**Version 1.2 1106**

*Changes include the amendment of section 5.7 for the 2007 (onwards) specification and some minor formatting changes. Please note that these are not side-barred.*

# GCE Mathematics *(6360)*

# Further Pure unit 3 *(MFP3)*

# Textbook

# Further Pure 3: Contents

# Further Pure 3: Introduction

The aim of this text is to provide a sound and readily accessible account of the items comprising the Further Pure Mathematics unit 3.

The chapters are arranged in the same order as the five main sections of the unit. The first chapter is therefore concerned with series expansions and the evaluation of limits and improper integrals. The second covers polar coordinates and their use in curve sketching and evaluation of areas.

The subject of differential equations forms a major part of this unit and Chapters 3, 4 and 5 are devoted to this topic. Chapter 3 introduces the subject and deals mainly with analytical methods for solving differential equations of first order linear form. In addition to the standard method of solution using an integrating factor, this chapter introduces the method based on finding a complementary function and a particular integral. This provides useful preparation for Chapter 5 where the same technique is used for solving second order differential equations.

With the advent of modern computers, numerical methods have become an essential practical tool for solving the many differential equations which cannot be solved by analytical methods. This important subject is covered in Chapter 4 in relation to differential equations of the form $y' = \mathrm{f}\left(x, y\right)$. It should be appreciated that, in practice, the numerical methods described would be carried out with the aid of a computer using an appropriate program. The purpose of the worked examples and exercises in this text is to exemplify the principles of the various methods and to show how these methods work. Relatively simple functions have been chosen, as far as possible, so that the necessary calculations with a scientific calculator are not unduly tedious.

Chapter 5 deals with analytical methods for solving second order differential equations and this requires some knowledge of complex numbers. Part of the required knowledge is included in the Further Pure 1 module, which is a prerequisite for studying this module, and the remainder is included in the Further Pure 2 module which is not a prerequisite. For both simplicity and completeness therefore, Chapter 5 begins with three short sections on complex numbers which cover, in a straightforward way, **all** that is required for the purpose of this chapter. These sections should not cause any difficulty and it is hoped that they will be found interesting as well as useful. Those who have already studied the topics covered can either pass over this work or regard it as useful revision.

The main methods for solving second order linear differential equations with *constant* coefficients are covered in Sections 5.5. and 5.6. These methods sometimes seem difficult when first met, but students should not be discouraged by this. Useful summaries are highlighted in the text and confidence should be restored by studying how these are applied in the worked examples and by working through the exercises.

The text concludes with a short section showing how some second order linear differential equations with *variable* coefficients can be solved by using a substitution to transform them to simpler forms.

# Chapter 1: Series and Limits



In this chapter, it is shown how series expansions are used to find limits and how improper integrals are evaluated. When you have completed it you will:

- have been reminded of the concept of a limit;
- have been reminded of methods for finding limits in simple cases;
- know about Maclaurin's series expansion;
- be able to use series expansions to find certain limits;
- know about the limits of $x^k e^{-x}$ as $x \to \infty$ and $x^k \ln x$ as $x \to 0$;
- know the definition of an improper integral;
- know how to evaluate improper integrals by finding a limit.

## 1.1 The concept of a limit

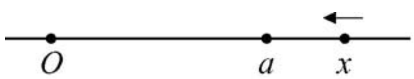
You will have already met the idea of a limit and be familiar with some of the notation used. If $x$ is a real number which varies in such a way that it gets closer and closer to a particular value $a$ , but is never equal to $a$ , then this is signified by writing $x \to a$. More precisely, the variation of $x$ must be such that for any positive number, $\delta$ , $x$ can be chosen so that $0 < |x-a| < \delta$ , no matter how small $\delta$ may be.

When a function $\mathrm{f}(x)$ is such that $\mathrm{f}(x) \to l$ when $x \to a$ , where $l$ is **finite**, the number $l$ is called the limit or limiting value of $\mathrm{f}(x)$ as $x \to a$ . This may be expressed as

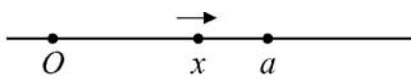
$$\lim_{x \to a} \mathrm{f}(x) = l \ .$$

The statements $x \to \infty$ and $\mathrm{f}(x) \to \infty$ mean that $x$ and $\mathrm{f}(x)$ increase indefinitely – i.e. that their values increase beyond any number we care to name, however large. Note that you should never write $x = \infty$ or $\mathrm{f}(x) = \infty$ , because $\infty$ is not a number.

Unless stated or implied otherwise, $x \to a$ means that $x$ can approach $a$ from either side. Occasionally however, it may be necessary to distinguish between $x$ approaching $a$ from the right, so that $x > a$ always and $x$ approaching $a$ from the left, so that $x < a$ always. The notation $x \to a+$ is used to signify that $x$ approaches $a$ from the right, and $x \to a-$ to signify that $x$ approaches $a$ from the left. The two cases are illustrated in the diagram below.




$x$ approaches $a$ from the right, so $x \to a+$




$x$ approaches $a$ from the left, so $x \to a-$

The distinction between the two cases is important, for instance, when we consider the behaviour of $\frac{1}{x}$ as $x \to 0$ . When $x \to 0+$, $\frac{1}{x} \to \infty$ ; but when $x \to 0-$, $\frac{1}{x} \to -\infty$ .

The use of $+$ or $-$ attached to $a$ is unneccessary when it is clear from the context that the approach to $a$ can only be from one particular side. For example, since $\ln x$ is defined only for $x > 0$ , one can write $\ln x \to -\infty$ as $x \to 0$ without ambiguity: the fact that $x$ approaches $0$ from the right is implied in this case so it is unneccessary to write $x \to 0+$ .

## 1.2 Finding limits in simple cases

In some simple cases, it is easy to see how a function $\mathrm{f}(x)$ behaves as $x$ approaches a given value and whether it has a limit. Here are three examples.

1. As $x \to 0$ , $\dfrac{1+x}{2-x} \to \dfrac{1}{2}$ because $1+x \to 1$ and $2-x \to 2$ as $x \to 0$ .
2. As $x \to \dfrac{\pi}{2}$ , $\dfrac{\sin x}{1-\cos x} \to 1$ , because $\sin x \to 1$ and $\cos x \to 0$ as $x \to \dfrac{\pi}{2}$ .
3. As $x \to 1+$ , $\dfrac{1+x}{1-x} \to -\infty$ , because $1+x \to 2+$ and $1-x \to 0-$ as $x \to 1+$ .

The first two of the above examples can be expressed as:

$$\lim_{x\to 0}\frac{1+x}{2-x} = \frac{1}{2} \quad \text{and} \quad \lim_{x\to \frac{\pi}{2}}\frac{\sin x}{1-\cos x} = 1$$

However, it would be wrong to state that

$$\lim_{x\to 1+}\frac{1+x}{1-x} = -\infty ,$$

because a limit has to be **finite**. The function $\dfrac{1+x}{1-x}$ does not have a limiting value as $x \to 1+$ .

Another example, not quite so straightforward as the examples above, is that of finding the limit as $x \to \infty$ of $\mathrm{f}(x) = \dfrac{1+x}{1-2x}$ .

As $x \to \infty$ , $1+x \to \infty$ and $1-2x \to -\infty$ , but the expression $\dfrac{\infty}{-\infty}$ is meaningless. This difficulty can be overcome by first dividing the numerator and denominator of $\mathrm{f}(x)$ by $x$ , giving

$$\mathrm{f}(x) = \frac{\dfrac{1}{x}+1}{\dfrac{1}{x}-2} .$$

It can now be seen that $\mathrm{f}(x) \to \dfrac{1}{-2}$ as $x \to \infty$ because $\dfrac{1}{x} \to 0$ . The limiting value of $\mathrm{f}(x)$ is therefore $-\dfrac{1}{2}$ .

The working for this example can be presented more concisely as follows.

$$\mathrm{f}(x) = \frac{1+x}{1-2x} = \frac{\frac{1}{x}+1}{\frac{1}{x}-2} \ .$$

As $x \to \infty$, $\frac{1}{x} \to 0$ . Hence

$$\lim_{x\to\infty} \mathrm{f}(x) = \frac{1}{-2} = -\frac{1}{2} \ .$$

There are many instances where the behaviour of a function is much more difficult to determine than in the cases considered above. For example, consider

$$\mathrm{f}(x) = x^2 \ln x$$

as $x \to 0$ . When $x \to 0$ , $x^2 \to 0$ and $\ln x \to -\infty$ . It is not obvious therefore what happens to the product of $x^2$ and $\ln x$ as $x \to 0$ .

Consider also the function

$$\mathrm{f}(x) = \frac{x}{1-\sqrt{1-x}} \ .$$

When $x \to 0$ , $\mathrm{f}(x) \to \frac{0}{0}$ which is an indeterminate form having no mathematical meaning. Although $\mathrm{f}(x)$ does not have a value at $x = 0$ , it does approach a limiting value as $x \to 0$ . Investigating with a calculator will produce the following results, to five decimal places.

$$\begin{aligned} \mathrm{f}(0.1) = 1.94868 \ , \ \mathrm{f}(-0.1) = 2.04881 \ , \\ \mathrm{f}(0.01) = 1.99499 \ , \ \mathrm{f}(-0.01) = 2.00499 \ , \\ \mathrm{f}(0.001) = 1.99950 \ , \ \mathrm{f}(-0.001) = 2.00050 \ . \end{aligned}$$

It can be seen from these results that $\mathrm{f}(x)$ appears to be approaching the value 2 as $x \to 0+$ and $x \to 0-$ ; the limit is, in fact, exactly 2 as you would expect.

However, no matter how convincing the evidence may seem, a numerical investigation of this kind does not constitute a satisfactory mathematical proof. Limits in these more difficult cases can often be found with the help of series expansions. The series expansions that we shall use are introduced in the next three sections of this chapter.

## Exercise 1A

1. Write down the missing values, indicated by $*$ , in the following.

   (a) As $x \to 0$ , $\dfrac{2+x}{2-x} \to *$ , (b) As $x \to 2$ , $\dfrac{1+x}{1-x} \to *$ ,

   (c) As $x \to \dfrac{\pi}{2}-$ , $\dfrac{x}{\tan x} \to *$ , (d) As $x \to \dfrac{\pi}{2}+$ , $\dfrac{x}{\tan x} \to *$ .

2. Explain why $\ln x$ does not have a limiting value as $x \to 0$ .

3. Write down the values of the following limits.

   (a) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{x}{1+\sin x}$ , (b) $\lim\limits_{x \to 1} \dfrac{\ln x}{1+\ln x}$ .

4. Find the values of the following.

   (a) $\lim\limits_{x \to \infty} \dfrac{3+2x}{2+3x}$ , (b) $\lim\limits_{x \to \infty} \dfrac{x}{2x+3}$ ,

   (c) $\lim\limits_{x \to \infty} \dfrac{1+x-x^2}{1-x+2x^2}$ , (d) $\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^3+3}$ .

5. Use a calculator to investigate the behaviour of $x^{0.01} \ln x$ as $x \to 0$ . You will find it difficult to guess from this investigation what happens as $x \to 0$ . Later in this chapter, it will be shown that the function tends to zero.

## 1.3 Maclaurin's series expansion

You will already be familiar with the binomial series expansion for $(1+x)^n$ :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \ .$$

Many other functions, such as $e^x$ , $\ln(1+x)$ , $\sin x$ and $\cos x$ can also be expressed as series in ascending powers of $x$ . Such expansions are called **Maclaurin series**.

The Maclaurin series for a function $f(x)$ is given by:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots \ ,$$

where $f'$ , $f''$ , $f'''$ , … denote the first, second, third, … derivatives of f , respectively, and $f^{(r)}$ is the general derivative of order $r$ .

To derive this, the following assumptions are made.

(i) The function $f(x)$ can be expressed as a series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_r x^r + \dots \ ,$$

where $a_0$ , $a_1$ , $a_2$ , $a_3$ , $a_4$ , … $a_r$ , …are constants.

(ii) The series can be differentiated term by term.

(iii) The function $f(x)$ and all its derivatives exist at $x = 0$ .

Succesive differentiations of each side of the equation under (i) gives

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \ ,$$
$$f''(x) = 2a_2 + 2\times 3a_3 x + 3\times 4a_4 x^2 + \dots \ ,$$
$$f'''(x) = 2\times 3a_3 + 2\times 3\times 4a_4 x + \dots \ ,$$

and, in general,

$$f^{(r)}(x) = 2\times 3\times \dots \times (r-1)\times r a_r + \text{ terms in } x \ , x^2 \ , x^3 \dots \ .$$

Putting $x = 0$ in the expressions for $\mathrm{f}(x)$ and its derivatives, we see that

$$a_0 = \mathrm{f}(0)\ ,\quad a_1 = \mathrm{f}'(0)$$
$$a_2 = \frac{\mathrm{f}''(0)}{2} = \frac{\mathrm{f}''(0)}{2!}\ ,$$
$$a_3 = \frac{\mathrm{f}'''(0)}{2\times 3} = \frac{\mathrm{f}'''(0)}{3!}\ ,$$

and, in general,

$$a_r = \frac{\mathrm{f}^{(r)}(0)}{1.2.3...(r-1)r} = \frac{\mathrm{f}^{(r)}(0)}{r!}\ .$$

Substituting these values into the series in (i) above gives the Maclaurin series of $\mathrm{f}(x)$ .

The statement that $\mathrm{f}(x)$ can be expressed as a Maclaurin series, subject to conditions (i) – (iii) above being satisfied, is often referred to as **Maclaurin's theorem**.

The Maclaurin series has an interesting history. It is named in honour of Colin Maclaurin, a notable Scottish mathematician. Born in 1698, he was a child prodigy who entered university at the age of 11 and became a professor at the age of 19. He was personally acquainted with Newton and made significant contributions to the development of Newton's pioneering work in Calculus. The series which bears Maclaurin's name was not discovered by Maclaurin – a fact that he readily acknowledged – but is a special case of a more general expansion called Taylor's series (see exercise 1B, question 6).

## Example 1.3.1

Use Maclaurin's theorem to obtain the expansion of $\ln(1+x)$ as a series in ascending powers of $x$ .

**Solution**

In this case

$$f(x) = \ln(1+x) \Rightarrow f(0) = \ln 1 = 0 \ .$$

Also,

$$\begin{aligned}
f'(x) &= \frac{1}{1+x} \Rightarrow f'(0) = 1 \ , \\
f''(x) &= -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1 \ , \\
f'''(x) &= \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2 \ , \\
f^{(4)}(x) &= -\frac{2\times 3}{(1+x)^4} \Rightarrow f^{(4)}(0) = -2\times 3 \ , \\
f^{(5)}(x) &= \frac{2\times 3\times 4}{(1+x)^5} \Rightarrow f^{(5)}(0) = 2\times 3\times 4 \ ,
\end{aligned}$$

and so on.

Substituting these values into the general form of the Maclaurin series gives

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2!} + \frac{2}{3!}x^3 - \frac{2\times 3}{4!}x^4 + \frac{2\times 3\times 4}{5!}x^5 - \ldots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \ .
\end{aligned}$$

The general term of this series is most readily obtained by inspecting the first few terms. It is $(-1)^{r+1}\frac{x^r}{r}$ , where $r=1$ gives the first term, $r=2$ gives the second term, $r=3$ gives the third term, and so on. Hence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \ ,$$

where $r$ can take the values 1, 2, 3… .

## Example 1.3.2

Obtain the Maclaurin series expansion of $\sin x$.

**Solution**

In this case

$$\begin{aligned}
\mathrm{f}(x) = \sin x &\Rightarrow \mathrm{f}(0) = 0 \ , \\
\mathrm{f}'(x) = \cos x &\Rightarrow \mathrm{f}'(0) = 1 \ , \\
\mathrm{f}''(x) = -\sin x &\Rightarrow \mathrm{f}''(0) = 0 \ , \\
\mathrm{f}'''(x) = -\cos x &\Rightarrow \mathrm{f}'''(0) = -1 \ , \\
\mathrm{f}^{(4)}(x) = \sin x &\Rightarrow \mathrm{f}^{(4)}(0) = 0 \ , \\
\mathrm{f}^{(5)}(x) = \cos x &\Rightarrow \mathrm{f}^{(5)}(0) = 1 \ ,
\end{aligned}$$

and so on.

Substituting these values into the general form of the Maclaurin series gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots \ .$$

By inspecting the first few terms of the series, the general term can be identified. It is

$$(-1)^r \frac{x^{2r+1}}{(2r+1)!}$$

where $r = 0$ gives the first term, $r = 1$ gives the second term, $r = 3$ gives the third term, and so on. Hence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ \ldots \ ,$$

where $r$ can take the values 0, 1, 2, … .

The general term could also be expressed as

$$(-1)^{r-1} \frac{x^{2r-1}}{(2r-1)!} \ ,$$

but with this form $r = 1$ gives the first term, $r = 2$ gives the second term, $r = 3$ gives the third term and so on. The admissible values of $r$ for this form of the general term are therefore 1, 2, 3, … .

*Whenever the general term of a series is given, the admissible values of $r$ should be stated.*

## Exercise 1B

1. Obtain the Maclaurin series expansion of $e^x$ , up to and including the term in $x^3$ . Write down an expression for the general term of the series.

2. By replacing $x$ by $-x$ in the series expansion for $e^x$ , obtained in the previous question, write down the Maclaurin series for $e^{-x}$ , up to and including the term in $x^3$ . Show that the general term is given by $(-1)^r \dfrac{x^r}{r!}$ , where $r = 0, 1, 2 \ldots$ .

3. Use Maclaurin's theorem to show that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots .$$

Write down an expression for the general term of the series.

4. Use Maclaurin's theorem to show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots .$$

5. Use Maclaurin's theorem to obtain the first three non-zero terms in the expansion of

$$f(x) = \frac{e^x + e^{-x}}{2} .$$

Give the general term.

