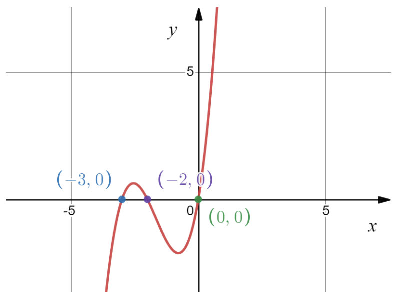
To solve these inequalities:

- We first multiply the whole inequality with the square of the denominator, which is $(x+3)^2$ in this case. We use the squared denominator to ensure that it is always positive, as multiplying the whole inequality by a negative number will cause the inequality sign to be turned, and wrong intervals will be chosen.
- Then, we rearrange the terms so that all terms are on one side of the sign and the other side is 0.
- Next, we solve the inequality for the critical values.
- Using the critical values and by sketching a graph, we can find the intervals which fulfil the inequality.

Strict and Non-strict Inequalities

- Strict inequalities are in the form of $a < b$ or $a > b$. This means that a is either lesser than or greater than b, but a and b are not equal.
- Non-strict inequalities are in the form of $a \leq b$ or $a \geq b$. This means that a is "lesser than or equals to" b or "greater than or equals to b".
- For strict inequalities, our critical values are not included in the solution, so we use the signs "<" or ">".
- For non-strict inequalities, our critical values are also included in the intervals, so we use the signs "≤" or "≥".

Example 1: Solve $\frac{x}{x+3} < x, x \neq -3$.

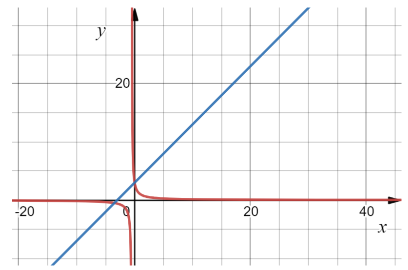
Multiply both sides of the inequality with the squared denominator, $(x+3)^2$	$\frac{x}{x+3}(x+3)^2 < x(x+3)^2$
Simplify the terms on each side	$x(x+3) < x(x+3)^2$
Move all terms to one side	$0 < x(x+3)^2 - x(x+3)$
Expand the brackets	$0 < x(x^2+6x+9) - (x^2+3x)$ $0 < x^3+6x^2+9x-x^2-3x$
Simplify the inequality	$0 < x^3+5x^2+6x$
Factorise to find the critical values	$0 < x(x^2+5x+6)$ $0 < x(x+2)(x+3)$ $\therefore$ CV $= -3, -2, 0$
Sketch the graph for the equation $y = x(x+2)(x=3)$	
Select the appropriate intervals. We are looking for values of x for which $y > 0$. Since we only want values for "greater than" and not "equal to", our critical values are not in the interval.	$-3 < x < -2$ or $x > 0$

If there are more than one algebraic fraction in the inequality, multiply both sides with the square denominators of each algebraic fractions.

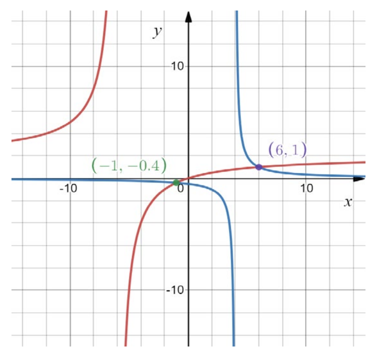
Graphical method

For an inequality $f(x) < g(x)$, you can sketch the graphs for $y = f(x)$ and $y = g(x)$ on the same set of axes, and then observe which parts of the graph satisfies the inequality. The critical values can be found by equating the left-hand side to right-hand side and solving the equation. Make sure to check if the question specifies whether an algebraic or graphical method should be used!

Example 2: Solve $\frac{3}{2x+1} \geq x+3$ using the graphical method.

