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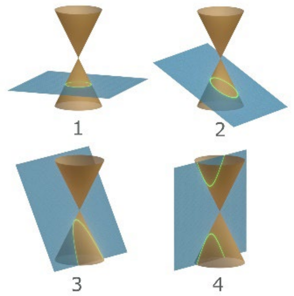
Conic Sections 1 Cheat Sheet

Edexcel FP1

This chapter aims to build upon the graph sketching skills you developed in year 1. As described by the name, conic sections are planes produced by slicing two cones placed point to point- more mathematically, the intersection of cones with planes are the graphs that you will see in this chapter. Conic sections are used in modelling: planets travel in ellipses, the mirrors in solar power stations are parabolas and a variety of conic sections are used in engineering. In the set of diagrams to the right, diagram 1 shows how a circle is obtained from a conic section, 2 shows an ellipse, 3 shows a parabola and 4 shows a hyperbola.



Parametric equations.

Unlike the Cartesian form of equations that you will have seen throughout school, parametric equations are given in terms of an independent variable, called a parameter- most of the time the parameter will be denoted 't', and the equations are given in the form $x = p(t), y = q(t)$. By substituting in specific values of t you can find the co-ordinates of a point on the curve. To convert parametric equations to Cartesian, you must eliminate the parameter between the equations- this can be often done by normal substitution, but keep an eye out for different trigonometric identities that can eliminate the parameter:

Example 1: A Curve has parametric equations $x = r\sin(t)$ and $y = r\cos(t), t \in \mathbb{R}$ and r is a positive constant. Find the Cartesian equation of the curve.

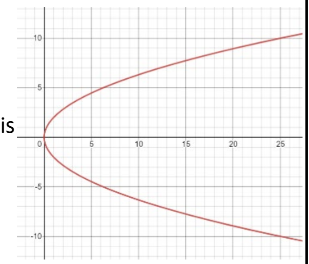
Notice that by squaring each equation, we can use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.	$x^2 = r^2\sin^2 t, y^2 = r^2\cos^2 t$
Add the equations together to put in the form of the identity we're using.	$x^2 + y^2 = r^2(\sin^2 t + \cos^2 t)$ $x^2 + y^2 = r^2$
We now have an equation that involves x and y only. r is a constant- notice that the equation we have found is the equation of a circle centred at (0,0) with radius r.	

You should be able to sketch a curve given its parametric equations- most of the time the easiest way to do this will be to convert to Cartesian form and then sketch, but if you are really struggling remember you can substitute values of t in and plot some points.

Parabolas

You have seen parabolas throughout mathematics, but you need to be able to identify and be confident working with the parametric form. The graph to the right is the graph of $y^2 = 4ax$, where a is a positive constant.

At first, this seems like a complicated way of defining a parabola, but this format allows the focus-directrix properties to appear much simpler.



- A parabola has Cartesian equation $y^2 = 4ax$, where a is a Positive constant. It has parametric equations

$$x = at^2, y = 2at,\ t \in \mathbb{R}.$$

Clearly this curve is symmetric about the x-axis, and a generic point can be defined as (x, y)or $(at^2, 2at)$.

As well as being viewed as graphs of functions, parabolas, and all conic sections, can be defined geometrically. A parabola is the locus (set of all points), whose distance from the focus (a certain point) is equal to the distance from the directrix (a certain line). For a parabola with cartesian equation $y^2 = 4ax$:

- The focus, denoted S, has co-ordinates $(a, 0)$
- The directrix has equation $x + a = 0$
- The vertex is at the origin, (0,0).

Example 2: Find the co-ordinates of the focus and the equation for the directrix of a parabola with equation $y^2 = \sqrt{80}x$.

Put the equation into the form $y^2 = 4ax$.	$y^2 = 4\sqrt{5}x$, so, $a = \sqrt{5}$
Define the focus and directrix using the forms above.	Focus: $(a, 0) \Rightarrow (\sqrt{5}, 0)$ Directrix: $x + a = 0 \Rightarrow x + \sqrt{5} = 0$

You also need to be able to find the equation of a parabola from the focus and directrix:

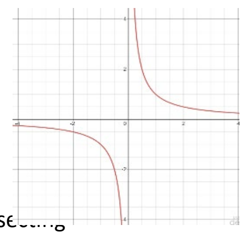
Example 3: Find the equation of the parabola with focus (3,0) and directrix $x = -3$

Put the equation for the directrix in the form $x + a = 0$.	$x = -3 \Rightarrow x + 3 = 0$
Define a.	Focus: (3,0), directrix: $x + 3 = 0$, so $a = 3$
Define the parabola- remember it is of the form $y^2 = 4ax$.	$y^2 = 12x$

Questions involving parabolas will not just require you to find the focus and directrix, you may also be required to use skills you already know to find intersection points, lengths and perpendicular bisectors

Rectangular hyperbolas

Hyperbolas are obtained by slicing through both of the cones that are placed vertex to vertex. A rectangular hyperbola is a specific type of hyperbola with asymptotes that meet at right angles. The curve to the right is a rectangular hyperbola. Again, you must be able to identify and work with both the Cartesian and parametric forms.