6. Suppose that $f(x+a)$, where $a$ is a constant, can be expanded as a series in ascending powers of $x$. Suppose also that the series can be differentiated term by term and that f and all its derivatives exist at $x = a$. By using a similar method to that used to derive the Maclaurin series of $f(x)$, show that

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \ldots .$$

[The above expansion of $f(x+a)$ is called **Taylor's series**. It is named after Brook Taylor, an eminent English mathematician who was a close contemporary of Maclaurin. Maclaurin's series can be obtained immediately from Taylor's series by putting $a = 0$ . An application of Taylor's series will be found later in section 4.5].

## 1.4 Range of validity of a series expansion

The Maclaurin series expansion of a function $\mathrm{f}(x)$ is not necessarily valid for all values of $x$. A simple example will show this. Consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

(see exercise 1B, question 4). For $x = 2$, the left-hand side of this equation has the value -1. However, for $x = 2$, the right-hand side is $1 + 2 + 2^2 + 2^3 + \dots$, which is clearly not equal to -1. The above expansion is therefore not valid for $x = 2$.

To determine the values of $x$ for which the expansion is valid is not too difficult in this case. As you may have noticed already, the expansion is an infinite geometric series with the first term 1 and common ratio $x$. The sum, $S_n(x)$, of the first $n$ terms is given by

$$S_n(x) = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}.$$

Hence,
$$\frac{1}{1-x} = S_n(x) + \frac{x^{n+1}}{1-x}.$$

For the expansion to be valid, $S_n(x)$ must have the same value as $\frac{1}{1-x}$ in the limit when $n \to \infty$. The requirement for this is that $\frac{x^{n+1}}{1-x} \to 0$ as $n \to \infty$, and this occurs only when $|x| < 1$. We conclude therefore that the Maclaurin series expansion of $\frac{1}{1-x}$ is valid provided $|x| < 1$.

In general, suppose that the Maclaurin series of $\mathrm{f}(x)$ is given by

$$\mathrm{f}(x) = S_n(x) + R_n(x),$$

where $S_n(x)$ is the sum of the first $n$ terms of the series and $R_n(x)$ is the sum of all the remaining terms. For a particular value of $x$, the series expansion will be valid provided that $R_n(x) \to 0$ as $n \to \infty$.

In the case when

$$f(x) = \frac{1}{1-x} ,$$

it was fairly easy to show that

$$R_n(x) = \frac{x^{n+1}}{1-x} ;$$

this enabled the values of $x$ for which the series expansion of $\frac{1}{1-x}$ is valid to be determined. Usually, however, finding an expression for $R_n(x)$ is much more difficult and beyond the scope of an A-level course. In what follows therefore, ranges of validity will be stated without proof.

*Use of series expansions in approximations*

For values of $x$ within the range of validity of a series expansion of a function, an approximation to the value of the function can be obtained by using just the first few terms of the series. For example, for the expansion of $\frac{1}{1-x}$ discussed above, substitution of $x = 0.2$ into the series and using the first five terms gives

$$\frac{1}{1-x} \approx 1 + 0.2 + 0.04 + 0.008 + 0.0016 = 1.2496 .$$

Substituting $x = 0.2$ into $\frac{1}{1-x}$ , the exact value is found to be 1.25. The error in the approximate value is therefore small, and it can be made smaller still by using more terms of the series.

## 1.5 The basic series expansions

The following expansions are considered basic ones from which others can be derived.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (r = 0, 1, 2, \ldots)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad (r = 0, 1, 2, \ldots)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad (r = 0, 1, 2, \ldots)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \frac{x^r}{r} + \ldots \qquad (r = 1, 2, 3 \ldots)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \binom{n}{r}x^r + \ldots \qquad (r = 0, 1, 2, \ldots)$$

The first three of the above expansions are valid for all real values of $x$. The expansion of $\ln(1+x)$ is valid only when $-1 < x \le 1$.

The expansion of $(1+x)^n$ is the binomial series and is valid for any real value of $n$ when $-1 < x < 1$. However, when $n$ is a positive integer, the series is finite, being a polynomial of degree $n$; it is therefore valid for all real values of $x$ in this case. One other special case, worthy of mention, is that when $x = 1$ the binomial expansion is still valid if $n \ge -\frac{1}{2}$.

All the above basic series, together with ranges of values of $x$ for which they are valid, are given in the AQA formulae booklet.

It is important to appreciate that in the series expansions for $\sin x$ and $\cos x$, **$x$ is a real number, not an angle.** However, if the values of these trigonometric functions are needed for any particular real value of $x$, they can be found using a calculator set to radian mode. For example, you will find that $\sin 1 = 0.84147\ldots$ .

The basic expansions are particularly useful in finding the Maclaurin series expansions of other, related functions, such as $\ln(1-2x)$ and $e^x \cos x$. This usually proves easier than using Maclaurin's theorem directly because finding the required derivatives can be troublesome. In general, it is advisable to use Maclaurin's theorem only when specifically requested to do so. The examples which follow show how the basic expansions can be used.

## Example 1.5.1

Obtain the first three non-zero terms in the series expansions of

(a) $x\sin x^2$, (b) $\dfrac{1}{(1+2x)^{\frac{1}{3}}}$ .

In each case, give the range of values of $x$ for which the expansion is valid.

**Solution**

(a) Using the series for $\sin x$ with $x$ replaced by $x^2$ gives

$$
\begin{aligned}
x\sin x^2 &= x\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \dots\right) \\
&= x^3 - \frac{x^7}{6} + \frac{x^{11}}{120} - \dots \ .
\end{aligned}
$$

The expansion is valid for all values of $x^2$ and hence for all $x$.

(b)
$$
\begin{aligned}
\frac{1}{(1+2x)^{\frac{1}{3}}} &= (1+2x)^{-\frac{1}{3}} \\
&= 1 - \frac{1}{3}(2x) + \frac{-\frac{1}{3}\times -\frac{4}{3}}{1\times 2}(2x)^2 + \dots \\
&= 1 - \frac{2}{3}x + \frac{8}{9}x^2 + \dots \ .
\end{aligned}
$$

The expansion is valid for $-1 < 2x < 1$ , which gives $-\frac{1}{2} < x < \frac{1}{2}$ .

## Example 1.5.2

(a) Expand $\ln(1-2x)$ as a series in ascending powers of $x$, up to and including the term in $x^3$.

(b) Determine the range of validity of this series.

**Solution**

(a) Replacing $x$ by $-2x$ in the series for $\ln(1+x)$ gives

$$\ln(1-2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \ldots$$
$$= -2x - 2x^2 - \frac{8}{3}x^3 - \ldots\ .$$

(b) Since the series for $\ln(1+x)$ is valid for $-1 < x \le 1$, the series expansion above will be valid when $-1 < -2x \le 1$.

Now $-1 < -2x \Rightarrow 2x < 1 \Rightarrow x < \frac{1}{2}$, and $-2x \le 1 \Rightarrow -1 \le 2x \Rightarrow -\frac{1}{2} \le x$.

Hence the range of validity of the series expansion of $\ln(1-2x)$ is $-\frac{1}{2} \le x < \frac{1}{2}$.

## Example 1.5.3

Obtain the expansion of $e^x \cos x$ up to, and including, the term in $x^3$.

**Solution**

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(1 - \frac{x^2}{2!} + \ldots\right)$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{x^2}{2} + \ldots\right)$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) - \frac{x^2}{2}\left(1 + x + \frac{x^2}{2} + \ldots\right)$$
$$= 1 + x - \frac{x^3}{3} + \ldots\ .$$

## Example 1.5.4

The function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = a(1+2x)^{\frac{1}{2}} - \ln(1+3x) \ ,$$

where $a$ is a constant. When $\mathrm{f}(x)$ is expanded as a series in ascending powers of $x$, there is no term in $x$.

(a) Find the value of $a$.

(b) Obtain the first two non-zero terms in the expansion.

(c) Determine the range of values of $x$ for which the expansion of $\mathrm{f}(x)$ is valid.

**Solution**

(a) Using the standard expansions,

$$\begin{aligned}
\mathrm{f}(x) &= a\left(1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(2x)^2+\ldots\right)-\left(3x-\frac{(3x)^2}{2}+\ldots\right) \\
&= \left(a+ax-\frac{1}{2}ax^2+\ldots\right)-\left(3x-\frac{9x^2}{2}+\ldots\right) \\
&= a+(a-3)x+\left(\frac{9}{2}-\frac{1}{2}a\right)x^2+\ldots \ .
\end{aligned}$$

Since there is no term in $x$, $a = 3$ .

(b) Putting $a = 3$ in the above expansion,

$$\mathrm{f}(x) = 3 + 3x^2 + \ldots \ .$$

(c) The expansion of $(1+2x)^{\frac{1}{2}}$ is valid for $-1 < 2x < 1$ , which gives $-\frac{1}{2} < x < \frac{1}{2}$. The expansion of $\ln(1+3x)$ is valid for $-1 < 3x \le 1$ , which gives $-\frac{1}{3} < x \le \frac{1}{3}$.

For the expansion of $\mathrm{f}(x)$ to be valid, the expansions of **both** $(1+2x)^{\frac{1}{2}}$ **and** $\ln(1+3x)$ must be valid. Hence the required range of validity is $-\frac{1}{3} < x \le \frac{1}{3}$ . Any $x$ in this interval will also be within the interval $-\frac{1}{2} < x < \frac{1}{2}$ .

## Exercise 1C

1. Use a calculator to evaluate $\cos 1$ to six decimal places.

2. (a) Use a calculator to evaluate $\sin 0.5$ to four decimal places.

   (b) Verify that using the first three terms of the series expansion for $\sin x$ with $x = 0.5$ gives the same value to four decimal places.

3. Obtain the first three non-zero terms of the series expansions of

$$\text{(a)}\ \frac{\sin 2x}{x}(x \neq 0)\ , \quad \text{(b)}\ \cos 3x\ , \quad \text{(c)}\ \frac{x}{(1+2x)^{\frac{2}{3}}}\ ,$$

$$\text{(d)}\ \frac{1}{e^{3x}}\ , \qquad \text{(e)}\ \ln(1+x+x^2)\,.$$

4. Expand each of the following functions as series in ascending powers of $x$ , up to and including the term in $x^3$.

$$\text{(a)}\ \ln(1-x), \quad \text{(b)}\ e^x(1-2x)^3, \quad \text{(c)}\ e^{-2x}+2\sin x, \quad \text{(d)}\ \frac{\ln(1+x)}{1+3x}\,.$$

5. For each of the series expansions in question 4, determine the range of values of $x$ for which the expansion is valid.

## 1.6 Use of series expansions to find limits

In this section, we shall show by means of examples how series expansions can be used to find limits.

Consider first the problem mentioned in section 1.2 of obtaining the limit, as $x \to 0$, of the function

$$\mathrm{f}(x) = \frac{x}{1-\sqrt{1-x}.}$$

Using the binomial expansion,

$$\begin{aligned}\sqrt{1-x} &= (1-x)^{\frac{1}{2}}\\ &= 1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(-x)^2+\frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{3!}(-x)^3+\ldots\\ &= 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+\ldots.\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{f}(x) &= \frac{x}{1-\left(1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+\ldots\right)}\\ &= \frac{x}{\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots}\\ &= \frac{1}{\frac{1}{2}+\frac{1}{8}x+\frac{1}{16}x^2+\ldots}.\end{aligned}$$

It can now be seen that when $x \to 0$, all the terms in the denominator of the above expression, except the first, tend to zero. Hence,

$$\lim_{x\to 0}\mathrm{f}(x) = \frac{1}{\frac{1}{2}} = 2.$$

Notice that after simplifying the denominator in the expression for $\mathrm{f}(x)$, the common factor $x$ in the numerator and denominator was cancelled – **this is a key step** common to many problems in which series expansions are used to determine limits. If common factors are not cancelled, then as $x \to 0$ both numerator and denominator will tend to zero giving the indeterminate form $\frac{0}{0}$.

**Example 1.6.1**

Find $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{\cos x-\cos 2x}$.

**Solution**

$$\frac{2\sin x-\sin 2x}{\cos x-\cos 2x}=\frac{2\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dots\right)-\left(2x-\dfrac{(2x)^3}{3!}+\dfrac{(2x)^5}{5!}-\dots\right)}{\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dots\right)-\left(1-\dfrac{(2x)^2}{2!}+\dfrac{(2x)^4}{4!}-\dots\right)}$$

$$=\frac{\left(2x-\dfrac{1}{3}x^3+\dfrac{1}{60}x^5-\dots\right)-\left(2x-\dfrac{4}{3}x^3+\dfrac{4}{15}x^5-\dots\right)}{\left(1-\dfrac{x^2}{2}+\dfrac{x^4}{24}+\dots\right)-\left(1-2x^2+\dfrac{2}{3}x^4-\dots\right)}$$

$$=\frac{x^3+\text{ terms in }x^5\text{ and higher powers}}{\dfrac{3}{2}x^2+\text{ terms in }x^4\text{ and higher powers}}$$

$$=\frac{x+\text{ terms in }x^3\text{ and higher powers}}{\dfrac{3}{2}+\text{ terms in }x^2\text{ and higher powers}}.$$

As $x\to 0,$ the numerator tends to zero and the denominator tends to $\dfrac{3}{2}$. Hence,

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{\cos x-\cos 2x}=\frac{0}{\frac{3}{2}}=0.$$

**Example 1.6.2**

(a) Find the first three non-zero terms in the expansion of $\dfrac{x}{\ln(1+x)}$ as a series in ascending powers of $x$.

(b) Hence find $\lim\limits_{x\to 0}\left(\dfrac{1}{\ln(1+x)}-\dfrac{1}{x}\right)$.

**Solution**

**(a)**

$$\begin{aligned}
\frac{x}{\ln(1+x)} &= \frac{x}{x-\frac{x^2}{2}+\frac{x^3}{3}-\dots} \\
&= \frac{1}{1-\frac{x}{2}+\frac{x^2}{3}-\dots} \\
&= \left(1-\frac{x}{2}+\frac{x^2}{3}-\dots\right)^{-1} \\
&= 1-\left(-\frac{x}{2}+\frac{x^2}{3}-\dots\right)+\frac{-1\times-2}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\dots\right)^2+\dots \\
&= 1+\frac{x}{2}-\frac{x^2}{3}+\frac{x^2}{4}+\dots \\
&= 1+\frac{x}{2}-\frac{x^2}{12}+\dots.
\end{aligned}$$

**(b)**

$$\begin{aligned}
\frac{1}{\ln(1+x)}-\frac{1}{x} &= \frac{1}{x}\left(\frac{x}{\ln(1+x)}-1\right) \\
&= \frac{1}{x}\left(1+\frac{x}{2}-\frac{1}{12}x^2+\dots-1\right) \\
&= \frac{1}{x}\left(\frac{x}{2}-\frac{1}{12}x^2+\dots\right) \\
&= \frac{1}{2}-\frac{1}{12}x+\dots \\
&\to \frac{1}{2} \text{ as } x\to 0.
\end{aligned}$$

It is interesting to note that when $x\to 0+$, $\dfrac{1}{\ln(1+x)}\to\infty$ and $\dfrac{1}{x}\to\infty$. Also, when $x\to 0-$, $\dfrac{1}{\ln(1+x)}\to-\infty$ and $\dfrac{1}{x}\to-\infty$. This example shows that, in both cases, the **difference** between $\dfrac{1}{\ln(1+x)}$ and $\dfrac{1}{x}$ tends to the finite value $\dfrac{1}{2}$.

**Exercise 1D**

1. Use series expansions to determine the following limits:

   (a) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$, (b) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$, (c) $\lim\limits_{x \to 0} \dfrac{x^2}{1 - \cos 2x}$, (d) $\lim\limits_{x \to 0} \dfrac{x \ln(1+x)}{1 - \cos x}$.

2. (a) Show that $\ln(1 + \sin x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dots$.

   (b) Hence find $\lim\limits_{x \to 0} \dfrac{\ln(1+\sin x) - x}{x^2}$.

3. Find $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x} - \sqrt{2}}{x}$.

4. (a) By using the identity $2^x \equiv e^{x \ln 2}$, obtain the first three terms in the expansion of $2^x$ as a series in ascending powers of $x$. Give the coefficients of $x$ and $x^2$ in terms of $\ln 2$.

   (b) Find $\lim\limits_{x \to 0} \dfrac{2^x - 1}{3^x - 1}$.

5. Find $\lim\limits_{x \to \infty} \left[ \left(x^2 + 3x\right)^{\frac{1}{2}} - x \right]$.

## 1.7 Two important limits

Two interesting limits are introduced in this section. They are of some importance because they occur quite frequently in applications of mathematics.

- The limit of $x^k e^{-x}$ as $x \to \infty$

Consider first the function $x^2 e^{-x}$. When $x$ becomes very large, $x^2$ becomes very large and $e^{-x}$ becomes very small. It is not immediately obvious what will happen to the product $x^2 e^{-x}$, but if its behaviour is investigated numerically using a calculator it will be seen that $x^2 e^{-x}$ is very small for large values of $x$. Evidently, therefore, the effect of $e^{-x}$ in making $x^2 e^{-x}$ smaller is stronger than the effect of $x^2$ making it larger. This is a particular case of the following general result:

$$\boxed{\text{when } x \to \infty,\ x^k e^{-x} \to 0 \text{ for any real number } k}$$

To prove this, note first that when $k = 0$ and $x \neq 0$ the expression $x^k e^{-x}$ becomes simply $e^{-x}$, which tends to zero as $x \to \infty$. The result therefore holds in this case. Also, when $k < 0$, both $x^k$ and $e^{-x}$ tend to zero as $x \to \infty$ so the product $x^k e^{-x}$ must also tend to zero. Therefore the result holds in this case too.

