Vectors Cheat Sheet

Edexcel FP1

This chapter aims to build upon the vectors you learnt in Core Pure 1. We will look at the cross product and its applications, vector equations of lines, direction ratios and cosines. Vectors are used in modelling in a variety of ways – from 3D printing to locating and modelling speeds.

The vector product

From Core Pure 1 you should recall the scalar (or dot) product of two vectors. It takes two vectors **a** and **b** and is defined as:

- $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

Where θ is the angle between the vectors **a** and **b**. As expected from its name, the scalar product always outputs a scalar (number). It is also useful to have a product that outputs a vector.
For two vectors **a** and **b**, the vector (or cross) product is defined as:

- $\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\hat{\boldsymbol{n}}$

Again, θ denotes the angle between **a** and **b** and $\hat{\boldsymbol{n}}$ denotes a unit vector perpendicular to both **a** and **b** (remember this means that the dot products **a**.$\hat{\boldsymbol{n}}$ and **b**.$\hat{\boldsymbol{n}}$ will always be 0!).

- Intuitively, as we know $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ is a positive scalar quantity (since $0 \le \theta \le 180°$) and $\hat{\boldsymbol{n}}$ is a vector, the cross product will also be a vector and will have a magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.
- It is incredibly important to note that unlike the scalar product, the vector product is not commutative. Most of the time, $\boldsymbol{a}\times\boldsymbol{b}$ will not equal $\boldsymbol{b}\times\boldsymbol{a}$. However, $\boldsymbol{b}\times\boldsymbol{a} = -\boldsymbol{a}\times\boldsymbol{b}$. To convince ourselves of this, if a screw is turned from **a** to **b** then it would move in the direction of $\hat{\boldsymbol{n}}$. If it is reversed, and the screw is turned from **b** to **a** then it would move in the direction of -$\hat{\boldsymbol{n}}$.

Calculating the vector product

Although the vector product can be found by simply substituting **a** and **b** into the relevant formula in the formula book, we will prove the formula to aid understanding.

Example 1: For $\boldsymbol{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ prove that $\boldsymbol{a}\times\boldsymbol{b} = (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

	$\boldsymbol{a}\times\boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})\times(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$
Using the distributive property of the vector product: $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b}) + (\boldsymbol{b}\times\boldsymbol{c})$	$\boldsymbol{a}\times\boldsymbol{b} = a_1b_1(\boldsymbol{i}\times\boldsymbol{i}) + a_1b_2(\boldsymbol{i}\times\boldsymbol{j}) + a_1b_3(\boldsymbol{i}\times\boldsymbol{k}) + a_2b_1(\boldsymbol{j}\times\boldsymbol{i}) + a_2b_2(\boldsymbol{j}\times\boldsymbol{j}) + a_2b_3(\boldsymbol{j}\times\boldsymbol{k}) + a_3b_1(\boldsymbol{k}\times\boldsymbol{i}) + a_3b_2(\boldsymbol{k}\times\boldsymbol{j}) + a_3b_3(\boldsymbol{k}\times\boldsymbol{k})$
Simplify the vector products of the unit vectors along the axis- parallel vectors have a vector product of 0 (as sin(0°) = 0). Use the right-hand rule to find which direction $\hat{\boldsymbol{n}}$ acts in, as it is not always intuitive.	$= a_1b_2\boldsymbol{k} + a_1b_3(-\boldsymbol{j}) + a_2b_1(-\boldsymbol{k}) + a_2b_3(\boldsymbol{i}) + a_3b_1(\boldsymbol{j}) + a_3b_2(-\boldsymbol{i})$
Collecting like terms	$= (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$
An alternative method can also be used:	
Determinant method:	$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
Expand the determinant of the 3x3 matrix	$\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$
Expand the determinants of the 2x2 matrices	$= (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

Example 2: Evaluate $\begin{pmatrix}3\\4\\6\end{pmatrix}\times\begin{pmatrix}2\\5\\4\end{pmatrix}$

	$(3\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k})\times(2\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k})$
Using the distributive property of the vector product: $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b}) + (\boldsymbol{b}\times\boldsymbol{c})$	$= 6(\boldsymbol{i}\times\boldsymbol{i}) + 15(\boldsymbol{i}\times\boldsymbol{j}) + 12(\boldsymbol{i}\times\boldsymbol{k}) + 8(\boldsymbol{j}\times\boldsymbol{i}) + 20(\boldsymbol{j}\times\boldsymbol{j}) + 16(\boldsymbol{j}\times\boldsymbol{k}) + 12(\boldsymbol{k}\times\boldsymbol{i}) + 30(\boldsymbol{k}\times\boldsymbol{j}) + 24(\boldsymbol{k}\times\boldsymbol{k})$
Simplify and collect like terms	$= -14\boldsymbol{i} + 7\boldsymbol{k}$

