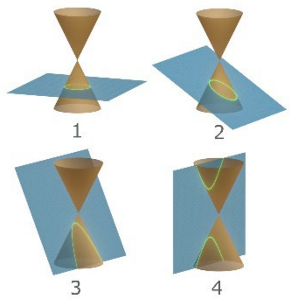
Conic Sections 2 Cheat Sheet

Edexcel FP1

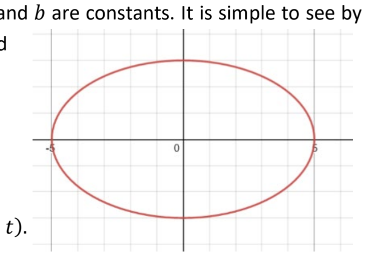
This chapter aims to build upon the graph sketching skills you developed in year 1 and the content covered in the previous chapter, conic sections 1. As described by the name, conic sections are planes produced by slicing two cones placed point to point- more mathematically, the intersection of cones with planes are the graphs that you will see in this chapter. Conic sections are used in modelling: planets travel in ellipses, the mirrors in solar power stations are parabolas and a variety of conic sections are used in engineering. In the set,of diagrams to the right, diagram 1 shows how a circle is obtained from a conic section, 2 shows an ellipse, 3 shows a parabola and 4 shows a hyperbola.



Ellipses

If you slice a cone such to produce a closed curve, the curve you have obtained is called an ellipse- a circle is a special case of an ellipse.

- An ellipse has the Cartesian equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are constants. It is simple to see by substituting in $x=0$ that the y-intercepts are at $y=\pm b$, and by substituting in $y=0$ the x-intercepts are at $x=\pm a$.
- Ellipses can also be defined using parametric equations. An ellipse has parametric equations $x=a\cos t, y=b\sin t, 0\le t<2\pi$. This can be derived by substituting $x=a\cos t$ and $y=b\sin t$ into the Cartesian equation. A general point P on an ellipse has co-ordinates $(a\cos t, b\sin t)$.



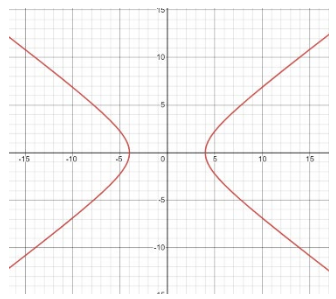
<u>Example 1:</u> Find the parametric equations for the ellipse with Cartesian co-ordinates $9x^2+25y^2=225$

Rewrite the equation given in the question to the general form for ellipses.	$9x^2+25y^2=225$ $\frac{x^2}{25}+\frac{y^2}{9}=1$
From the general form, identify a and b and use this to find the parametric forms.	$a=5, b=3$ So the parametric equations are $x=5\cos t, y=3\sin t$

Again, lots of questions in exams will build on pervious knowledge- make sure you are confident with finding points of intersections and distances!

Hyperbolas

We have already seen rectangular hyperbolas in the previous chapter, conic sections 1. However, it isn't necessary for hyperbolas to have perpendicular asymptotes, and we can find both Cartesian and parametric equations for standard hyperbolas



- A standard hyperbola has Cartesian equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Note that the fractions are not added- don't get them confused with ellipses! By substituting in $y=0$, we can see that $x^2=a^2$. To investigate the asymptotes, we can look at see what happens to the equation as x and y tend to infinity. As they tend to infinity, $\frac{x^2}{a^2}\approx\frac{y^2}{b^2}$, and thus the asymptotes are $y=\pm\frac{b}{a}x$. The equations of the asymptotes can also be found in the formula booklet. These asymptotes are perpendicular to one another, however, don't confuse all hyperbolas with perpendicular asymptotes as rectangular- only the curves of the form $xy=c^2$are.
- The parametric equations of the hyperbolas are connected to the hyperbolic trigonometric functions and their identities. The standard hyperbola has parametric equations $x=\pm a\cosh t, y=b\sinh t, t\in\mathbb{R}$, or equivalently, $x=a\sec\theta, y=b\tan\theta, -\pi\le\theta<\pi, \theta\ne\pm\frac{\pi}{2}$.