Now suppose that $k > 0$. Let $n$ be an integer such that $n > k$. Using the series expansion for $e^x$,

$$\begin{aligned}
x^k e^{-x} &= \frac{x^k}{e^x} \\
&= \frac{x^k}{1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \ldots} \\
&= \frac{x^{k-n}}{x^{-n} + x^{1-n} + \frac{x^{2-n}}{2!} + \ldots + \frac{1}{n!} + \frac{x}{(n+1)!} + \frac{x^2}{(n+2)!} + \ldots}.
\end{aligned}$$

But $n > k$, so $k - n$ is negative. Hence, when $x \to \infty$, $x^{k-n} \to 0$. In the denominator of the expression above, all the terms are positive and all those after the $\frac{1}{n!}$ term tend to infinity as $x \to \infty$. Hence, when $x \to \infty$ the denominator of the expression tends to infinity. It follows, therefore, that

$$x^k e^{-x} \to \frac{0}{\infty} = 0.$$

Note that no matter how large the number $k$ may be, $x^k e^{-x} \to 0$ as $x \to \infty$. For large values of $x$, the influence of $e^{-x}$ is therefore stronger than any power of $x$.

- The limit of $x^k \ln x$ as $x \to 0$

Consider the function $x^k \ln x$ where $k > 0$. Here **$x$ must be restricted to positive values** because otherwise $\ln x$ is not defined.

When $x \to 0$, $x^k \to 0$ but $\ln x \to -\infty$. Therefore, it is not obvious what happens to the product $x^k \ln x$ as $x \to 0$. It will be proved that:

$$\boxed{\text{when } x \to 0,\ x^k \ln x \to 0 \text{ for all } k > 0}$$

Let $x = e^{-\frac{y}{k}}$. Then when $y \to \infty$, $x \to 0$. Also $x^k = e^{-y}$ and $\ln x = -\frac{y}{k}$. Hence,

$$\lim_{x\to 0}\left(x^k \ln x\right) = \lim_{y\to\infty}\left(-\frac{y}{k}e^{-y}\right)$$
$$= -\frac{1}{k}\lim_{y\to\infty}\left(y\,e^{-y}\right).$$

Using the limit established for $x^k e^{-x}$ with $k = 1$ (and $x$ replaced by $y$, of course),

$$\lim_{y\to\infty}\left(y\,e^{-y}\right) = 0.$$

Hence, $\lim_{x\to 0}\left(x^k \ln x\right) = 0$, as required.

Note that for small values of $x$, $x^k \ln x$ will be negative because $\ln x < 0$ for $0 < x < 1$. Hence the limit zero is approached through negative values. The result can therefore be expressed more fully as:

$$\boxed{x^k \ln x \to 0- \text{ when } x \to 0+ \quad (k > 0)}$$

Note that as $x$ approaches zero, the effect of $\ln x$ making $x^k \ln x$ large and negative is weaker than the effect of $x^k$ making the product $x^k \ln x$ smaller, no matter how small $k$ may be.

**Example 1.7.1**

Show that $\lim_{x\to\infty}\left(x^2 e^{-2x}\right) = 0.$

**Solution**

Put $2x = y$. Then $x \to \infty$ corresponds to $y \to \infty$. Hence,

$$\begin{aligned}\lim_{x\to\infty}\left(x^2\,\mathrm{e}^{-2x}\right) &= \lim_{y\to\infty}\left(\frac{y^2}{4}\,\mathrm{e}^{-y}\right)\\ &= \frac{1}{4}\lim_{y\to\infty}\left(y^2\,\mathrm{e}^{-y}\right)\\ &= 0.\end{aligned}$$

**Example 1.7.2**

Find $\lim_{x\to 0}\left[(1+x)^2-1\right]\ln x$, where $x>0$.

**Solution**

$$\begin{aligned}\left[(1+x)^2-1\right]\ln x &= \left(2x+x^2\right)\ln x\\ &= 2x\ln x + x^2\ln x.\end{aligned}$$

When $x\to 0$, $x\ln x\to 0$ and $x^2\ln x\to 0$. Hence,

$$\lim_{x\to 0}\left[(1+x)^2-1\right]\ln x = 0.$$

**Example 1.7.3**

(a) Express $x^x$ in the form $\mathrm{e}^a$, where $x>0$.

(b) Hence show that $x^x\to 1$ as $x\to 0$.

**Solution**

**(a)** Let $x^x=\mathrm{e}^a$. Then

$$\ln x^x = \ln \mathrm{e}^a$$
$$\Rightarrow x\ln x = a.$$

Therefore $x^x = \mathrm{e}^{x\ln x}$.

**(b)** When $x\to 0$, $x\ln x\to 0$. Hence, as $x\to 0$, $x^x\to \mathrm{e}^0 = 1$.

**Exercise 1E**

1. Find the following limits.

   (a) $\lim\limits_{x\to\infty} \dfrac{1+x}{e^x}$, (b) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^{2x}}$, (c) $\lim\limits_{x\to\infty} \left[(1+x)^3 - 1\right]e^{-x}$, (d) $\lim\limits_{x\to-\infty} x^{10}e^x$,

   (e) $\lim\limits_{x\to 0} x\ln 2x$, (f) $\lim\limits_{x\to 0+} x\ln\left(x+x^2\right)$, (g) $\lim\limits_{x\to 1-} (1-x)\ln(1-x)$.

2. By setting $x = e^y$, show that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$.

3. The function f is defined by

$$f(x) = \frac{1}{x^3(1+\ln x)}, \qquad x > 0.$$

   Show that $f(x) \to -\infty$ as $x \to 0$.

4. (a) Show that the curve with equation $y = x\,e^{-x}$ has a stationary point at $\left(1, e^{-1}\right)$.

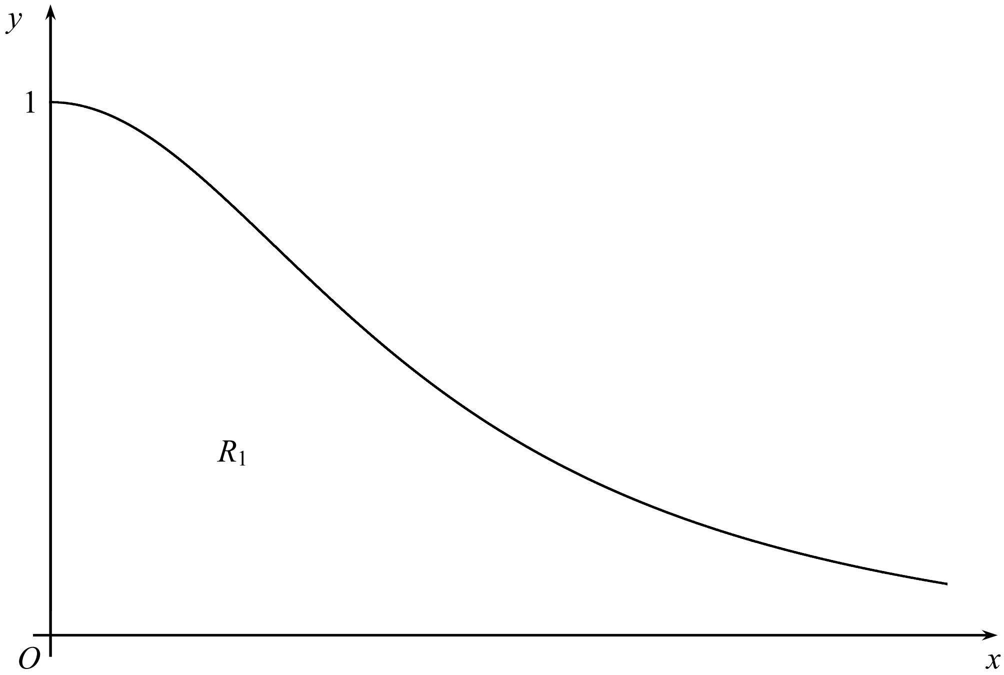
   (b) Sketch the curve.

5. (a) Show that the curve with equation $y = x\ln x$, where $x > 0$, has a stationary point at $\left(e^{-1}, -e^{-1}\right)$.

   (b) Sketch the curve.

## 1.8 Improper integrals

Consider the integral $I_1 = \int_0^\infty \frac{1}{1+x^2}\,dx$. This gives the area $A$ of the region $R_1$ in the first quadrant enclosed by the curve $y = \frac{1}{1+x^2}$, the $x$-axis and the $y$-axis. The region $R_1$ is shown in the diagram below.



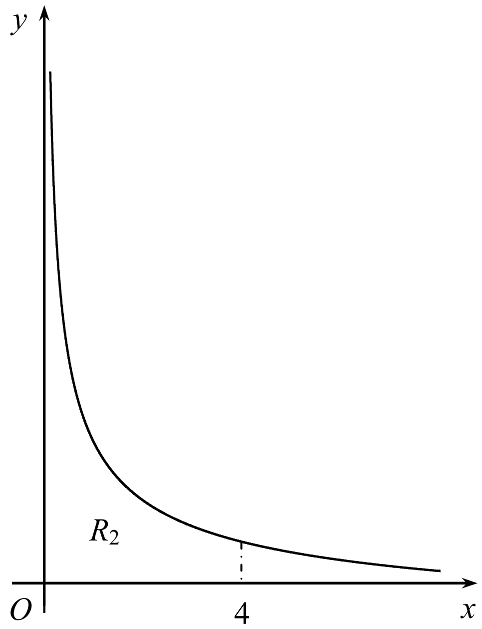

However, because the upper limit of the integral is infinite, $R_1$ is unbounded and it is not clear that $A$ will have a finite value.

To investigate this, the upper limit of $I_1$ is replaced by $c$. Then

$$\begin{aligned} I_1 &= \int_0^c \frac{1}{1+x^2}\,dx \\ &= \left[\tan^{-1} x\right]_0^c \\ &= \tan^{-1} c. \end{aligned}$$

Now let $c \to \infty$ and it can be seen that $I_1 \to \frac{\pi}{2}$ because $\tan^{-1} c \to \frac{\pi}{2}$. The area $A$ of the region $R_1$ is therefore finite having the value $\frac{\pi}{2}$.

Consider next the integral $I_2 = \int_0^4 \frac{1}{x^{\frac{1}{2}}}\,\mathrm{d}x$. This gives the area of the region $R_2$ enclosed by the curve $y = \frac{1}{x^{\frac{1}{2}}}$, the $x$-axis, the $y$-axis and the ordinate $x = 4$. The region $R_2$ is shown in the diagram below.




However, $R_2$ is also an unbounded region because $y \to \infty$ as $x \to 0$. To determine whether or not the area of $R_2$ is finite, first replace the lower limit by $c$. Then

$$\begin{aligned} I_2 &= \int_c^4 \frac{1}{x^{\frac{1}{2}}}\,\mathrm{d}x \\ &= \left[2x^{\frac{1}{2}}\right]_c^4 \\ &= 4 - 2c^{\frac{1}{2}}. \end{aligned}$$

Now let $c \to 0$ and it can be seen that $I_2 \to 4$. The area of $R_2$ is therefore equal to 4.

The two integrals $I_1$ and $I_2$ are special cases of what are called **improper integrals**. The formal definition is as follows.

> The integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is said to be improper if
>
> (1) the interval of integration is infinite,
>
> or (2) $\mathrm{f}(x)$ is not defined at one or both of the end points $x = a$ and $x = b$,
>
> or (3) $\mathrm{f}(x)$ is not defined at one or more interior points of the interval $a \le x \le b$.

In this chapter, only improper integrals of types (1) and (2) will be considered.

The integral $I_1$ above is of type (1) because the interval of integration is infinite. The integral $I_2$ is of type (2) because $\frac{1}{x^{\frac{1}{2}}}$ is not defined at $x = 0$.

The integrals $I_1$ and $I_2$ were evaluated by finding appropriate limits and a similar procedure is used for all improper integrals. However, in some cases it will be found that no limit exists. In such cases, it is said that the integral is **divergent** or **does not exist**.

**Exercise 1F**

1. Explain why each of the following integrals is improper.

(a) $\int_{-\infty}^{1} \frac{1}{1+x^2}\,\mathrm{d}x$, (b) $\int_{0}^{1} \frac{\ln x}{x^{\frac{1}{2}}}\,\mathrm{d}x$, (c) $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$.

**Example 1.8.1**

Show that none of the following integrals exists.

(a) $I = \int_{1}^{\infty} \frac{1}{x}\,\mathrm{d}x$, (b) $J = \int_{0}^{1} \frac{1}{(1-x)^2}\,\mathrm{d}x$, (c) $K = \int_{0}^{\infty} \cos x\,\mathrm{d}x$.

**Solution**

**(a)** Replacing the upper limit in $I$ by $c$ gives $\int_{1}^{c} \frac{1}{x}\,\mathrm{d}x = [\ln x]_1^c = \ln c$. When $c \to \infty$, $\ln c \to \infty$ and therefore $I$ does not exist.

[Remember that $\infty$ does not qualify as a limiting value because limiting values must be finite. Read the definition given in Section 1.1 again.]

**(b)** The integral $J$ is improper because $\frac{1}{(1-x)^2}$ is not defined at $x = 1$. Consider therefore

$$\int_{0}^{c} \frac{1}{(1-x)^2}\,\mathrm{d}x = -\left[\frac{1}{1-x}\right]_0^c = -\frac{1}{1-c} + 1.$$

When $c \to 1$, $\frac{1}{1-c} \to \infty$. Therefore the integral $J$ does not exist.

**(c)** Consider $\int_{0}^{c} \cos x\,\mathrm{d}x = [\sin x]_0^c = \sin c$. When $c \to \infty$, $\sin c$ oscillates between –1 and +1. Hence there is no limiting value and $K$ does not exist.

**Example 1.8.2**

(a) Explain why $\int_0^{\mathrm{e}} x \ln x \, \mathrm{d}x$ is an improper integral.

(b) Show that the integral exists and find its value.

**Solution**

**(a)** The integral is improper because $x \ln x$ is not defined at $x = 0$.

**(b)** Let $I = \int_c^{\mathrm{e}} x \ln x \, \mathrm{d}x$. Integrating by parts,

$$\begin{aligned} I &= \left[\frac{1}{2}x^2 \ln x\right]_c^{\mathrm{e}} - \int_c^{\mathrm{e}} \frac{1}{2}x^2 \times \frac{1}{x} \, \mathrm{d}x \\ &= \frac{1}{2}\mathrm{e}^2 - \frac{1}{2}c^2 \ln c - \left[\frac{x^2}{4}\right]_c^{\mathrm{e}} \\ &= \frac{1}{4}\mathrm{e}^2 - \frac{1}{2}c^2 \ln c + \frac{1}{4}c^2. \end{aligned}$$

When $c \to 0$, $c^2 \ln c \to 0$ and $c^2 \to 0$. Hence the given integral exists and its value is $\frac{1}{4}\mathrm{e}^2$.

**Exercise 1G**

1. (a) Show that one of the following integrals exists and that the other does not:

$$\int_1^{\infty} \frac{1}{x^3} \, \mathrm{d}x, \qquad \int_0^1 \frac{1}{x^3} \, \mathrm{d}x.$$

(b) Evaluate the one that does exist.

2. Evaluate the following improper integrals, showing in each case the limiting process used.

(a) $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x$, (b) $\int_0^{\infty} \frac{1}{(1+x)^2} \, \mathrm{d}x$, (c) $\int_0^{\infty} x \, \mathrm{e}^{-x} \, \mathrm{d}x$,

(d) $\int_{-\infty}^0 \frac{1}{(4-x)^{\frac{3}{2}}} \, \mathrm{d}x$, (e) $\int_0^1 x^2 \ln x \, \mathrm{d}x$, (f) $\int_0^{\mathrm{e}} \ln x \, \mathrm{d}x$.

3. (a) Explain why each of the following integrals is improper:

(i) $\int_0^{\infty} \frac{1}{\sqrt{1+x}} \, \mathrm{d}x$, (ii) $\int_0^1 \frac{x}{1-x^2} \, \mathrm{d}x$.

(b) Show that neither integral exists.

**Miscellaneous exercises 1**

1. Use Maclaurin's theorem to show that

$$\tan\left(x+\frac{\pi}{4}\right)=1+2x+2x^2+\dots\ .$$

2. Explain why $\ln x$ cannot have a Maclaurin expansion.

3. Use Maclaurin's theorem to show that

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\ .$$

4. (a) Expand $(1+2x)^{\frac{1}{2}}\sin x$ as a series in ascending powers of $x$, up to and including the term in $x^3$.

   (b) Determine the range of values of $x$ for which the expansion is valid.

5. (a) Obtain the first two non-zero terms in the expansion of

$$e^{3x}+\ln(1-3x)\,.$$

   (b) Determine the range of values of $x$ for which the expansion is valid.

6. (a) Obtain the first three non-zero terms in the expansions in ascending powers of $x$ of (i) $x^2e^x$, (ii) $\cos 2x$.

   (b) Hence find $\lim\limits_{x\to 0}\dfrac{x^2e^x}{\cos 2x-1}$.

[AQA, 1999]

7. By means of the substitution $x=y+3$, or otherwise, evaluate

$$\lim_{x\to 3}\frac{\sqrt{(4-x)}-1}{\sqrt{(1+x)}-2}.$$

[JMB, 1984]

8. (a) Find $\lim\limits_{x\to 0}\left[1-\dfrac{1}{(1+x)^2}\right]\ln x$.

(b) Find $\lim\limits_{x\to\infty}\dfrac{x+e^x}{x-e^x}$.

9. The function f is defined

$$f(x)=\frac{e^x}{1-x}, \qquad x\neq 1.$$

Show that $f(x)\to -\infty$ as $x\to\infty$.

10.(a) Use integration by parts to evaluate

$$\int_a^1 \ln x\,dx, \qquad a>0.$$

(b) Explain why $\int_0^1 \ln x\,dx$ is an improper integral. Determine whether the integral exists or not, giving a reason for your answer.