Example 3: Find a unit vector perpendicular to $(5\boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k})$ and $(\boldsymbol{i} + 7\boldsymbol{j} + 2\boldsymbol{k})$

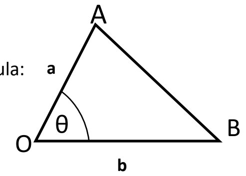
We know that the vector product results in a perpendicular vector so we must evaluate that first	$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 5 & 3 & 8\\ 1 & 7 & 2\end{vmatrix} = \boldsymbol{i}\begin{vmatrix}3 & 8\\ 7 & 2\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}5 & 8\\ 1 & 2\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}5 & 3\\ 1 & 7\end{vmatrix}$
	$= (6-56)\boldsymbol{i} - (10-8)\boldsymbol{j} + (35-3)\boldsymbol{k}$ $= -50\boldsymbol{i} - 2\boldsymbol{j} + 32\boldsymbol{k}$
To obtain a unit vector (a vector with a magnitude of 1) with the same direction as this vector we must divide the vector by its magnitude	$\lvert -50\boldsymbol{i} - 2\boldsymbol{j} + 32\boldsymbol{k}\rvert = \sqrt{(-50)^2 + (-2)^2 + 32^2}$ $= 42\sqrt{2}$
Thus the unit vector is	$\frac{1}{42\sqrt{2}}(-50\boldsymbol{i} - 2\boldsymbol{j} + 32\boldsymbol{k})$

Example 4: Using the vector product, find the sin of the acute angle between $\boldsymbol{a} = \begin{pmatrix}3\\4\\6\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}2\\5\\4\end{pmatrix}$

Rearrange the cross product formula to find the angle	$\boldsymbol{a}\times\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta\,\hat{\boldsymbol{n}}$ $\sin\theta = \frac{\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert}{\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert}$
Calculate the magnitudes- since we know the cross product from example 2 we can substitute it in here	$\lvert\boldsymbol{a}\rvert = \sqrt{3^2+4^2+6^2} = \sqrt{61}$ $\lvert\boldsymbol{b}\rvert = \sqrt{2^2+5^2+4^2} = 3\sqrt{5}$ $\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert = \sqrt{(-14)^2+7^2} = 7\sqrt{5}$
Substitute into our formula for sinθ	$\sin\theta = \frac{7\sqrt{5}}{\sqrt{61}\times 3\sqrt{5}}$ $= 0.2988$

Finding areas

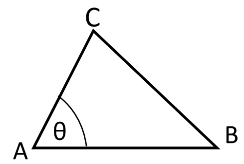
The vector product can also be used to find the areas of triangles and parallelograms.



The simplest triangles to consider are those with a vertex at the origin. Considering the formula: **Area of OAB=** $\frac{1}{2}ab\sin\theta$, we can rewrite this using vectors with |**a**| and |**b**| denoting the lengths of sides OA and OB respectively. Thus, we obtain:

Area of triangle OAB $= \frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}|\sin\theta = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$

However, not all triangles have a vertex at the origin, so we need to develop a formula that works for any triangle. Let a triangle have its vertices at points A, B and C, with position vectors **a**, **b** and **c** respectively. Again, using the normal formula for area of a triangle, Area of ABC = $\frac{1}{2}|\boldsymbol{AB}||\boldsymbol{AC}|\sin\theta$. Rewriting this using vectors (the length of **AB** is the same as |**b-a**|), we obtain:

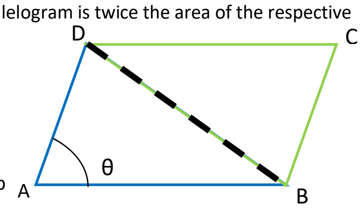


$$\text{Area of ABC} = \frac{1}{2}|\boldsymbol{b}-\boldsymbol{a}||\boldsymbol{c}-\boldsymbol{a}|\sin\theta$$
$$= \frac{1}{2}|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})|$$
$$= \frac{1}{2}|(\boldsymbol{b}\times\boldsymbol{c}) - (\boldsymbol{b}\times\boldsymbol{a}) - (\boldsymbol{a}\times\boldsymbol{c}) + (\boldsymbol{a}\times\boldsymbol{a})|$$
$$= \frac{1}{2}|(\boldsymbol{b}\times\boldsymbol{c}) + (\boldsymbol{c}\times\boldsymbol{a}) + (\boldsymbol{a}\times\boldsymbol{b})|$$
$$= \frac{1}{2}|(\boldsymbol{a}\times\boldsymbol{b}) + (\boldsymbol{b}\times\boldsymbol{c}) + (\boldsymbol{c}\times\boldsymbol{a})|$$

Areas of parallelograms can be calculated in a similar way. Any parallelogram can be split up into two congruent triangles (there is a line of symmetry on the diagonal), so the area of the parallelogram is twice the area of the respective triangle.