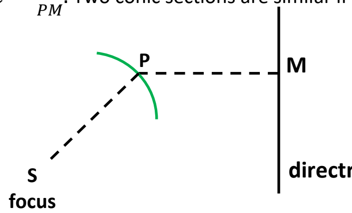
<u>Example 2:</u> Find the Cartesian equation of the hyperbola given by $x=\pm 5\cosh t, y=3\sinh t$.

Use the hyperbolic trigonometric identity $\cosh^2\theta-\sinh^2\theta=1$ to eliminate the parameter t	$x^2=25\cosh^2 t$ $y^2=9\sinh^2 t$ $\frac{x^2}{25}-\frac{y^2}{9}=\cosh^2 t-\sinh^2 t$ $\frac{x^2}{25}-\frac{y^2}{9}=1$

Eccentricity

Like shown with the parabola in the previous topic, conic sections can be defined in terms of their focus-directrix properties. To do this we need to consider the eccentricity of a conic section.

- For all points, P, on a conic section, the eccentricity of the curve is defined as the ratio of the distance of P from the focus, denoted S, to the shortest distance from P to the directrix (the point on the directrix is denoted M). The ratio is denoted e and is a unique non-negative real number that characterises its shape. In the diagram below, we can see that $e=\frac{PS}{PM}$. Two conic sections are similar if and only if they have the same eccentricity.



- If $0<e<1$, the point P is on an ellipse
- If $e=1$, the point P is on a parabola
- If $e>1$, the point P describes a hyperbola

If $e=0$, the curve is a circle and if $e=\infty$, the curve is a straight line, but this is beyond the concepts in this chapter.

For an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with $a>b$,

- As seen before, $0<e<1$, with the specific value of the eccentricity being given by $b^2=a^2(1-e^2)$
- The foci are at $(\pm ae, 0)$
- The directrices are given by the equations $x=\pm\frac{a}{e}$

As $a>b$, the ellipse is 'longer' in the x-direction. In this case, we call the x-axis the 'major axis'. If $b>a$, then the y-axis is the major axis, the foci will be on the y-axis at the points $(0,\pm be)$, directrices will be given by the equations $y=\pm\frac{b}{e}$ and the specific value of the eccentricity will be given by $a^2=b^2(1-e^2)$.

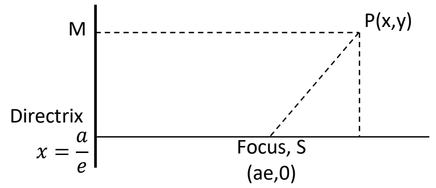
<u>Example 3:</u> Find the foci, equations of the directrices and state the major axis of the ellipse with the equation $4x^2+y^2=36$.

Put the equation into the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. In this case, we divide both sides of the equation by 36.	$4x^2+y^2=36$ $\frac{4x^2}{36}+\frac{y^2}{36}=1$ $\frac{x^2}{9}+\frac{y^2}{36}=1$
Notice that $b>a$, and state the major axis. We do not consider the negative square roots.	$a^2=9\Rightarrow a=3$ $b^2=36\Rightarrow b=6$ $b>a$, so the y-axis is the major axis.
In the case where the y-axis is the major axis, the eccentricity is given by $a^2=b^2(1-e^2)$.	$a^2=b^2(1-e^2)$ $9=36(1-e^2)$ $\frac{1}{4}=1-e^2$ $e^2=\frac{3}{4}\Rightarrow e=\frac{\sqrt{3}}{2}$
The foci are at $(0,\pm be)$, and the directrices are given by the equations $y=\pm\frac{b}{e}$	Foci are at $(0,\pm 3\sqrt{3})$ Directrices are given by $y=\pm\frac{12}{\sqrt{3}}$

For a hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

- As see before, $e>1$, with the specific value of the eccentricity being given by $b^2=a^2(e^2-1)$
- The foci are at $(\pm ae, 0)$
- The directrices are given by the equations $x=\pm\frac{a}{e}$