[NEAB, 1995]

11. (a) Use the expansion of $\cos x-1$ to obtain the expansion of $e^{\cos x-1}$ in a series in ascending powers of $x$, up to and including the term in $x^4$.

(b) Evaluate $\lim\limits_{x\to 0}\dfrac{e-e^{\cos x}}{x^2}$.

[JMB, 1988]

12 (a) Write down the value of $\lim\limits_{x\to\infty}\dfrac{x}{2x+1}$.

(b) Evaluate

$$\int_1^\infty\left(\frac{1}{x}-\frac{2}{2x+1}\right)dx$$

giving your answer in the form $\ln k$, where $k$ is a constant to be determined.
Explain why this is an improper integral.

[NEAB, 1997]

13. The function f is defined by

$$\mathrm{f}(x) = \left(\frac{1+x}{1-3x}\right)^{\frac{1}{2}}, \qquad -1 \le x < \frac{1}{3}.$$

(a) Expand $\mathrm{f}(x)$ as a series in ascending powers of $x$, up to and including the term in $x^2$.

(b) By expressing $\ln\left[\mathrm{f}(x)\right]$ in terms of $\ln(1+x)$ and $\ln(1-3x)$, expand $\ln\left[\mathrm{f}(x)\right]$ as a series in ascending powers of $x$, up to and including the term in $x^3$.

(c) Find $\displaystyle\lim_{x\to 0} \frac{\mathrm{f}(x)-1}{\ln\left[\mathrm{f}(x)\right]}$.

[NEAB, 1998]

14. Show that one of the following integrals exists and that the other does not. Evaluate the one that exists, showing the limiting process used.

(a) $I = \displaystyle\int_0^1 \frac{1}{x} \ln x \, \mathrm{d}x$ (b) $J = \displaystyle\int_0^1 \frac{1}{\sqrt{x}} \ln x \, \mathrm{d}x$

[JMB, 1990]

15. A curve $C$ has the equation

$$y = x^2 \mathrm{e}^{-x}.$$

(a) Show that $C$ has stationary points at the origin and at the point $\left(2, 4\mathrm{e}^{-2}\right)$.

(b) Sketch $C$, indicating the asymptote clearly.

(c) The area of the region in the first quadrant bounded by $C$, the positive $x$-axis and the ordinate $x = a$ is $A$.

(i) Show that $A = 2 - 2\mathrm{e}^{-a} - 2a\mathrm{e}^{-a} - a^2\mathrm{e}^{-a}$.

(ii) Hence obtain the area of the whole of the region in the first quadrant bounded by $C$ and the positive $x$-axis.

16. (a) (i) Expand $\left(1+x+x^2\right)^{\frac{1}{2}}$ as a series in ascending powers of $x$, up to and including the term in $x^2$.

(ii) Hence, or otherwise, show that

$$\left(1-x+x^2\right)^{\frac{1}{2}}=1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots.$$

(b) Find $\lim\limits_{x\to 0}\dfrac{\left(1+x+x^2\right)^{\frac{1}{2}}-1}{\left(1-x+x^2\right)^{\frac{1}{2}}-1}$.

(c) (i) Express $\left(\dfrac{1}{x^2}+\dfrac{1}{x}+1\right)^{\frac{1}{2}}$ in the form

$$\frac{1}{x^p}\left(1+x+x^2\right)^{\frac{1}{2}},$$

where $p$ is a number to be determined.

(ii) Find $\lim\limits_{x\to 0}\left[\left(\dfrac{1}{x^2}+\dfrac{1}{x}+1\right)^{\frac{1}{2}}-\left(\dfrac{1}{x^2}-\dfrac{1}{x}+1\right)^{\frac{1}{2}}\right]$.

[AQA, 1999]

17. (a) (i) Obtain, in simplified form, the first three non-zero terms in the expansions in ascending powers of $x$ of each of

$$\sin 2x \quad\text{and}\quad 1-\mathrm{e}^{-x}.$$

(ii) Hence show that $\lim\limits_{x\to 0}\dfrac{\sin 2x}{1-\mathrm{e}^{-x}}=2$.

(b) (i) Show that the expansion of $\dfrac{1}{\sin 2x}$ in ascending powers of $x$ begins with the terms

$$\frac{1}{2x}+\frac{1}{3}x+\frac{7}{45}x^3.$$

(ii) Find the first three non-zero terms in the expansion of $\dfrac{1}{1-\mathrm{e}^{-x}}$ in ascending powers of $x$.

(iii) Hence find $\lim\limits_{x\to 0}\left(\dfrac{2}{\sin 2x}-\dfrac{1}{1-\mathrm{e}^{-x}}\right)$.

[AQA, 2001]

# Chapter 2: Polar Coordinates

2.1 Cartesian and polar frames of reference

2.2 Restrictions on the values of $\theta$

2.3 The relationship between Cartesian and polar coordinates

2.4 Representation of curves in polar form

2.5 Curve sketching

2.6 The area bounded by a polar curve

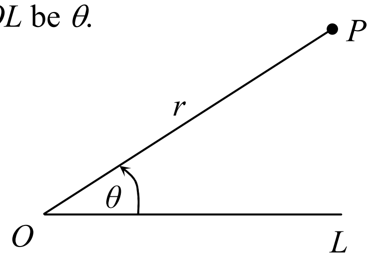
This chapter introduces polar coordinates. When you have completed it, you will:

- know what is meant by polar coordinates;
- know how polar coordinates are related to Cartesian coordinates;
- know that equations of curves can be expressed in terms of polar coordinates;
- be able to sketch curves of equations given in polar form;
- be able to find areas by integration using polar coordinates.

## 2.1 Cartesian and polar frames of reference

You should already be familiar with the use of rectangular Cartesian coordinate axes as a frame of reference for labelling points in a plane and for investigating the properties of curves given in Cartesian form. When fixed axes $Ox$ and $Oy$ have been chosen, the position of any point $P$ in the plane $Oxy$ can be specified by its coordinates $(x, y)$ relative to those axes.

This is not the only way in which points in a plane may be labelled. Let $O$ be a fixed point and $OL$ a fixed line in the plane. For any point $P$, let the distance of $P$ from $O$ be $r$ and the angle that $OP$ makes with $OL$ be $\theta$.




Then $r$ and $\theta$ are called the **polar coordinates** of $P$: when their values are known, the position of $P$ is also known.

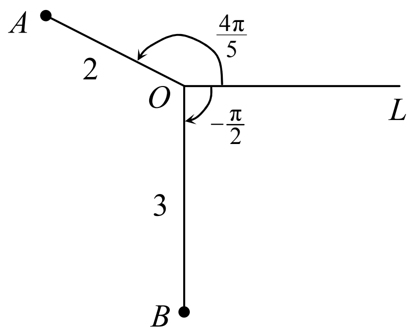
> The point $O$ is called the **pole** and $OL$ is called the **initial line**. The angle $\theta$ is measured in **radians**. Positive values of $\theta$ correspond to an anticlockwise rotation from $OL$, and negative values to a clockwise rotation. The plane containing $OL$ and $OP$ is called the $r$–$\theta$ plane.

Note that $r \geq 0$ because $r$ is defined here as the **distance** of $P$ from $O$, which is necessarily non-negative. In some textbooks, $r$ is defined in such a way that negative values are permissible.

**Example 2.1.1**

Draw a diagram which shows the points $A$ and $B$ with polar coordinates $\left(2, \frac{4\pi}{5}\right)$ and $\left(3, -\frac{\pi}{2}\right)$, respectively.

**Solution**



**Exercise 2A**

1. Show on a diagram the points $A$, $B$, $C$ and $D$ which have polar coordinates $\left(1, \frac{\pi}{5}\right)$, $(2, 0)$, $\left(3, -\frac{3\pi}{4}\right)$ and $\left(3, \frac{5\pi}{4}\right)$, respectively.

2. The points $A$ and $B$ have polar coordinates $\left(2, \frac{\pi}{6}\right)$ and $\left(3, -\frac{\pi}{2}\right)$, respectively.
   (a) Find the angle between $OA$ and $OB$, where $O$ is the pole.
   (b) Use the cosine rule to find the distance between $A$ and $B$.

3. Sketch the regions of the $r$–$\theta$ plane for which (a) $1 \le r \le 2$, (b) $-\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$.

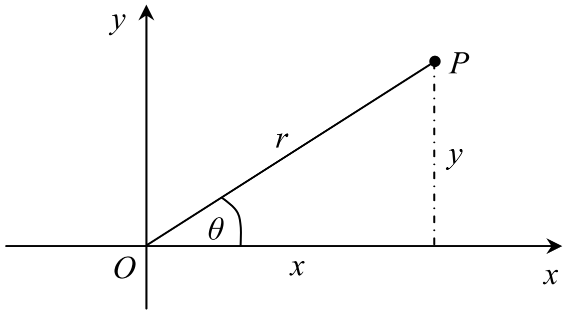
## 2.2 Restrictions on the values of $\theta$

In answering Question 1 in Exercise 2A you will have noticed that $C$ and $D$ are the same point even though they have different polar coordinates.

To ensure that each point in a plane, other than the pole $O$, has one and only one pair of polar coordinates, the values that $\theta$ can take will sometimes be restricted to the interval $-\pi < \theta \le \pi$ or $0 \le \theta < 2\pi$. The pole is an exceptional point: it is defined by $r = 0$ without reference to $\theta$.

## 2.3 The relationship between Cartesian and polar coordinates

The diagram below shows the Cartesian coordinate axes $Ox$ and $Oy$ together with a polar coordinate system in which $O$ is the pole and $Ox$ is the inital line.



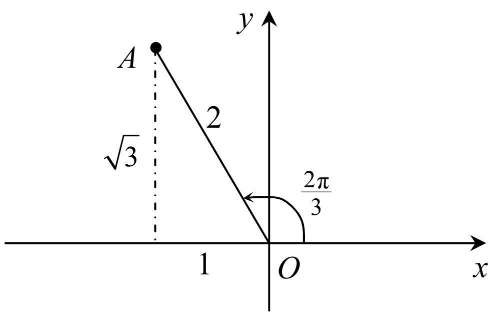

When the two systems are superimposed in this way, there are simple relationships between the Cartesian and polar coordinates of $P$. It can be seen from the diagram that

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

The first two of these relationships hold for all values of $\theta$, always giving the correct signs for $x$ and $y$. For example, the point $A$ with polar coordinates $\left(2, \frac{2\pi}{3}\right)$ will lie in the second quadrant as shown in the diagram below.

The Cartesian coordinates are

$$x = 2\cos\frac{2\pi}{3} = -1,$$
$$y = 2\sin\frac{2\pi}{3} = \sqrt{3}.$$




**Exercise 2B**

1. The points $A$ and $B$ have polar coordinates $\left(3, \frac{\pi}{6}\right)$ and $\left(4, -\frac{\pi}{3}\right)$, respectively.
   (a) Show that $AB = 5$.
   (b) Find the Cartesian coordinates of $A$ and $B$.
   (c) Use the Cartesian coordinates of $A$ and $B$ to verify that $AB = 5$.

2. Find the polar coordinates of the points with Cartesian coordinates
   (a) $(2, 2)$, (b) $(-1, \sqrt{3})$, (c) $(-3, -4)$.

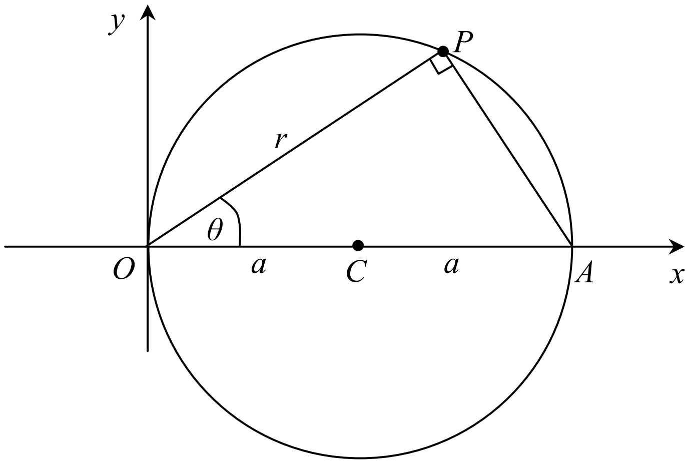
3. The points $A$ and $B$ have polar coordinates $\left(\sqrt{3}, \frac{\pi}{3}\right)$ and $\left(1, -\frac{\pi}{6}\right)$, respectively. Show that $AB$ is perpendicular to the initial line, and find the length of $AB$.

## 2.4 Representation of curves in polar form

If a point $P$ with Cartesian coordinates $(x, y)$ moves on a circle of radius $a$ and centre $C$ $(a, 0)$, then, for all positions of $P$,

$$(x-a)^2 + y^2 = a^2.$$

This is the Cartesian equation of the circle.




Now let $P$ have polar coordinates $(r, \theta)$, as shown in the diagram above. The circle cuts the positive $x$-axis at the point $A$ $(2a, 0)$. Hence, from the triangle $OAP$,

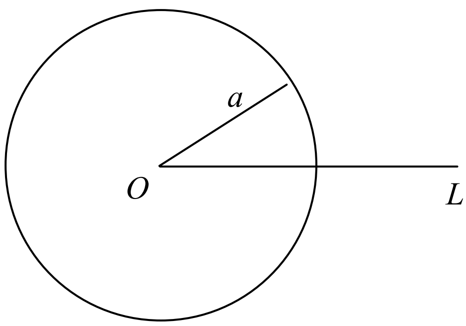
$$r = 2a\cos\theta.$$

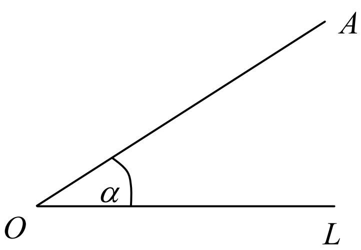
This is called the **polar equation** of the circle. Note that the equation is valid for negative values of $\theta$ because $\cos(-\theta) = \cos\theta$.

There are many examples of curves whose properties are more easily investigated using a polar equation rather than a Cartesian equation. The following two, particularly simple, cases of polar equations should be noted:

- the equation $r = a$ represents a circle centred at $O$ and of radius $a$;
- the equation $\theta = \alpha$ represents a semi-infinite straight line $OA$ radiating from the origin and making an angle of $\alpha$ with the initial line $OL$.

These loci are shown in the diagrams below.



**Exercise 2C**

1. Use a method similar to that used at the beginning of Section 2.4 to find the polar equation of the circle of radius $a$ whose centre has polar coordinates $\left(a, \frac{\pi}{2}\right)$.

2. Use the relationships between Cartesian and polar coordinates, given in Section 2.3, to obtain the polar equation of the circle with Cartesian equation

$$(x-a)^2 + y^2 = a^2.$$

3. Find the polar equation of the straight line which is perpendicular to the initial line and at a perpendicular distance $a$ from the pole.

## 2.5 Curve sketching

In general, an equation connecting $r$ and $\theta$ represents a curve. To discover the shape of the curve, values of $r$ for convenient values of $\theta$ can be tabulated to give the polar coordinates of a number of points on the curve. Plotting these points and joining them up will give a good indication of the shape of the curve, but it may be necessary to investigate a little further to determine the shape near some points with more certainty. The following considerations will often prove helpful.

1. There may be some symmetry. For example, if $r$ can be expressed as a function of $\cos\theta$ only, the curve will be symmetrical about the initial line $\theta = 0$ because $\cos(-\theta) = \cos\theta$. Also, if $r$ can be expressed as a function of $\sin\theta$ only, the curve will be symmetrical about the line $\theta = \frac{\pi}{2}$ because $\sin(\pi - \theta) = \sin\theta$.
2. If $r \to 0$ as $\theta \to \alpha$, then the line $\theta = \alpha$ will be a tangent to the curve at the pole $O$.
3. Negative values of $r$ are not allowed. If values of $\theta$ in the interval $\alpha \le \theta \le \beta$ give $r < 0$, then there is no curve in the region $\alpha \le \theta \le \beta$.

**Example 2.5.1**

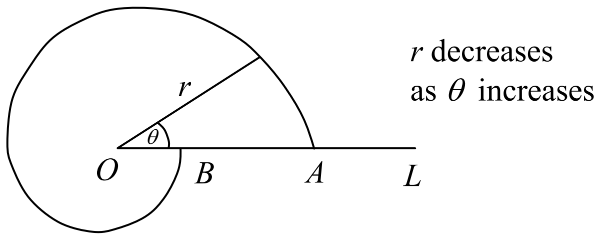
Sketch the curve with polar equation $r = \dfrac{3\pi}{\pi + \theta}$, $\quad 0 \le \theta \le 2\pi$.

**Solution**

This is a relatively easy curve to sketch. It can be seen that $r$ decreases steadily as $\theta$ increases from 0 to $2\pi$. Also, when $\theta = 0, r = 3$ and when $\theta = 2\pi, r = 1$. The curve must therefore be roughly as shown in the diagram below.

Point $A$ has polar coordinates (3, 0).

Point $B$ has polar coordinates $(1, 2\pi)$.

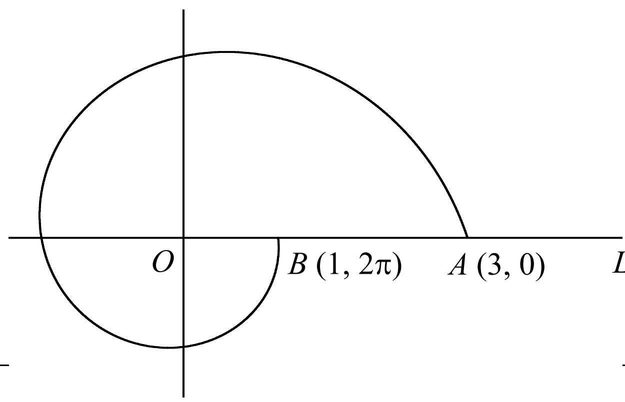



To obtain a more accurate sketch, a few more values of $r$ are needed.