Sketch the graphs $y = \frac{3}{2x+1}$ and $y = x+3$ on the same scale	
Find the points of intersection by equating LHS to RHS	$\frac{3}{2x+1} = x+3$
Solve the equation to find the critical values	$3 = (x+3)(2x+1)$ $3 = 2x^2+x+6x+3$ $2x^2+7x = 0$ $x(2x+7) = 0$ $\therefore$ CV $= 0, -\frac{7}{2}$
Note the vertical asymptotes	$x = 0$ and $x = -\frac{1}{2}$
Select the intervals according to the graph. We want to find the interval for which the red curve, $y = \frac{3}{2x+1}$, is above the blue line, $y = x+3$	$x < -\frac{7}{2}$ or $-\frac{1}{2} \leq x \leq 0$

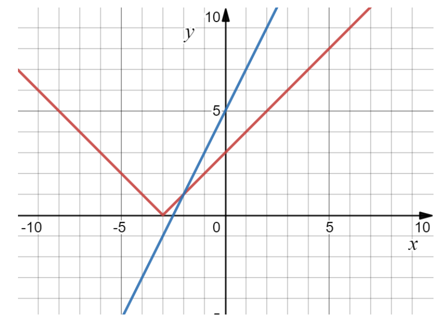
Example 3: Solve $\frac{2x}{x+6} > \frac{2}{x-4}$ using the graphical method.

Sketch the graphs $y = \frac{2x}{x+6}$ and $y = \frac{2}{x-4}$ on the same scale	
Find the points of intersection by equating LHS to RHS	$\frac{2x}{x+6} = \frac{2}{x-4}$
Solve the equation to find the critical values	$2x(x-4) = 2(x+6)$ $2x^2-8x = 2x+12$ $2x^2-10x-12 = 0$ $x^2-5x-6 = 0$ $(x-6)(x+1) = 0$ $\therefore$ CV $= -1, 6$
Note the vertical asymptotes	$x = -6$ and $x = 4$
Select the intervals according to the graph. We want to find the interval for which the red curve, $y = \frac{2x}{x+6}$, is above the blue curve, $y = \frac{2}{x-4}$.	$x < -6, -1 < x < 4$ or $x > 6$

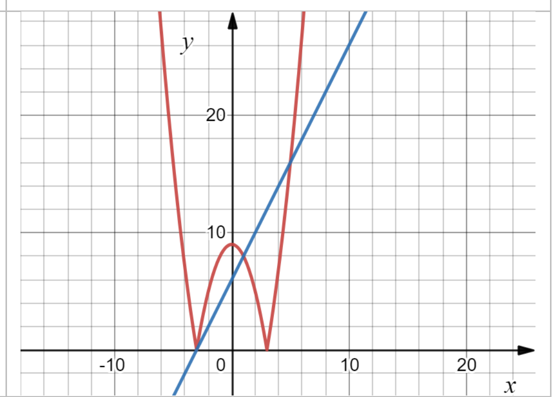
Modulus inequalities

For inequalities involving a modulus sign, it may be helpful to use the graphical approach.

Example 4: Solve $|x+3| < 2x+5$

Sketch $y = \|x+3\|$ and $y = 2x+5$	
Find the critical values by solving both $x+3 = 2x+5$ and $-(x+3) = 2x+5$	$x+3 = 2x+5$ $x = -2$ $-(x+3) = 2x+5$ $-x-3 = 2x+5$ $3x = -8$ $x = -\frac{8}{3}$ $\therefore$ CV $= -2, -\frac{8}{3}$
Using your graph and critical values found, you can identify the x-coordinate of the intersection point and find out what values of x satisfy the inequality. Here, we want the interval in which the red line is below the blue.	$x \geq -2$

Example 5: Solve $|x^2-9| - 2 > 2x+4$. Write you answer in set notation.

Rearrange the terms so that the modulus is on one side, and other terms are on the other side of the inequality sign	$\|x^2-9\| > 2x+6$
Sketch $y = \|x^2-9\|$ and $y = 2x+6$	
Find the critical values by solving $x^2-9 = 2x+6$	$x^2-9 = 2x+6$ $x^2-2x-15 = 0$ $(x-5)(x+3) = 0$ $\therefore$ CV $= -3, 5$
Find the remaining critical values by solving $-(x^2-9) = 2x+6$	$-(x^2-9) = 2x+6$ $x^2+2x-3 = 0$ $(x+3)(x-1) = 0$ $\therefore$ CV $= -3, 1$
Select the appropriate intervals. We are looking for values of x for which the red curve, $y = \|x^2-9\|$, is above the blue line, $y = 2x+6$.	$x < -3, -3 < x < 1$ or $x > 5$
Write your answer in set notation	$\{x: x < -3\} \cup \{x: -3 < x < 1\} \cup \{x: x > 5\}$