<u>Example 4:</u> Show that for $e>1$, the hyperbola with foci at $(\pm ae, 0)$ and directrices at $x=\pm\frac{a}{e}$ has equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

For questions like this it is often really helpful to sketch a diagram- it also helps the examiner know what you mean a lot more easily.	
Use the definition for eccentricity to find expressions for PS^2 and PM^2 in terms of a, e and x, y. The expression for PS^2 is found by using Pythagoras' theorem	$\frac{PS}{PM}=e\Rightarrow PS^2=e^2PM^2$ $PS^2=(x-ae)^2+y^2$ $PM^2=(x-\frac{a}{e})^2=\frac{(ex-a)^2}{e^2}$
Substitute these expressions into $PS^2=e^2PM^2$	$x^2-2aex+a^2e^2+y^2=e^2x^2-2aex+a^2$ $a^2(e^2-1)=x^2(e^2-1)-y^2$ $1=\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}$ So if $b^2=a^2(e^2-1)$, we have the standard equation of a hyperbola

Similar reasoning can be used to prove the equation of an ellipse given the foci and directrices.

Tangents and normals

Similarly to Conic Section 1, we can use parametric or implicit differentiation to find the gradient of an ellipse or hyperbola at a point, and therefore find the equations of the tangent and normal at that point. Don't try and memorise the general form for the normal or the tangent- they are very similar, easily confused and you may be asked to prove your result, the equations are easily derived!

<u>Example 5:</u> Find the equation of the normal to the ellipse with the equation $\frac{x^2}{9}+\frac{y^2}{36}=1$ at the point $P(3\cos t, 6\sin t)$.

Differentiate to find the gradient. You can either differentiate parametrically or implicitly- both are demonstrated.	$y=6\sin t, x=3\cos t$ $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6\cos t}{-3\sin t}=\frac{-2\cos t}{\sin t}$ Differentiating implicitly: $\frac{x^2}{9}+\frac{y^2}{36}=1$ $\frac{2}{9}x+\frac{1}{18}y\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{4x}{y}$
Use $y=mx+c$ or $y-y_1=m(x-x_1)$ to find the equation of the normal.	Gradient of the normal $=\frac{\sin t}{2\cos t}$ or $\frac{y}{4x}$ $6\sin t=\frac{\sin t}{2\cos t}(3\cos t)+c$ $6\sin t=\frac{3\sin t}{2}+c$ $c=\frac{9}{2}\sin t$ $y=\frac{\sin t}{2\cos t}x+\frac{9}{2}\sin t$

Similar reasoning can be used to find the tangent to an ellipse at a point. Exam questions may also draw upon previous knowledge, such as finding intersection points between lines and ellipses and integration- integrating the equation for an ellipse will require a substitution!

<u>Example 6:</u> Show that the equation of the tangent to the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\cosh t, b\sinh t)$ can be written as $bx\cosh t-ay\sinh t=ab$.

Differentiate either parametrically or implicitly to find the gradient at a point.	$x=a\cosh t, y=b\sinh t$ $\frac{dx}{dt}=a\sinh t, \frac{dy}{dt}=b\cosh t$ $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{b\cosh t}{a\sinh t}$
Use $y=mx+c$ or $y-y_1=m(x-x_1)$ to find the equation of the tangent.	$y-b\sinh t=\frac{b\cosh t}{a\sinh t}(x-a\cosh t)$ $ay\sinh t-ab\sinh^2 t=bx\cosh t-ab\cosh^2 t$ $ay\sinh t+ab(\cosh^2 t-\sinh^2 t)=bx\cosh t$ $bx\cosh t-ay\sinh t=ab$

Loci

Again, each of the conic sections can also be thought of as a locus of points

- For a point P, the locus of points such that the sum of distance between P and the two foci (denoted A and B) is constant defines an ellipse: $AP+BP=c$.
- For a point P, the locus of the points such that the difference between the distance between P and the two foci is constant defines a hyperbola: $AP-BP=c$.

Exam questions may also ask you to find the locus of midpoints of tangents for example. To do this it is always easier to sketch a diagram- keep an eye out for trigonometric simplifications too as you may be asked to put the locus in a specific form.