Area of parallelogram ABCD, with position vectors **a**, **b**, **c** and **d**
=Area of ABD + Area of BCD
= 2 x ABD
= |(**a** × b) + (b × d) + (d × **a**)|

Area of parallelogram OABC, A and B have position vectors a and b
=| **a** × **b** |



The scalar triple product

The triple product is used to find volumes of 3D shapes, namely a parallelepiped (solid with 6 parallelogram shaped faces that looks like a sheared cube) and a tetrahedron. For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ and $\boldsymbol{c} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$, the scalar triple product is defined as:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$
$$= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Example 5: For $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}, \boldsymbol{b} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = 4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, find $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$

Calculate $\boldsymbol{b}\times\boldsymbol{c}$. You could also calculate the scalar triple product as a determinant, shown above	$\boldsymbol{b}\times\boldsymbol{c} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 3 & 2 & -1\\ 4 & 1 & 3\end{vmatrix} = 7\boldsymbol{i} - 13\boldsymbol{j} - 5\boldsymbol{k}$
Find the scalar product of the vector just found with **a**.	$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = (2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}).(7\boldsymbol{i} - 13\boldsymbol{j} - 5\boldsymbol{k})$ $= 14 + 26 - 25 = 15$

There are two more important points to note about the scalar triple product:

- $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ - the scalar triple product is cyclic
- $\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{p}) = \boldsymbol{a}\cdot(\boldsymbol{p}\times\boldsymbol{a}) = 0$ for any vector **p**

If three non-parallel sides of a parallelepiped are given by vectors **a**, **b** and **c**, then the **volume of the parallelepiped** is given by **|a · (b × c)|.**
This is because the volume of the parallelepiped is given by $(area\ of\ base)\times h$, where h is the perpendicular distance between the base and top of the shape. The area of the base is $|\boldsymbol{b}\times\boldsymbol{c}|$, as shown in a previous section, and h is given by, $OA\cos\theta$, where θ is the angle between the base's perpendicular and **a**. The volume is $|\boldsymbol{b}\times\boldsymbol{c}|OA\cos\theta$, which is equivalent to $|\boldsymbol{b}\times\boldsymbol{c}||\boldsymbol{a}|\cos\theta$ and by the definition of the scalar product is $|\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})|$.

If three non-coplanar sides of a tetrahedron are given by **a**, **b**, and **c** then the **volume of the tetrahedron** is given by $\frac{1}{6}|\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})|$. This is derived in a similar way to the parallelepiped.

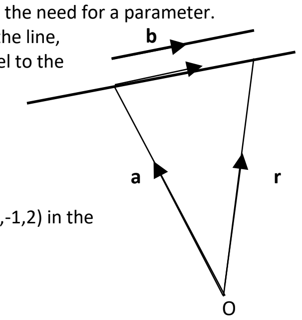
Example 6: Find the volume of the parallelepiped ABCDEFGH where vertices A, B, D, E have coordinates (2,-1,0), (-2,-1,-2), (1,-1,1) and (1,0,1) respectively.

Find the vectors **AB**, **AD** and **AE**	**AB**= $(-2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j}) = -4\boldsymbol{i} - 2\boldsymbol{k}$ **AD**=$(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j}) = -\boldsymbol{i} - \boldsymbol{k}$ **AE**=$(\boldsymbol{i} + \boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j}) = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
Substitute into the triple scalar product formula: $\lvert\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\rvert$	Volume=$\lvert(-4\boldsymbol{i} - 2\boldsymbol{k})\cdot((-\boldsymbol{i} - \boldsymbol{k})\times(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}))\rvert$ $= \lvert -2\rvert$ = 2 units³

Straight lines

The vector product can also be used to write a vector equation of a line without the need for a parameter. To form the vector equation of a line, we need a position vector **a** of a point of the line, A, a position vector **r** to a generic point on the line, R, and a vector that is parallel to the line, **b**. The vector **AR** is parallel to **b**, so $\boldsymbol{AR}\times\boldsymbol{b} = \boldsymbol{0}$. AR is found by **r-a**, so the line is denoted by the equation:

- $(\boldsymbol{r}-\boldsymbol{a})\times\boldsymbol{b} = \boldsymbol{0}$
- Or, equivalently, $\boldsymbol{r}\times\boldsymbol{b} = \boldsymbol{a}\times\boldsymbol{b}$



Example 7: find the vector equation of the line through the points (3,7,3) and (2,-1,2) in the form $(\boldsymbol{r}-\boldsymbol{a})\times\boldsymbol{b} = \boldsymbol{0}$

Find the direction vector of the line- any multiple (including 1) is parallel to the line we are finding	$(2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) - (3\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}) = (-\boldsymbol{i} - 8\boldsymbol{j} - \boldsymbol{k})$
Substitute into the equation- you can use either of the vectors given in the question as the position vector	$\left(r - \begin{pmatrix}3\\7\\3\end{pmatrix}\right)\times\begin{pmatrix}-1\\-8\\-1\end{pmatrix} = 0$

As seen in Year 2 Pure, the direction vector can be used to find the angles that the line makes with each of the axes, α, β and γ denote the angles with the x, y and z axes respectively. If a line is parallel to a vector $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then the direction ratios of the line are in the form x:y:z, and the direction cosines of the line are:
$\cos\alpha = \frac{x}{|\boldsymbol{a}|}$, $\cos\beta = \frac{y}{|\boldsymbol{a}|}$ and $\cos\gamma = \frac{z}{|\boldsymbol{a}|}$.
These are denoted l, m and n. $l^2 + m^2 + n^2 = 1$

Example 8: Find the direction cosines l, m and n of the line $\left(r - \begin{pmatrix}2\\5\\7\end{pmatrix}\right)\times\begin{pmatrix}-4\\3\\1\end{pmatrix} = 0$

Using the direction vector of the line (in this case $\begin{pmatrix}-4\\3\\1\end{pmatrix}$), substitute into the equations for the direction cosines	l= $\frac{-4}{\sqrt{(-4)^2+3^2+1^2}} = \frac{-4}{\sqrt{26}}$ m= $\frac{3}{\sqrt{(-4)^2+3^2+1^2}} = \frac{3}{\sqrt{26}}$ n= $\frac{1}{\sqrt{(-4)^2+3^2+1^2}} = \frac{1}{\sqrt{26}}$

Solving geometric problems

There is a wide range of questions that can be asked of you in exams, which will involve both planes and lines in 3D. These will be multi-step problems but will only require the skills that you already know- read the question carefully and work out what you can.

Example 9: Find the shortest distance between the skew lines with equations $r_1 = \begin{pmatrix}-2\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $r_2 = \begin{pmatrix}1\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\4\end{pmatrix}$

As in any distance problem, the shortest distance between two non-intersecting lines is a line that is perpendicular to both lines. When a question is asking for you to find a perpendicular to two vectors, the vector products should be the first thing you try.	$\boldsymbol{n} = \begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}-1\\2\\4\end{pmatrix} = \begin{pmatrix}14\\-7\\7\end{pmatrix}$
Now we have a direction vector for the shortest distance. Let Q and P denote the points of each line. We need to find the distance between them.	$\boldsymbol{QP} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}-2\\1\\-1\end{pmatrix} = \begin{pmatrix}3\\-2\\3\end{pmatrix}$
We have a direction vector for the vector showing the shortest distance (**n**) and a vector connecting the two lines (**QP**). We need to project the connecting vector **QP** in the direction of the shared perpendicular- to do this we find the scalar product of the vector **QP** with the UNIT vector in the direction of the common perpendicular- this way we are not affecting the length of the distance between the two points in our projection!	$\left\lvert\begin{pmatrix}14\\-7\\7\end{pmatrix}\right\rvert = \sqrt{14^2 + (-7)^2 + 7^2} = 7\sqrt{6}$ So, the unit vector is given by: $\boldsymbol{n} = \frac{1}{7\sqrt{6}}\begin{pmatrix}14\\-7\\7\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\1\end{pmatrix}$ And the projection (i.e. shortest distance) is given by: $\begin{pmatrix}3\\-2\\3\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\1\end{pmatrix} = \frac{11}{\sqrt{6}}$ units

As demonstrated in the example above, the shortest distance between two skew lines with equations $r = a + \lambda b, r = c + \mu d$, where λ and μ are scalars, is given by:

$$\left|\frac{(\boldsymbol{a}-\boldsymbol{c})\cdot(\boldsymbol{b}\times\boldsymbol{d})}{|\boldsymbol{b}\times\boldsymbol{d}|}\right|$$