The values shown are sufficient in this case.

| $\theta$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $r$ | 3 | 2 | 1.5 | 1.2 | 1 |

Plotting the five points with the coordinates given in the table above gives the curve shown in this diagram.



**Example 2.5.2**

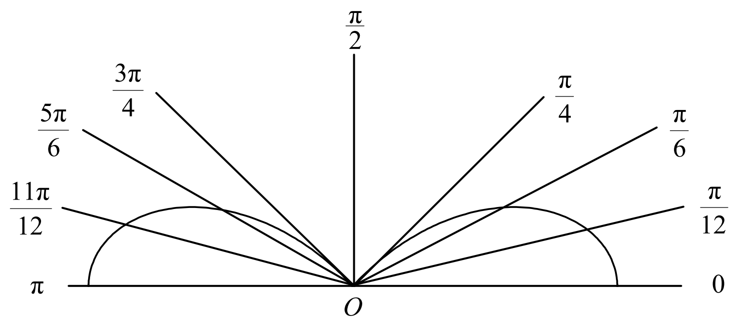
Sketch the curve $r = a\cos 2\theta$, where $a > 0$ and $-\pi < \theta \le \pi$.

**Solution**

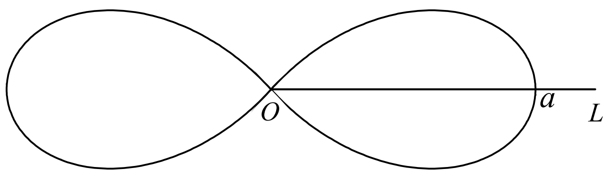
Note first that, because $\cos(-2\theta) = \cos 2\theta$, the curve will be symmetrical about the initial line. Therefore it is sufficient to tabulate values of $r$ for $0 \le \theta \le \pi$.

| $\theta$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ | $\frac{7\pi}{12}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{11\pi}{12}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r$ | $a$ | $\frac{a\sqrt{3}}{2}$ | $\frac{a}{2}$ | 0 | –ve | –ve | –ve | –ve | –ve | 0 | $\frac{a}{2}$ | $\frac{a\sqrt{3}}{2}$ | $a$ |

Because $r < 0$ when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, there is no curve in this region. Plotting the points and joining them up gives the curve shown in the diagram below.




The complete curve can now be obtained by reflecting this curve in the initial line. This is shown in the diagram below.




The curve consists of two equal loops. Note that the curve is also symmetrical about the line $\theta = \frac{\pi}{2}$. This could have been deduced from $r = a\cos 2\theta$ by expressing it as $r = a(1 - 2\sin^2\theta)$. As mentioned earlier, if $r$ can be expressed as a function of $\sin\theta$ only, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$. Note also, from $r = a\cos 2\theta$, that when $\theta \to \pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$, $r \to 0$. This indicates that the lines $\theta = \pm\frac{\pi}{4}$ and $\theta = \pm\frac{3\pi}{4}$ are tangents to the curve at the pole, as confirmed by the diagram.

**Exercise 2D**

1. A curve has the polar equation

$$r = 1 + \frac{\theta}{\pi}, \qquad 0 \le \theta \le 4\pi.$$

(a) Make a rough sketch of the curve by considering how $r$ varies as $\theta$ increases from 0 to $4\pi$.

(b) Tabulate the values of $r$ for $\theta = 0, \frac{\pi}{2}, \pi, \ldots, 4\pi$. Hence make a more accurate sketch of the curve.

2. A curve has the polar equation

$$r = 2 + \cos\theta, \text{ where } -\pi < \theta \le \pi.$$

(a) Tabulate the values of $r$ for $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$ and $\pi$.

(b) Sketch the curve.

3. (a) Sketch the curve with the polar equation

$$r = 1 - \sin\theta, \qquad -\pi < \theta \le \pi.$$

(b) State the polar equation of the tangent to the curve at the pole.

4. A curve $C$ has the polar equation

$$r = a\sin 2\theta, \text{ where } a > 0 \text{ and } -\pi < \theta \le \pi.$$

(a) Show that there is no part of $C$ in the regions $\frac{\pi}{2} < \theta < \pi$ and $-\frac{\pi}{2} < \theta < 0$.

(b) Sketch the curve.

5. (a) Sketch the curve with the polar equation

$$r = 2\sin 3\theta, \qquad -\pi < \theta \le \pi.$$

(b) Give the polar equations of the tangents to the curve at the pole.

6. Sketch the curve with the polar equation

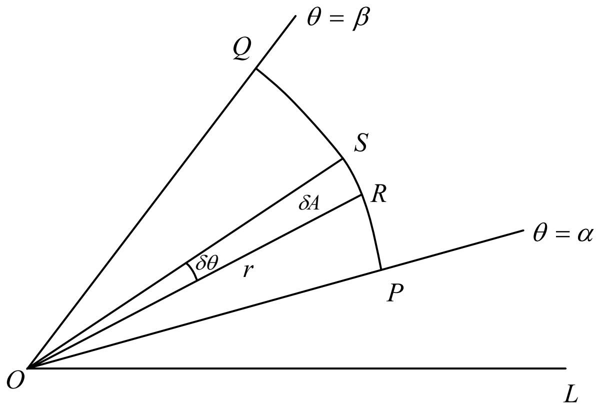
$$r = e^{\frac{1}{4}\theta}, \qquad 0 \le \theta \le 2\pi.$$

## 2.6 The area bounded by a polar curve

Consider the curve

$$r = \mathrm{f}(\theta), \qquad \alpha \le \theta \le \beta.$$

Suppose that $r \ge 0$ throughout the interval $\alpha \le \theta \le \beta$. Let $P$ and $Q$ be the points on the curve at which $\theta = \alpha$ and $\theta = \beta$, respectively.




A formula for the area $A$ bounded by the sector $OPQ$ can be found as follows.

Consider an elementary sector $ORS$, as shown in the diagram above, where $R(r, \theta)$ and $S(r + \delta r, \theta + \delta\theta)$ are neighbouring points on the curve. The area, $\delta A$, of this elementary sector is approximately the same as that of a circular sector of radius $r$ and angle $\delta\theta$, i.e.

$$\delta A \approx \frac{1}{2} r^2 \delta\theta.$$

This approximation will become increasingly accurate as $\delta\theta \to 0$. Forming the sum of the areas of all such elementary sectors between $\theta = \alpha$ and $\theta = \beta$, the total area, $A$, of the sector $OPQ$ is given by

$$A = \lim_{\delta\theta \to 0} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 \, \delta\theta.$$

Hence,

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, \mathrm{d}\theta}$$

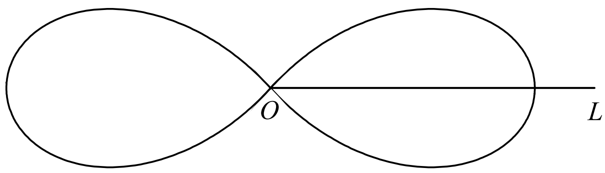
**When applying this formula, it is important to remember that $r$ must be defined and be non-negative throughout the interval $\alpha \le \theta \le \beta$.**

**Example 2.6.1**

Find the total area of the two loops of the curve $r = a\cos 2\theta$, where $a > 0$ and $-\pi < \theta \le \pi$.

**Solution**

The sketch of the curve was obtained earlier. For convenience, it is repeated here.




The two loops are reflections of each other in the line $\theta = \frac{\pi}{2}$ and the right-hand loop lies in the region $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. Hence the total area bounded by the two loops will be given by

$$A = 2\times\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}a^2\cos^2 2\theta\,\mathrm{d}\theta$$

$$= a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 4\theta)\mathrm{d}\theta$$

$$= \frac{1}{2}a^2\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}a^2\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$= \frac{\pi a^2}{4}.$$

**Exercise 2E**

1. Explain why it would be wrong to calculate the area of the curve in Example 2.6.1 by evaluating

$$\int_{-\pi}^{\pi} \frac{1}{2}a^2\cos^2 2\theta\,\mathrm{d}\theta.$$

2. (a) Write down the polar equation of a circle of radius $a$ with centre at the pole $O$.

   (b) Use the formula $A = \int_{\alpha}^{\beta} \frac{1}{2}r^2\,\mathrm{d}\theta$ to show that the area of the circle is $\pi a^2$.

**Example 2.6.2**

A curve is given in Cartesian form by the equation $x^2 + y^2 - 2x = (x^2 + y^2)^{\frac{1}{2}}$.

(a) Show that the polar equation of this curve is $r = 1 + 2\cos\theta$, where $-\pi < \theta \le \pi$.

(b) Show that, at the pole, the curve is tangential to the lines $\theta = \pm\dfrac{2\pi}{3}$.

(c) Sketch the curve.

(d) Show that the area enclosed by the curve is $2\pi + \dfrac{3\sqrt{3}}{2}$.

**Solution**

**(a)** Substituting $x^2 + y^2 = r^2$ and $x = r\cos\theta$ into $x^2 + y^2 - 2x = (x^2 + y^2)^{\frac{1}{2}}$ gives

$$r^2 - 2r\cos\theta = r.$$

Hence, $r = 0$ or $r = 1 + 2\cos\theta$.

The equation $r = 0$ simply shows that the pole $O$ lies on the curve. Because no restrictions were placed on $x$ or $y$, $\theta$ can take all values in an interval of length $2\pi$. The equation of the curve is therefore, as stated, $r = 1 + 2\cos\theta$, where $-\pi < \theta \le \pi$.

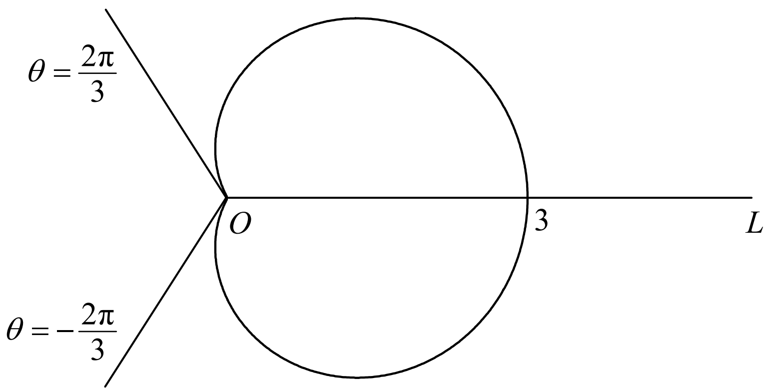
**(b)** When $r \to 0$, $\cos\theta \to -\dfrac{1}{2}$. Hence $\theta \to \pm\dfrac{2\pi}{3}$. The lines $\theta = \pm\dfrac{2\pi}{3}$ are therefore tangents to the curve at the pole.

**(c)** Because $r$ is a function of $\cos\theta$, the curve is symmetrical about the initial line. It is sufficient therefore to tabulate values in the interval $0 \le \theta \le \pi$.

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | 3 | 2.73 | 2 | 1 | 0 | –ve | –ve |

The table shows that there is no curve in the region $\dfrac{2\pi}{3} < \theta \le \pi$.

Plotting these points, and remembering that the curve is symmetrical about the initial line, the curve shown here is obtained.



**(d)** For all values of $\theta$ in the interval $0 \le \theta \le \frac{2\pi}{3}$, $r \ge 0$. Hence, the area enclosed by the curve is

$$A = 2\int_0^{\frac{2\pi}{3}} \tfrac{1}{2}(1+2\cos\theta)^2\,\mathrm{d}\theta$$
$$= \int_0^{\frac{2\pi}{3}} (1+4\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$$
$$= \int_0^{\frac{2\pi}{3}} \{1+4\cos\theta+2(1+\cos 2\theta)\}\,\mathrm{d}\theta$$
$$= \left[3\theta+4\sin\theta+\sin 2\theta\right]_0^{\frac{2\pi}{3}}$$
$$= 2\pi + \left(4\times\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}$$
$$= 2\pi + \frac{3}{2}\sqrt{3}.$$

**Exercise 2F**

1. (a) Sketch the curve with polar equation $r = \theta$, $\quad 0 \le \theta \le \pi$.
   (b) Find the area of the region bounded by the curve and the line $\theta = \pi$.

2. (a) Sketch the curve with polar equation $r = \mathrm{e}^{\frac{1}{4}\theta}$, $\quad 0 \le \theta \le \pi$.
   (b) Find the area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \pi$.

For the next three problems, you will need to recall some of the curves you obtained in Exercise 2D.

3. Find the area of the region enclosed by the curve with the polar equation

$$r = 2+\cos\theta, \qquad -\pi < \theta \le \pi.$$

4. Find the area enclosed by each of the loops of the curve with the polar equation

$$r = a\sin 2\theta, \text{ where } a > 0 \text{ and } -\pi < \theta \le \pi.$$

5. (a) Sketch, on the same diagram, the curve with the polar equation

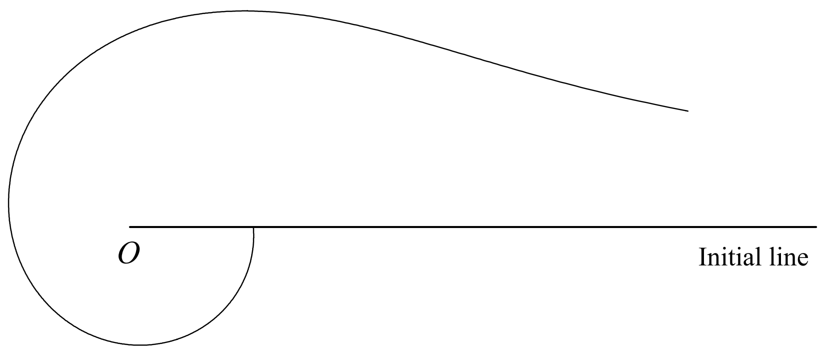
$$r = 1-\sin\theta, \qquad -\pi < \theta \le \pi$$

   and the circle $r = \frac{1}{2}$.
   (b) Find the polar coordinates of the points where the two curves intersect.
   (c) Find the total area of the region which lies inside both curves.

**Miscellaneous exercises 2**

1. The diagram shows a sketch of the curve whose polar equation is

$$r = \theta^{-\frac{1}{2}}, \qquad 0 < \theta \leq 2\pi.$$




Show that the area enclosed between the curve and the lines $\theta = \alpha$ and $\theta = 2\alpha$, where $0 < \alpha \leq \pi$, is independent of $\alpha$.

[AQA 1999]

2. A line $l$ and a curve $C$ have polar equations

$$r\sin\theta = 2, \qquad r = \frac{2}{1+\sin\theta}, \qquad 0 < \theta \leq \pi,$$

respectively.

(a) Sketch $l$ and $C$ on the same diagram.

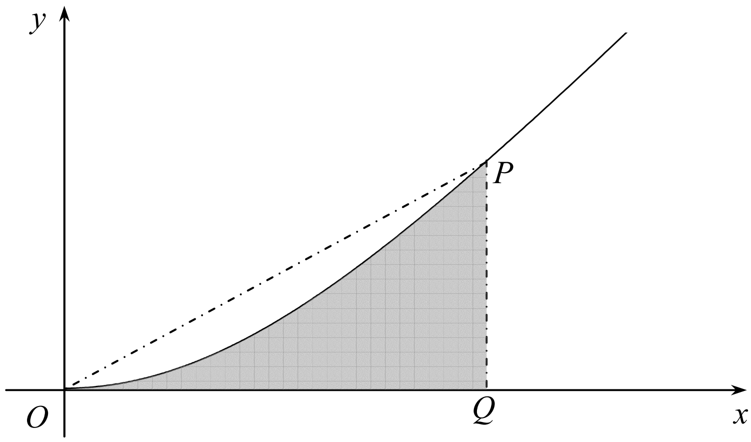
(b) The point $P$, with polar coordinates $(a, \phi)$, lies on $C$ and $O$ is the pole. The foot of the perpendicular from $P$ onto $l$ is $N$. Show that $OP = PN$.

3. In terms of polar coordinates $(r, \theta)$, the equation of a curve $C$ is

$$r = \tan 2\theta \quad , \quad 0 \le \theta < \frac{\pi}{4}.$$

(a) Write down expressions in terms of $\theta$ for the Cartesian coordinates $(x, y)$ of a general point on $C$.

(b)




The diagram shows a sketch of part of the curve $C$. The point $P$ lies on the curve and is such that $\angle POQ = \frac{\pi}{6}$, where $Q$ is the foot of the perpendicular from $P$ to the $x$-axis.

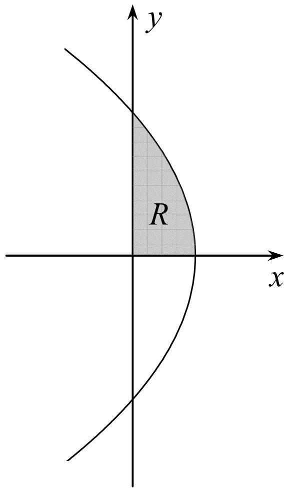
(i) Find the exact value of the area of the triangle $OPQ$.

(ii) Show that the area of the shaded region bounded by $OQ$, $PQ$ and the arc of the curve between $O$ and $P$ is

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8}.$$

[NEAB 1997]

4. The diagram shows a sketch of the curve $y^2 = 4(1-x)$.




(a) Show that the area of region $R$ bounded by the axes and the curve is $\frac{4}{3}$.

(b) (i) Show that the equation of the curve can be expressed as

$$x^2 + y^2 = (2-x)^2.$$

(ii) Hence, obtain the polar equation of the above curve in the form $r = \mathrm{f}(\theta)$.

(c) Hence, or otherwise, show that

$$\int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{(1+\cos\theta)^2} = \frac{2}{3}.$$

[AQA 2000]

5. The curve $C_1$ is given in polar coordinates, with origin $O$, by the equation

$$r = a(1+\cos\theta), \qquad -\pi < \theta \le \pi.$$

(a) Sketch the curve.