- A rectangular hyperbola has Cartesian equation $xy = c^2$, where c is a positive constant
- A rectangular hyperbola has parametric equations $x = ct, y = \frac{c}{t}, t \in \mathbb{R}, t \neq 0$.
- The asymptotes are the x-axis ($x = 0$) and the y-axis ($y = 0$), and a generic point can be defined as (x, y) or $(ct, \frac{c}{t})$

Basic exam questions with rectangular hyperbolas in will often be about intersections of lines, bisectors and finding lengths or midpoints, and therefore will only require skills you already know.

Tangents and normals

As you will have seen previously, being able to find the gradient of a curve is essential for being able to find its tangent or normal. You can use parametric differentiation or implicit differentiation, which were covered in pure year 2. For the general parabola $y^2 = 4ax$, the gradient is given by $\frac{dy}{dx} = \frac{2a}{y}$, which can be derived either parametrically or implicitly

The set of all of the tangents gives the envelope of a curve

Example 3: Derive the gradient of a general parabola $y^2 = 4ax$ or $x = at^2, y = 2at$ both parametrically and implicitly.

Parametric differentiation: Differentiate each parametric equation with respect to t.	$x = at^2 \Rightarrow \frac{dx}{dt} = 2at$ $y = 2at \Rightarrow \frac{dy}{dt} = 2a$
Use similar reasoning to the chain rule to find $\frac{dy}{dx}$, recalling that $\frac{dt}{dx} = 1/\frac{dx}{dt}$.	$\frac{dt}{dx} = \frac{1}{2at}$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2a}{2at} = \frac{1}{t}$
Implicit differentiation:	$y^2 = 4ax$ $2y\frac{dy}{dx} = 4a$
Rearrange to find $\frac{dy}{dx}$, notice that by substituting in t, the two gradients are the same, as expected.	$\frac{dy}{dx} = \frac{2a}{y}$

Similarly, the gradient of a rectangular hyperbola can be found by rearranging into the form $y = \frac{c^2}{x}$ and differentiating. If you are sitting the AS paper you **do not need to know** how to differentiate implicitly or parametrically, you will be given the result in the exam. From these results, you should be able to find a normal or tangent to a parabola or rectangular hyperbola.

Example 4: Find the normal to the parabola $y^2 = 16x$ that passes through the point P$(2, 4\sqrt{2})$.

Find the gradient of the parabola (in an exam you will have to derive the gradient as above)	$\frac{dy}{dx} = \frac{2a}{y} = \frac{8}{y}$
Find the gradient at the point P, and determine the gradient of the normal.	Gradient= $\frac{8}{4\sqrt{2}} = \sqrt{2}$ The gradient of the normal at a point is the negative reciprocal of the gradient at that point, so the gradient of the normal is $-\frac{1}{\sqrt{2}}$.
Use $y = mx + c$ or $y - y_1 = m_N(x - x_1)$ to find the equation of the normal.	$y = mx + c$ $4\sqrt{2} = \frac{-1}{\sqrt{2}}(2) + c$ $4\sqrt{2} + \frac{2}{\sqrt{2}} = c$ $c = 5\sqrt{2}$ Equation of the normal is $y = \frac{-1}{\sqrt{2}}x + 5\sqrt{2}$

Example 5: The point $P\left(4t, \frac{4}{t}\right), t \neq 0$ lies on the rectangular hyperbola H with equation $xy = 16$. Find a tangent to H at P.

Find the gradient of the hyperbola.	$xy = 16$ $y = \frac{16}{x}, \quad \frac{dy}{dx} = -\frac{16}{x^2}$
Find the gradient of the hyperbola at P by substituting in the x-value of the co-ordinate given.	The gradient at P is $\frac{-16}{16t^2} = \frac{-1}{t^2}$
Use $y = mx + c$ or $y - y_1 = m_N(x - x_1)$ to find the equation of the tangent.	$y = mx + c$ $\frac{4}{t} = \frac{-1}{t^2}(4t) + c$ $c = \frac{4}{t} + \frac{4}{t}$ $y = \frac{-1}{t^2}x + \frac{8}{t}$ $yt^2 = -x + 8t$

Loci

The focus directrix property of a parabola can be used to derive its general equation.

Example 6: The curve C is the locus of points that are equidistant from the line with equation $x + 3 = 0$ and the point (3,0). Prove that C is a parabola.

Set up an equation using the fact that a generic point P will be equidistant from the focus, S, and the directrix, X.	We know from the question that the locus satisfies $\boldsymbol{SP = XP}$.
When we talk about distances, Pythagoras' theorem is often useful, so we square the values of our generic point P to get our equation into a more useful form. We then express the distances between S and P and X and P - it is useful to sketch a diagram.	$\boldsymbol{SP^2 = XP^2}$ $(x-3)^2 + (y-0)^2 = (x+3)^2$
Expand and simplify our expression.	$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$ $y^2 = 12x$
State what you have found.	The curve C has an equation of the form $y^2 = 4ax$, in our case $a = 3$, so is therefore a parabola.

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