(b) A straight line through $O$ meets $C_1$ at the points $A$ and $B$, and $M$ is the mid point of $AB$. The line makes an angle $\phi$ with the initial line $\theta = 0$ and $\phi$ varies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

(i) Prove that $AB$ is of constant length.

(ii) Show that the locus of $M$ is the curve $C_2$ whose equation is

$$r = a\cos\theta \quad , \quad -\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi.$$

(iii) Sketch the curve $C_2$ on the same diagram as the curve $C_1$.

(c) Given that $S_1$ is the area of the region enclosed by $C_1$ and that $S_2$ is the area of the region enclosed by $C_2$, show that $S_1 = 6S_2$.

[JMB 1989]

6. The Cartesian equation of a curve $C$ is

$$\left(x^2+y^2\right)^2=2a^2xy,$$

where $a$ is a positive constant.

(a) Show that the equation of $C$ can be expressed in polar coordinates as

$$r=a\sqrt{\sin\theta}.$$

(b) (i) Write down the ranges of values of $\theta$ in the interval $-\pi<\theta\le\pi$ for which no part of $C$ exists, giving a reason for your answer.

(ii) Write down the polar coordinates of the points on $C$ which are furthest from the origin.

(iii) Sketch $C$.

(c) Find the area $A$ of that part of the interior of $C$ which lies in the region $0\le\theta\le\frac{1}{2}\pi.$

(d) The line $\theta=\beta,$ where $0<\beta<\frac{1}{4}\pi,$ divides $A$ into two parts which are in the ratio $1:3.$

Find the value of $\beta.$

[NEAB 1998]

# Chapter 3: Introduction to Differential Equations



This is the first of three chapters on differential equations. When you have completed it, you will:

- have been reminded of the basic concept of a differential equation;
- have been reminded of the method of separation of variables;
- have been reminded of the growth and decay equations;
- understand the terms order, linearity, families of solutions, general solutions, particular solutions, boundary conditions, end conditions and initial conditions;
- know how to solve first order linear differential equations using an integrating factor;
- know how to solve first order linear differential equations with constant coefficients by finding a complementary function and a particular integral;
- know how some first order non-linear differential equations can be solved by transforming them to linear form.

## 3.1 The concept of a differential equation: order and linearity

There are numerous applications of differential equations in modelling real world phenomena, especially in science and engineering. In this chapter, and those that follow, some of the simpler types of differential equations that occur will be introduced. Two distinct types of method for solving differential equations will be considered:

- *analytical methods*, in which exact solutions of explicit mathematical forms are found;
- *numerical methods*, which give approximate solutions to differential equations that cannot be solved using analytical methods.

You will already be familiar with the basic concept of a differential equation – it is one which involves the derivatives of a function. The function will usually be denoted by $y(x)$.
Particular examples are:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x^2 \qquad y\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0 \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin x.$$

When only the first order derivative, $\frac{\mathrm{d}y}{\mathrm{d}x}$, is involved (as in the first four examples above), the differential equation is said to be of **first order**. When the second order derivative, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, is involved (as in the last two examples), the differential equation is said to be of **second order**. Differential equations of order 3, 4, … are defined similarly.

A differential equation is said to be **linear** if it is linear in the dependent variable $y$ and the derivatives of $y$. The fourth example above contains contains two non-linear terms, $y\frac{\mathrm{d}y}{\mathrm{d}x}$ and $y^2$, and is therefore non-linear. All the other examples are linear.

Linearity can also be defined in another way: a differential equation is linear if the highest order derivative of the dependent variable $y$ can be expressed as a linear function of $y$ and the lower order derivatives. Hence, for a second order differential equation to be linear it must be possible to express $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{f}(x) + \mathrm{g}(x)y + \mathrm{h}(x)\frac{\mathrm{d}y}{\mathrm{d}x},$$

where f, g and h are functions of $x$ only.

**Exercise 3A**

1. Write down the order of each of these differential equations.

   (a) $x\dfrac{dy}{dx} + y = x^2$.  (b) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$.

   (c) $\dfrac{dy}{dx} = x^3 + y^3$.  (d) $x\left(\dfrac{dy}{dx}\right)^2 + y = 1$.

2. State which of the differential equations in Question 1 are linear.

## 3.2 Families of solutions, general solutions and particular solutions

Solving a differential equation is quite different to solving an algebraic equation. Finding the solution means finding the function $y(x)$ which satisfies the differential equation.

The differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x+1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$, listed in Section 3.1, are types that you should recognise from earlier studies and you will be familiar with the methods of solving them.

The first is solved by simply integrating each side with respect to $x$. This gives

$$y = x^2 + x + C,$$

where $C$ is an arbitrary constant.

The second equation can be solved by the method of separation of variables. The differential equation can be rewritten as

$$\int \frac{\mathrm{d}y}{y} = \int 2x \, \mathrm{d}x.$$

Performing the integrations gives

$$\ln|y| = x^2 + C,$$

where $C$ is an arbitrary constant. Hence,

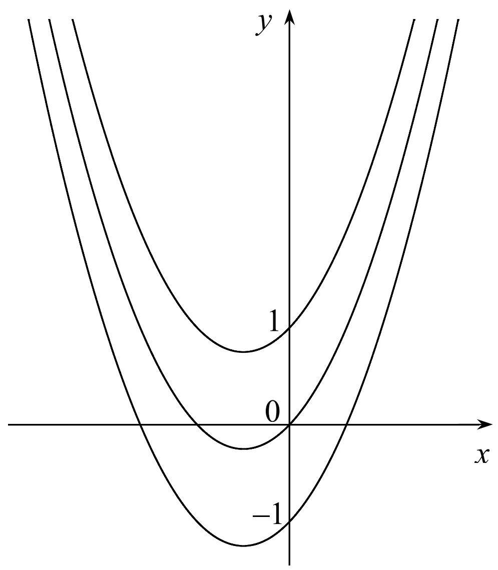
$$\begin{aligned} y &= \pm \mathrm{e}^{x^2+C} \\ &= \pm \mathrm{e}^{C} \mathrm{e}^{x^2} \\ &= A\mathrm{e}^{x^2}, \end{aligned}$$

where, for convenience, $\pm \mathrm{e}^{C}$ has been rewritten as the arbitrary constant $A$.

The set of all possible solutions of a differential equation is said to form a **family of solutions**. Particular members of the family of solutions of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x+1$ are

$$y = x^2 + x - 1,$$
$$y = x^2 + x,$$
$$y = x^2 + x + 1.$$

These are obtained by taking $C = -1, \quad 0$ and $1$, respectively, in the solution obtained above. The diagram alongside shows how the graphs of these three members of the family of solutions are related to each other – they differ only by a simple translation of 1 unit in the $y$-direction.



Solutions that involve arbitrary constants are called **general solutions** because they represent the whole family of possible solutions. A solution which satisfies the differential equation but contains no arbitrary constants is called a **particular solution**. General solutions of first order differential equations always contain exactly one arbitrary constant, as will be seen in the cases dealt with in this chapter.

For the differential equation $\frac{dy}{dx} = 2xy$, the general solution, as shown above, is $y = Ae^{x^2}$.

Examples of particular solutions are $y = e^{x^2}$ and $y = 2e^{x^2}$, obtained by taking $A = 1$ and $A = 2$, respectively.

In most of the applications of differential equations, a particular solution valid over some specified interval, such as $0 \le x \le 1$ or $x \ge 0$, is required. The required solution of a first order differential equation is often chosen in order to satisfy a given condition at an end point of an interval under consideration. For example, if the interval under consideration is $x \ge 0$, the given condition might be $y(0) = 2$.

It will be shown later that general solutions of second order linear differential equations contain two arbitrary constants and therefore two conditions need to be specified to identify a particular solution. Such conditions are called **boundary conditions** or **end conditions** or **initial conditions**. The term 'initial condition' is particularly appropriate in applications in which the independent variable is time $t$ and a solution valid for $t \ge 0$ is required. Then $t = 0$ marks the beginning of the period under consideration.

**Example 3.2.1**

The function $y(x)$ satisfies the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 0, \qquad x \geq 1.$$

(a) Find the general solution for $y(x)$.

(b) Hence find the particular solution satisfying the boundary condition $y(1) = \frac{1}{2}$.

**Solution**

**(a)** The differential equation can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^2}.$$

Separating the variables, this becomes

$$\int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{x^2},$$

giving
$$-\frac{1}{y} = -\frac{1}{x} + C,$$

where $C$ is an arbitrary constant.

Hence,
$$\frac{1}{y} = \frac{1 - Cx}{x}.$$

The general solution is therefore

$$y = \frac{x}{1 - Cx}.$$

**(b)** Applying the boundary condition, $y(1) = \frac{1}{2}$, gives

$$\frac{1}{2} = \frac{1}{1 - C}.$$

Hence, $C = -1$. The required particular solution is therefore

$$y = \frac{x}{1 + x}.$$

**Exercise 3B**

1. The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = ky,$ where $k$ is a constant, governs phenomena involving growth $(k > 0)$ or decay $(k < 0)$.
   Show that the general solution is $y = C\mathrm{e}^{kx},$ where $C$ is an arbitrary constant.

   **[You will find it useful to memorise this solution so that you can quote it when required.]**

2. (a) Obtain the general solution of the differential equation
   $$\frac{\mathrm{d}y}{\mathrm{d}x} + xy^2 = 0.$$
   (b) Find the particular solution which satisfies the condition $y(0) = 2.$

3. (a) Obtain an equation representing the family of solutions of the differential equation
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}, \qquad x > 0.$$
   (b) Find the equation of the member of this family whose graph passes through the point $(1, 0)$.
   (c) Sketch this graph.

4. (a) Solve the differential equation
   $$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \qquad 0 \le x \le 2,$$
   subject to the boundary condition $y(0) = 3.$
   (b) Verify that $y(2) \approx 0.406.$

5. A cyclist travelling on a level road stops pedalling and freewheels for 5 seconds. The distance travelled by the cyclist in $t$ seconds is $x$ metres. The relationship between $x$ and $t$ while the cyclist is freewheeling can be modelled by the differential equation
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{250}{(5+t)^2}.$$
   (a) Find the general solution of this differential equation.
   (b) (i) State the appropriate initial condition to be satisfied by $x(t)$.
   (ii) Find the particular solution satisfying this condition.
   (c) Deduce that the cyclist travels 25 metres while freewheeling.

## 3.3 Analytical solution of first order differential equations: integrating factors

The general first order linear differential equation may be expressed as $\frac{\mathrm{d}y}{\mathrm{d}x}+Py=Q$, where $P$ and $Q$ are functions of $x$. A simple example is $\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x}=x^2$, which was included in the list of differential equations in Section 3.1.

Except in the special cases when both $P$ and $Q$ are constants or when one of these two functions is zero, differential equations of this type cannot be expressed in variables separated form; the method of separation of variables is therefore not available. However, there are two other analytical methods of solution which can be used, the first of which is as follows.

If each side of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+Py=Q$$

is multiplied by $I$, where $I$ is a function of $x$, it becomes

$$I\frac{\mathrm{d}y}{\mathrm{d}x}+IPy=IQ.$$

The aim of this method is to choose $I$ so that

$$I\frac{\mathrm{d}y}{\mathrm{d}x}+IPy=\frac{\mathrm{d}}{\mathrm{d}x}(Iy).$$

Since $$\frac{\mathrm{d}}{\mathrm{d}x}(Iy)=I\frac{\mathrm{d}y}{\mathrm{d}x}+y\frac{\mathrm{d}I}{\mathrm{d}x},$$

$I$ must be chosen so that $\frac{\mathrm{d}I}{\mathrm{d}x}=IP$. Hence

$$\int\frac{\mathrm{d}I}{I}=\int P\,\mathrm{d}x,$$

which gives $\ln|I|=\int P\,\mathrm{d}x$ and therefore $I=\pm\mathrm{e}^{\int P\,\mathrm{d}x}$. It is usual to choose the positive sign (either will do) and hence

$$\boxed{I=\mathrm{e}^{\int P\,\mathrm{d}x}}$$

The function $I$ is obtainable provided that $P(x)$ can be integrated.

Assuming that $I$ has been found, the differential equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(Iy)=IQ.$$

Hence $$Iy=\int IQ\,\mathrm{d}x,$$

giving the solution

$$y(x)=\frac{1}{I}\int IQ\,\mathrm{d}x.$$

The function $I(x)$ is called an **integrating factor** because knowledge of this enables the equation to be solved.

**Example 3.3.1**

(a) Find the integrating factor of the differential equation $\frac{dy}{dx} - \frac{y}{x} = x^2$, where $x > 0$.

(b) Hence find

(i) the general solution,

(ii) the particular solution satisfying the condition $y(1) = 0$.

**Solution**

**(a)** In this case $P = -\frac{1}{x}$. Hence, using the formula obtained above, the integrating factor is

$$\begin{aligned} I &= e^{\int -\frac{1}{x}\,dx} \\ &= e^{-\ln|x|} \\ &= e^{\ln|x|^{-1}} \\ &= \frac{1}{|x|} \\ &= \frac{1}{x}, \qquad \text{since } x > 0. \end{aligned}$$

**(b)(i)** Multiplying each side of the differential equation by the integrating factor gives

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x,$$

which can be rewritten as

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x.$$

Note that the left-hand side of this equation is $\frac{d}{dx}(Iy)$, as it should be. The above equation can be integrated at once giving

$$\frac{y}{x} = \frac{1}{2}x^2 + C,$$

where $C$ is an arbitrary constant. The general solution of the differential equation is therefore

$$y = x\left(\frac{1}{2}x^2 + C\right).$$

**(ii)** Applying the condition $y(1) = 0$ gives

$$0 = \frac{1}{2} + C.$$

Hence $C = -\frac{1}{2}$ and $y = \frac{1}{2}x\left(x^2 - 1\right)$.

**There are some points to note about the application of this method.**

- If the differential equation considered in Example 3.3.1 had been given in the form $x\dfrac{dy}{dx} - y = x^3$, it would have been necessary, as a first step, to divide through by $x$ to express it in the standard form $\dfrac{dy}{dx} + Py = Q$. It is essential to do this before attempting to find the integrating factor.
- When the integrating factor $I$ has been found, it is the **standard form** that is multiplied through by $I$, **not the original form**.
- When finding $I$, it will often be necessary to use the fact that $e^{\ln f(x)} = f(x)$, as in Example 3.3.1.
- When the differential equation (in standard form) is multiplied through by the integrating factor $I$, the left-hand side should then be expressible as $\dfrac{d}{dx}(Iy)$.

**Example 3.3.2**

Find the solution of the differential equation

$$(\cos x)\frac{dy}{dx} + (\sin x)\,y = 1, \qquad 0 \le x < \frac{\pi}{2},$$

satisfying the boundary condition $y(0) = 1$.

**Solution**

In standard form, the differential equation is

$$\frac{dy}{dx} + (\tan x)\,y = \sec x.$$

The integrating factor is

$$I = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = |\sec x|.$$

Since $\sec x > 0$ when $0 \le x < \frac{\pi}{2}$, $\quad I = \sec x.$

Multiplying the differential equation (in standard form) throughout by $\sec x$ gives

$$(\sec x)\frac{dy}{dx} + (\sec x\,\tan x)\,y = \sec^2 x,$$

which can be written as $\dfrac{d}{dx}(y\sec x) = \sec^2 x.$

Hence, $\qquad y\sec x = \tan x + C,$

where $C$ is an arbitrary constant. In terms of $\sin x$ and $\cos x$, this can be expressed as

$$y = \sin x + C\cos x,$$

and this is the general solution for $y$.

Applying the boundary condition, $y(0) = 1$, gives $1 = C$. Hence

$$y = \sin x + \cos x.$$

**Exercise 3C**

1. (a) Find the integrating factor of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2y}{x} = 4x, \qquad x > 0.$

   (b) Hence find the general solution.

2. (a) Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^{2x}.$

   (b) Hence find the particular solution satisfying the condition $y(0) = 0.$

3. Solve the differential equation
$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + xy = 1,$$
   where $x > 0,$ subject to the condition that $y(1) = 0.$

4. Find the general solution of the differential equation
$$x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^3,$$
   where $x > 0.$

5. (a) Show that $\sin x$ is an integrating factor of the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} + (\cot x)\, y = x, \qquad 0 < x < \pi.$$

   (b) Hence find

   (i) the general solution,

   (ii) the particular solution satisfying the condition $y\left(\frac{\pi}{2}\right) = 0.$

6. Find the solution of the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} - (\tan x)\, y = 1, \qquad |x| < \frac{\pi}{2},$$
   which is such that $y(0) = 1.$

7. (a) Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x\,\mathrm{e}^{-2x}, \qquad x > 0.$

   (b) Hence find the particular solution which is such that $y$ remains finite as $x \to \infty.$

8. A curve is such that at any point $(x, y)$ on it, its gradient is $y - x.$ The curve passes through the point $(0, 2).$ Find the equation of the curve.

## 3.4 Complementary functions and particular integrals

Linear differential equations in which the coefficients of $y$ and its derivatives are constants occur frequently. The general first order differential equation of this type has the form

$$a\frac{\mathrm{d}y}{\mathrm{d}x}+by=\mathrm{f}\left(x\right),$$

where $a$ and $b$ are constants. Such equations can be solved by finding an integrating factor but there is an alternative method which is sometimes easier to apply. The alternative method also has the advantage that it can be used to solve higher order linear differential equations with constant coefficients. These will be dealt with in Chapter 5.

To understand the essentials of this alternative method, consider the particular case

$$\frac{\mathrm{d}y}{\mathrm{d}x}+2y=\mathrm{e}^{x}.$$

The first step is to replace the right-hand side by zero giving

$$\frac{\mathrm{d}y}{\mathrm{d}x}+2y=0.$$

This is called the **reduced equation**. Rewriting it as

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2y,$$

it can be seen that it is a standard decay equation for which the general solution is

$$y=C\mathrm{e}^{-2x},$$

where $C$ is an arbitrary constant. (See Exercise 3B, Question 1.)

The next step is to try to find, by trial or inspection, a particular solution of the complete equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+2y=\mathrm{e}^{x}.$$

**Any** function that satisfies this equation is sufficient. In this case, the form of the right-hand side suggests that $y=A\mathrm{e}^{x}$ is tried, where $A$ is a constant to be found. Substituting this into the differential equation gives

$$A\mathrm{e}^{x}+2A\mathrm{e}^{x}=\mathrm{e}^{x}.$$

The terms in $\mathrm{e}^{x}$ cancel giving $A=\frac{1}{3}$. Hence $y=\frac{1}{3}\mathrm{e}^{x}$ is a particular solution.

The final step is to add the general solution of the reduced equation and the particular solution of the complete equation. This gives

$$y=C\mathrm{e}^{-2x}+\frac{1}{3}\mathrm{e}^{x}.$$

It is easy to verify that this function satisfies the complete equation (check this). As it also contains an arbitrary constant it is the required general solution.

The general solution of the reduced equation is called the **complementary function**, and the particular solution of the complete equation is called a **particular integral**. The sum of the two gives the required **general solution** of the complete equation. These terms are often abbreviated to CF, PI and GS, respectively.

The steps in the procedure for solving differential equations of the form

$$a\frac{\mathrm{d}y}{\mathrm{d}x}+by=\mathrm{f}\left(x\right)$$

can be summarised as follows.

- Find the GS of the reduced equation $a\frac{\mathrm{d}y}{\mathrm{d}x}+by=0$. This solution is the CF, $y_\mathrm{C}$.
- Find a particular solution of the complete equation $a\frac{\mathrm{d}y}{\mathrm{d}x}+by=\mathrm{f}\left(x\right)$.

  This function is a PI, $y_\mathrm{P}$.
- The GS of the complete equation is then $y=y_\mathrm{C}+y_\mathrm{P}$.

In practice, finding a CF is straightforward, as it was in the example above. However, finding a PI is generally much more difficult and only when $\mathrm{f}\left(x\right)$ takes certain simple forms is it likely to be possible – this is the main drawback of this method. Fortunately, in applications $\mathrm{f}\left(x\right)$ is often of a simple exponential, trigonometric or polynomial form and in these cases a PI can be found by using an appropriate trial function, as follows.

- If $\mathrm{f}\left(x\right)$ is of the form $c\mathrm{e}^{kx}$, where *c* and *k* are constants, try a PI of the form $y=a\mathrm{e}^{kx}$, where *a* is a constant to be found. The example used at the beginning of this section (to explain the method) illustrates this case.

  This fails when the CF has the same exponential form as the right-hand side of the differential equation. However, it will then be found that the trial function $y=ax\mathrm{e}^{kx}$ will provide a PI. Example 3.4.3 covers this more difficult case.
- If $\mathrm{f}\left(x\right)$ is of the form $c\cos kx$ or $c\sin kx$, try a PI of the form $y=a\cos kx+b\sin kx$, where *a* and *b* are constants to be found.
- If $\mathrm{f}\left(x\right)$ is a polynomial of degree *n*, try a PI of the form $y=ax^n+bx^{n-1}+\cdots$, where *a*, *b*, … are constants to be found.

Only the above cases and simple combinations of them need to be considered for the purposes of this course.

**Example 3.4.1**

Find a PI of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sin x$.

**Solution**

The appropriate trial function is

$$y = a\cos x + b\sin x.$$

Substituting this into the differential equation gives

$$-a\sin x + b\cos x + a\cos x + b\sin x = \sin x,$$

i.e.

$$(b+a)\cos x + (b-a)\sin x = \sin x.$$

The constants $a$ and $b$ must be chosen so that $b + a = 0$ and $b - a = 1$. This gives $a = -\frac{1}{2}$ and $b = \frac{1}{2}$. Hence, the required PI is

$$y = -\frac{1}{2}\cos x + \frac{1}{2}\sin x.$$

**Example 3.4.2**

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+2y=2x^2+3,$ given that $y(0)=5.$

**Solution**

The reduced equation is $\frac{\mathrm{d}y}{\mathrm{d}x}+2y=0$ for which the GS is $y=C\mathrm{e}^{-2x},$ where $C$ is an arbitrary constant. This is the CF.

The right-hand side of the complete equation is a polynomial of degree 2, so the appropriate trial function for a PI is

$$y=ax^2+bx+c.$$

Differentiating this gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2ax+b.$$

Substituting these expressions for $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ into the complete equation gives

$$2ax+b+2ax^2+2bx+2c=2x^2+3.$$

This may be expressed more conveniently as

$$2ax^2+2(a+b)x+b+2c=2x^2+3.$$

Hence, comparing coefficients,

$$\begin{aligned}2a&=2,\\2(a+b)&=0,\\b+2c&=3.\end{aligned}$$

Thus $a=1,\quad b=-1$ and $c=2.$

The PI is therefore $\quad y=x^2-x+2.$

Adding the CF and the PI, the GS of the complete equation is therefore

$$y=C\mathrm{e}^{-2x}+x^2-x+2.$$

Applying the condition $y(0)=5$ gives $5=C+2$ and hence $C=3.$ The required solution is therefore

$$y=3\mathrm{e}^{-2x}+x^2-x+2.$$

**Note** that a common mistake is to apply the end condition $\left[y(0)=5 \text{ in the example above}\right]$ to the CF (i.e. to the GS of the reduced equation). It is essential to apply the end condition to the GS of the **complete equation**.

**Example 3.4.3**

Find the CF and a PI of the differential equation $2\frac{\mathrm{d}y}{\mathrm{d}x} - y = 3\mathrm{e}^{\frac{1}{2}x}$. Hence write down the GS.

**Solution**

The reduced equation is $2\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$,

i.e. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}y.$$

The GS of this is $y = C\mathrm{e}^{\frac{1}{2}x}$ and this is the required CF of the complete equation.

To find a PI, note first that the CF has the same exponential form as the right-hand side of the given differential equation. The appropriate trial function for the PI is therefore not $a\mathrm{e}^{\frac{1}{2}x}$ but

$$y = ax\mathrm{e}^{\frac{1}{2}x}.$$

Differentiating this using the product rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}ax\mathrm{e}^{\frac{1}{2}x} + a\mathrm{e}^{\frac{1}{2}x}.$$

Substituting $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ into the complete differential equation,

$$ax\mathrm{e}^{\frac{1}{2}x} + 2a\mathrm{e}^{\frac{1}{2}x} - ax\mathrm{e}^{\frac{1}{2}x} = 3\mathrm{e}^{\frac{1}{2}x}.$$

The terms in $ax\mathrm{e}^{\frac{1}{2}x}$ cancel and it can be seen that $a = \frac{3}{2}$. Hence the PI is

$$y = \frac{3}{2}x\mathrm{e}^{\frac{1}{2}x}.$$

The GS of the complete equation is therefore

$$y = C\mathrm{e}^{\frac{1}{2}x} + \frac{3}{2}x\mathrm{e}^{\frac{1}{2}x}.$$

**Note**: It is interesting to see why the trial function $y = a\mathrm{e}^{\frac{1}{2}x}$ will not provide a PI in this case. Substituting this into the differential equation gives

$$2\left(\frac{1}{2}a\mathrm{e}^{\frac{1}{2}x}\right) - a\mathrm{e}^{\frac{1}{2}x} = 3\mathrm{e}^{\frac{1}{2}x}.$$

However, the left-hand side is zero so there is no value of $a$ for which the differential equation can be satisfied. Of course, the left-hand side must be zero because $a\mathrm{e}^{\frac{1}{2}x}$ is a solution of the reduced equation!

**Exercise 3D**

1. Find a particular integral of each of these differential equations.

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3y = 9x^2 + 1$ (b) $2\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = x - 3$

   (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3y = \sin x + 2\cos x$ (d) $2\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = 2\cos x$

   (e) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3y = 3\mathrm{e}^{3x}$ (f) $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y = 3\mathrm{e}^{3x}$

   (g) $2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x + \mathrm{e}^{-x}$

2. (a) Find the complementary function and a particular integral of the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 6.$$
   (b) Hence obtain the solution satisfying the condition $y(1) = 0$.

3. (a) Find the complementary function and a particular integral of the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \mathrm{e}^{2x}.$$
   (b) Hence obtain the solution satisfying the condition $y(0) = 2$.

4. Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sin 2x,$$
   subject to the condition $y(0) = 1$.

## 3.5 Transformations of non-linear differential equations to linear form

It is sometimes possible to transform a first order non-linear differential equation into one of first order linear form by means of a suitable substitution. If the solution of the transformed equation can be found, the solution of the non-linear equation can be deduced from it. The example which follows shows how this technique may be used.

**Example 3.5.1**

The function $y(x)$ satisfies the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2y}{x} + 2x\sqrt{y}, \quad x \geq 1.$

(a) Show that the substitution $y = u^2$ transforms the differential equation into $\dfrac{\mathrm{d}u}{\mathrm{d}x} - \dfrac{u}{x} = x.$

(b) Find the general solution for $u$.

(c) Hence obtain the solution for $y$ satisfying the boundary condition $y(1) = 0.$

**Solution**

**(a)** Differentiating each side of $y = u^2$ with respect to $x$ gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2u\dfrac{\mathrm{d}u}{\mathrm{d}x}.$
Hence the differential equation transforms to

$$2u\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2u^2}{x} + 2xu,$$

which can be written as $\dfrac{\mathrm{d}u}{\mathrm{d}x} - \dfrac{u}{x} = x.$

**(b)** The integrating factor for the differential equation above is

$$\mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln|x|} = \frac{1}{|x|} = \frac{1}{x} \qquad (x \geq 1 \Rightarrow |x| = x).$$

Multiplying the differential equation throughout by this factor gives

$$\frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{x^2} = 1.$$

Hence, $\quad \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{x}\right) = 1,$

which gives $\quad \dfrac{u}{x} = x + C,$

where $C$ is an arbitrary constant. The general solution for $u$ is therefore

$$u = x(x + C).$$

**(c)** Since $y = u^2$, the general solution for $y$ is

$$y = x^2(x + C)^2.$$

Applying the boundary condition, $y(1) = 0,$ gives $0 = (1 + C)^2$. Hence $C = -1$ and therefore $\quad y = x^2(x - 1)^2.$

**Exercise 3E**

1. (a) Show that the substitution $y = \frac{1}{u}$ transforms the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = -2y^2,$$

where $x > 0$, into

$$\frac{du}{dx} + \frac{u}{x} = 2.$$

(b) Obtain the general solution for $u$ and hence the general solution for $y$.

2. (a) Show that the substitution $y - x = u$ transforms the differential equation

$$\frac{dy}{dx} + e^{y-x} = 1$$

into

$$\frac{du}{dx} + e^{u} = 0.$$

(b) Find the general solution for $u$.

(c) Hence obtain the solution for $y$ satisfying the condition $y(0) = 0$.

(d) State the range of values of $x$ for which the solution in part (c) is valid.

3. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} + \frac{e^y}{x^2} = \frac{2}{x}, \qquad x \geq 1.$$

(a) Show that the substitution $y = -\ln u$ transforms the differential equation into

$$\frac{du}{dx} + \frac{2u}{x} = \frac{1}{x^2}.$$

(b) Obtain the general solution for $u$.

(c) Hence obtain the solution for $y$ satisfying the boundary condition $y(1) = 0$.

**Miscellaneous exercises 3**

1. (a) Show that the integrating factor of the differential equation

$$\frac{dy}{dx} + (\tan x)y = 2x\cos x, \qquad 0 \le x < \frac{\pi}{2},$$

is $\dfrac{1}{\cos x}$.

(b) Hence obtain the solution for $y$ satisfying the boundary condition that $y(0) = 3$.

[AQA 1999]

2. (a) For the differential equation $\dfrac{dy}{dx} + y = \sin x$, find

(i) the complementary function,

(ii) a particular integral.

(b) Hence, or otherwise, solve the equation given that $y = 1$ when $x = 0$.

[JMB 1991]

3. A curve $C$ in the $x$–$y$ plane passes through the point $(1, 0)$. At any point $(x, y)$ on $C$,

$$\frac{dy}{dx} + y = e^{-x}.$$

(a) Find the general solution of this differential equation.

(b) (i) Hence find the equation of $C$, giving your answer in the form $y = f(x)$.

(ii) Write down the equation of the asymptote of $C$.

[NEAB 1998]

4. (a) Show that the integrating factor of the differential equation

$$\frac{dy}{dx} - \frac{1}{x-1}y = x, \qquad x > 1,$$

is $\dfrac{1}{x-1}$.

(b) Find the solution of this differential equation, given that $y(2) = 1$. Express your answer in the form $y = f(x)$.

[AQA 2001]

5. The differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=8x^3$ has a particular integral of the form

$$ax^3+bx^2+cx+d,$$

where $a$, $b$, $c$ and $d$ are constants.

(a) Find the values of $a$, $b$, $c$ and $d$.

(b) Find the complementary function.

(c) Obtain the solution of the differential equation which satisfies the condition $y(0)=0$.

6. A curve in the region $x>0$ of the $x$–$y$ plane is such that the tangent at every point $(x, y)$ on it intersects the $y$-axis at $(0, x)$.

(a) Show that $x\dfrac{\mathrm{d}y}{\mathrm{d}x}=y-x$.

(b) Hence find the equation of the curve which possesses this property and which passes through the point $(1,-2)$.

(c) Show that this curve approaches the origin as $x\to 0$.

[NEAB 1994]

7. (a) Show that the substitution $y=\dfrac{1}{u}$ transforms the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+y=y^2\ln x, \qquad x>0,$$

into the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{u}{x}=-\frac{\ln x}{x}.$$

(b) Find the general solution of the equation for $u$.

(c) Hence find $y$ in terms of $x$, given that $y=\dfrac{1}{2}$ when $x=1$.

[JMB 1988]

8. (a) Show that the substitution $y=u^{-\frac{1}{3}}$ transforms the differential equation

$$y-x\frac{\mathrm{d}y}{\mathrm{d}x}=2x^3y^4,$$

where $x>0$, into

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{3}{x}u=6x^2.$$

(b) Use the above result to find the general solution of the differential equation for $y$.

[JMB 1990]

# Chapter 4: Numerical Methods for the Solution of First Order Differential Equations



This chapter gives an introduction to numerical methods for solving first order differential equations. When you have completed it, you will:

- be familiar with the standard notation used;
- be familiar with the methods which use Euler's formula, the mid-point formula and Euler's improved formula;
- know how the above formulae can be derived both geometrically and analytically;
- be aware of the principal sources of error in the methods described;
- know how the accuracy of a numerical solution can be estimated.

## 4.1 Introduction

Although many of the differential equations which result from modelling real-world problems can be solved analytically, there are many others which cannot. In general, when the modelling leads to a **linear** differential equation, the prospects of obtaining an exact mathematical solution are good. However, non-linear differential equations present much greater difficulty and exact solutions can seldom be obtained. There is a need, therefore, for numerical methods that can provide approximate solutions to problems which would otherwise be intractable. The advent of powerful computers capable of performing calculations at very high speed has led to a rapid development in this area and there are now many numerical methods available.

This chapter is concerned with numerical methods for solving differential equations of the form

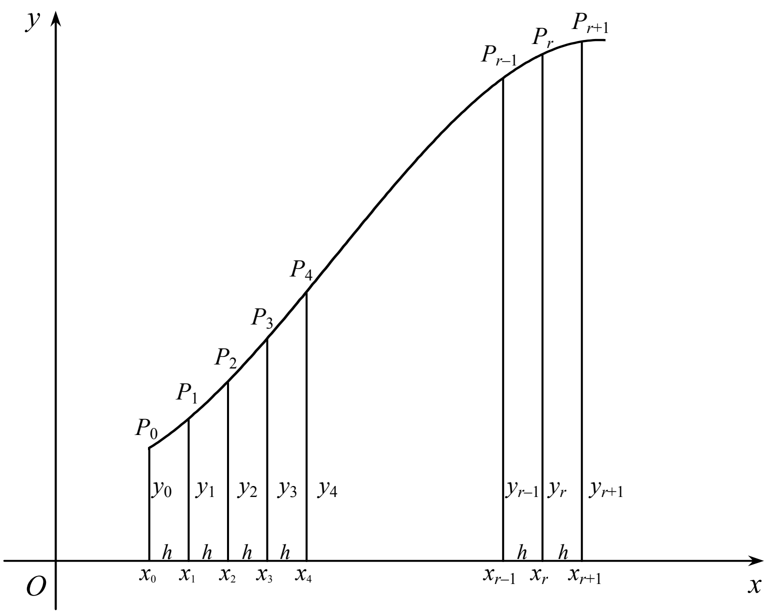
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y),$$

subject to boundary conditions of the form

$$y(x_0) = y_0.$$

In each case, the solution $y(x)$, valid for $x \geq x_0$, is required.

Suppose that the graph of the exact solution is as shown in the diagram below.




Let $x_0, x_1, x_2, \cdots$ be points on the $x$-axis which are equally spaced at a distance $h$ apart. Denote the values of $y(x_0), y(x_1), y(x_2), \cdots$ by $y_1, y_2, y_3, \cdots$ and let $P_0$, $P_1$, $P_2$, … be the points on the curve with coordinates $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots$, respectively. In general therefore, $y(x_r) = y_r$, where $x_r = x_0 + rh$, and $P_r$ is the point with coordinates $(x_r, y_r)$.

The basic idea of all the numerical methods is to use a step-by-step procedure to obtain **approximations** to the values of $y_1, y_2, y_3, \cdots$ successively. The interval, $h$, between successive $x$-values is called the **step length**. Some of the methods of obtaining these approximations are suggested by geometrical considerations of the graph. Three such methods will be described in the sections that follow. Later in the chapter, consideration will be given to the accuracy of the values obtained.

It is important to become thoroughly familiar with the standard notation introduced above. Exercise 4A will help with this.

**Exercise 4A**

Suppose that the differential equation to be solved is

$$\frac{dy}{dx} = x^3 + y^3, \qquad x \ge 1,$$

and that the boundary condition is $y(1) = 2$.

Suppose also that the values of $y(1.1), y(1.2), \cdots, y(2.0)$ are required. In this case:

(a) state the values of $x_0, y_0, h, x_1$ and $x_2$;

(b) state which of the values of $y$ that are required correspond to $y_1$ and $y_{10}$;

(c) evaluate $f(x_0, y_0)$.

## 4.2 Euler's formula

Consider again the diagram in Section 4.1. One simple way of obtaining an approximation to the value of $y_1$ is to assume that the part of the curve between $P_0$ and $P_1$ is a straight line segment with gradient equal to the gradient of the curve at $P_0$. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}\left(x, y\right)$, the gradient of the curve at $P_0$ is $\mathrm{f}\left(x_0, y_0\right)$. Hence, with this approximation,

$$\frac{y_1 - y_0}{h} = \mathrm{f}\left(x_0, y_0\right),$$

giving

$$y_1 = y_0 + h\,\mathrm{f}\left(x_0, y_0\right).$$

Using this approximation to obtain the value of $y$ at $P_1$, the process can be repeated, assuming that the part of the curve between $P_1$ and $P_2$ is a straight line segment with gradient equal to the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $P_1$. This gives

$$y_2 = y_1 + h\,\mathrm{f}\left(x_1, y_1\right).$$

Continuing in this way gives, in general,

$$y_{r+1} = y_r + h\,\mathrm{f}\left(x_r, y_r\right), \qquad r = 0, 1, 2, \cdots.$$

This is **Euler's formula**. Successive calculation of values of $y$ using this formula is known as **Euler's method**. It is clear from the nature of the linear approximation on which this method is based that the step length $h$ needs to be fairly small to achieve reasonable accuracy.

**Example 4.2.1**

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x+y)$$

and the condition $y(1) = 0$.

Use Euler's formula to estimate the values of $y(1.1)$, $y(1.2)$ and $y(1.3)$, giving the answers to four decimal places.

**Solution**

In this case, $x_0 = 1$, $y_0 = 0$, $h = 0.1$ and $\mathrm{f}(x, y) = \sin(x+y)$.

Also $y(1.1) = y_1$, $y(1.2) = y_2$ and $y(1.3) = y_3$.

Using Euler's formula:

$$\begin{aligned} y_1 &= y_0 + h\sin(x_0 + y_0) \\ &= 0.1\sin 1 \\ &\approx 0.084147 \\ &= 0.0841 \text{ to four d.p.} \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h\sin(x_1 + y_1) \\ &= 0.084147 + 0.1\sin(1.1 + 0.084147) \\ &\approx 0.176765 \\ &= 0.1768 \text{ to four d.p.} \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h\sin(x_2 + y_2) \\ &= 0.176765 + 0.1\sin(1.2 + 0.176765) \\ &\approx 0.274889 \\ &= 0.2749 \text{ to four d.p.} \end{aligned}$$

Note that at each stage after the first in the above calculations, the previous value of $y$ to six decimal places is used instead of the four-decimal place rounded answer. If this is not done, accuracy to four decimal places is not guaranteed. For example, the use of $y_1 = 0.0841$ would have given $y_2 = 0.1767$. It is recommended that working is always carried out to two decimal places more than the number of decimal places required in the final answer.

**Exercise 4B**

1. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln(x+y)$$

and the condition $y(1)=1$.

Use Euler's formula with a step length of 0.1 to obtain approximations to the values of $y(1.1)$, $y(1.2)$, $y(1.3)$, $y(1.4)$ and $y(1.5)$. Give your answers to three decimal places.

2. The function $y(x)$ satisfies the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = x+y$$

and the condition $y(0)=1$.

Use Euler's method with a step length of 0.25 to obtain an estimate of the value of $y(1)$. Give your answer to three decimal places.

3. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x,y)$$

where $$\mathrm{f}(x,y) = \left(\mathrm{e}^x + \mathrm{e}^y\right)^2.$$

The value of $y(0)$ is zero.

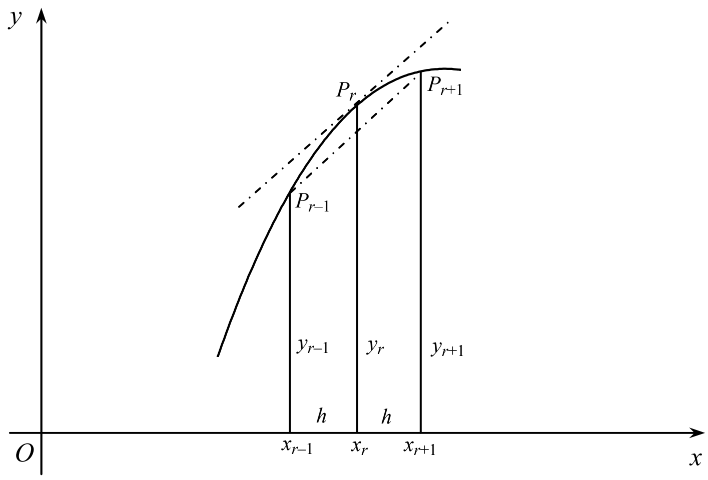
(a) Use of the Euler formula gave $y_1 = 0.05$. Determine the value that was used for the step length $h$.

(b) Show that, with this value for $h$, the Euler formula gives $y_2 = 0.103$ to three decimal places.

(c) Calculate the value of $y_3$ to three decimal places.

## 4.3 The mid-point formula

The diagram below shows three points, $P_{r-1}$, $P_r$ and $P_{r+1}$, on part of the curve representing the solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$.




Provided that the three points are reasonably close together, the gradient of the line segment joining $P_{r-1}$ and $P_{r+1}$ will be approximately the same as the gradient of the tangent to the curve at $P_r$. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x_r, y_r)$ at $P_r$, this approximation gives

$$\frac{y_{r+1} - y_{r-1}}{2h} = \mathrm{f}(x_r, y_r).$$

Hence,
$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r).$$

This is the **mid-point formula**.

As with Euler's formula, the mid-point formula can be used to calculate values of $y$ successively, but at each stage the previous **two** values of $y$ are required. In particular, putting $r = 1$ gives

$$y_2 = y_0 + 2h\,\mathrm{f}(x_1, y_1).$$

Since only the value of $y_0$ is known initially, it is necessary to calculate $y_1$ by some other method before the application of the mid-point formula can begin. Euler's formula may be used for this purpose.

It will be shown later that, for a given step length $h$, the mid-point formula is more accurate than Euler's formula.

**Example 4.3.1**

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \sqrt{xy}$$

and the condition $y(0) = 1$.

Use the mid-point formula with a step length of 0.25 to obtain an approximate value for $y(1)$ to three decimal places. Take $y(0.25)$ to be the value given by Euler's formula.

**Solution**

In this case, $x_0 = 0,\ y_0 = 1,\ h = 0.25$ and $\mathrm{f}(x, y) = 1 + \sqrt{xy}$.

Using Euler's formula,

$$\begin{aligned} y(0.25) = y_1 = y_0 + h\left(1 + \sqrt{x_0 y_0}\right) \\ &= 1 + (0.25 \times 1) \\ &= 1.25. \end{aligned}$$

The mid-point formula gives:

$$\begin{aligned} y(0.5) = y_2 &= y_0 + 2h\left(1 + \sqrt{x_1 y_1}\right) \\ &= 1 + 0.5\left(1 + \sqrt{0.25 \times 1.25}\right) \\ &\approx 1.77951 \end{aligned}$$

$$\begin{aligned} y(0.75) = y_3 &= y_1 + 2h\left(1 + \sqrt{x_2 y_2}\right) \\ &= 1.25 + 0.5\left(1 + \sqrt{0.5 \times 1.77951}\right) \\ &\approx 2.22163 \end{aligned}$$

$$\begin{aligned} y(1) = y_4 &= y_2 + 2h\left(1 + \sqrt{x_3 y_3}\right) \\ &= 1.77951 + 0.5\left(1 + \sqrt{0.75 \times 2.22163}\right) \\ &\approx 2.92492 \\ &= 2.925 \text{ to three decimal places.} \end{aligned}$$

**Exercise 4C**

1. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^3 + y^3$$

and the condition $y(1) = 1$.

(a) Using a step length of 0.1 in the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r),$$

obtain estimates of the values of $y$ when $x = 1.1$ and $x = 1.2$.

(b) Use the mid-point formula,

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r),$$

with $h = 0.1$ and your value of $y_1$ obtained in part (a), to calculate an improved estimate of the value of $y$ when $x = 1.2$.

2. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \cos xy$$

and the condition $y(1) = 2$.

(a) Verify that the Euler formula with a step length of 0.25 gives $y(1.25) = 1.89596$ to five decimal places.

(b) Use the mid-point formula with a step length of 0.25 to obtain an estimate of the value of $y(2)$. Give your answer to three decimal places.

3. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \left(1 + x^2 + y^2\right)^{\frac{1}{2}}$$

and the condition $y(0) = 0$.

(a) Given that in this case $y_{-1} = -y_1$, show that the mid-point formula gives $y_1 = h$, where $h$ is the step length.

(b) Use the mid-point formula with $h = 0.1$ to obtain an approximation to the value of $y(0.5)$. Give your answer to three decimal places.

## 4.4 The improved Euler formula

The Euler formula derived in Section 4.2 is based on the assumption that the gradient of the line segment joining any two successive points, $P_r$ and $P_{r+1}$, on the graph of $y(x)$ is equal to the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $P_r$. Another formula, which is considerably more accurate than Euler's, can be obtained by assuming that the gradient of $P_r P_{r+1}$ is equal to the average of the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $P_r$ and $P_{r+1}$. With this assumption,

$$\frac{y_{r+1}-y_r}{h} = \frac{1}{2}\left[\mathrm{f}\left(x_r, y_r\right) + \mathrm{f}\left(x_{r+1}, y_{r+1}\right)\right].$$

Hence,
$$y_{r+1} = y_r + \frac{h}{2}\left[\mathrm{f}\left(x_r, y_r\right) + \mathrm{f}\left(x_{r+1}, y_{r+1}\right)\right].$$

This formula cannot be used directly to calculate $y_{r+1}$ because $y_{r+1}$ is needed in order to evaluate $\mathrm{f}\left(x_{r+1}, y_{r+1}\right)$ on the right-hand side. To overcome this problem, $y_{r+1}$ on the right-hand side is replaced by a first estimate, denoted by $y^*_{r+1}$, giving

$$y_{r+1} = y_r + \frac{h}{2}\left[\mathrm{f}\left(x_r, y_r\right) + \mathrm{f}\left(x_{r+1}, y^*_{r+1}\right)\right].$$

There are a number of ways of obtaining a value for $y^*_{r+1}$. The simplest is to use Euler's formula, giving

$$y^*_{r+1} = y_r + h\,\mathrm{f}\left(x_r, y_r\right).$$

Using this expression together with the previous one for $y_{r+1}$ constitutes the **improved Euler method**. It is sometimes called a predictor–corrector method because Euler's formula gives an initial prediction of $y_{r+1}$ and this is followed by the calculation of a corrected value. Note, however, that 'corrected' here should not be interpreted literally – the new value of $y_{r+1}$ will almost invariably be more accurate than the first estimate, but as the method is based on approximations there will still be some error involved.

**Example 4.4.1**

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln(x+y), \qquad x \geq 1.$$

The boundary condition is $y(1)=1$. Use the improved Euler method with a step length of 0.25 to estimate the value of $y(1.5)$ to three decimal places.

**Solution**

In this case, $x_0 = 1, y_0 = 1, h = 0.25$ and $\mathrm{f}(x, y) = \ln(x+y)$.
Using Euler's formula, the first estimate of $y_1$ is

$$\begin{aligned} y^*_1 &= y_0 + h\ln(x_0 + y_0) \\ &= 1 + 0.25\ln 2 \\ &\approx 1.17329. \end{aligned}$$

Hence, using the expression derived above for $y_{r+1}$, with $r = 0$,

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}\left[\ln(x_0 + y_0) + \ln\left(x_1 + y^*_1\right)\right] \\ &= 1 + 0.125\left[\ln 2 + \ln(1.25 + 1.17329)\right] \\ &\approx 1.19728. \end{aligned}$$

Observe that the 'corrected' value, $y_1$, differs from the first estimate, $y^*_1$, as is to be expected.

The procedure can now be repeated to obtain $y_2$ (i.e. the value of $y(1.5)$). Using Euler's formula, the initial estimate is

$$\begin{aligned} y^*_2 &= y_1 + h\ln(x_1 + y_1) \\ &= 1.19728 + 0.25\ln(1.25 + 1.19728) \\ &\approx 1.42102. \end{aligned}$$

Hence,

$$\begin{aligned} y_2 &= y_1 + \frac{h}{2}\left[\ln(x_1 + y_1) + \ln\left(x_2 + y^*_2\right)\right] \\ &= 1.19728 + 0.125\left[\ln(1.25 + 1.19728) + \ln(1.5 + 1.42102)\right] \\ &\approx 1.443. \end{aligned}$$

Again, note that $y_2$ differs slightly from $y^*_2$.

**Alternative way of applying Euler's improved method**

The formulae comprising the improved Euler method can be expressed in a different way. Substituting the expression obtained for $y^*_{r+1}$ into the expression for $y_{r+1}$ gives

$$y_{r+1} = y_r + \frac{h}{2}\left[\mathrm{f}\left(x_r, y_r\right) + \mathrm{f}\left(x_{r+1}, y_r + h\,\mathrm{f}\left(x_r, y_r\right)\right)\right].$$

Noting that $x_{r+1} = x_r + h,$ this can be expressed as

$$y_{r+1} = y_r + \frac{1}{2}\left(k_1 + k_2\right),$$

where $k_1 = h\,\mathrm{f}\left(x_r, y_r\right)$

and $k_2 = h\,\mathrm{f}\left(x_r + h, y_r + k_1\right).$

These expressions are quoted in the AQA *Formulae Booklet*, together with Euler's formula. The example which follows shows how these expressions can be used as an alternative to the procedure used in Example 4.4.1 .

**Example 4.4.2**

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}\left(x, y\right),$$

where $\mathrm{f}\left(x, y\right) = 2x^2 + 3 - 2y, \qquad x \geq 0.$

The boundary condition is $y(0) = 5.$

(a) Use the improved Euler method, with $h = 0.25,$ to calculate approximations to the values of $y(0.25)$ and $y(0.5)$.

(b) The exact solution of the differential equation (obtained in Chapter 3, example 3.4.2) is

$$y(x) = 3\mathrm{e}^{-2x} + x^2 - x + 2.$$

Calculate, to two significant figures, the percentage error in the value of $y(0.5)$ obtained in part (a).

**Solution**

**(a)** The solution can be conveniently set out in tabular form.

| $r$ | $x_r$ | $y_r$ | $k_1$ | $x_r + h$ | $y_r + k_1$ | $k_2$ | $y_{r+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 5 | –1.75 | 0.25 | 3.25 | –0.84375 | 3.703125 |
| 1 | 0.25 | 3.703125 | –1.0703125 | 0.5 | 2.6328125 | –0.44140625 | 2.947266 |

The entries on the first row are obtained as follows. It is given that $y(0)=5$, which implies that $x_0=0$ and $y_0=5$. The formula for $k_1$, with $r=0$, gives

$$\begin{aligned} k_1 &= h\,\mathrm{f}\left(x_0, y_0\right) \\ &= 0.25\left(2x_0{}^2+3-2y_0\right) \\ &= -1.75. \end{aligned}$$

The next two entries in the table are $x_0+h=0.25$ and $y_0+k_1=3.25$. The formula for $k_2$, with $r=0$, gives

$$\begin{aligned} k_2 &= 0.25\,\mathrm{f}\left(0.25, 3.25\right) \\ &= 0.25\left\{2\left(0.25\right)^2+3-\left(2\times 3.25\right)\right\} \\ &= -0.84375. \end{aligned}$$

The formula for $y_{r+1}$, with $r=0$, gives

$$\begin{aligned} y_1 &= y_0+\frac{1}{2}\left(k_1+k_2\right) \\ &= 5+\frac{1}{2}\left(-1.75-0.84375\right) \\ &= 3.703125. \end{aligned}$$

On the second row, the first two entries are $x_1=x_0+h=0.25$ and $y_1=3.703125$ (transferred from row 1). Then

$$\begin{aligned} k_1 &= h\,\mathrm{f}\left(x_1, y_1\right) \\ &= 0.25\left\{2\times\left(0.25\right)^2+3-\left(2\times 3.703125\right)\right\} \\ &= -1.0703125. \end{aligned}$$

The entries for $x_1+h$ and $y_1+k_1$ are straightforward with the values as shown. These give

$$\begin{aligned} k_2 &= 0.25\,\mathrm{f}\left(0.5, 2.6328125\right) \\ &= 0.25\left[\left(2\times 0.5^2\right)+3-\left(2\times 2.6328125\right)\right] \\ &= -0.44140625. \end{aligned}$$

Finally,

$$\begin{aligned} y_2 &= y_1+\frac{1}{2}\left(k_1+k_2\right) \\ &= 3.703125+\frac{1}{2}\left(-1.51171875\right) \\ &= 2.947266 \qquad \text{to six d.p.} \end{aligned}$$

**(b)** The exact solution quoted gives

$$\begin{aligned} y_2 = y\left(0.5\right) &= 3\mathrm{e}^{-1}+0.5^2-0.5+2 \\ &= 2.853638 \qquad \text{to six d.p.} \end{aligned}$$

Hence the error in the value obtained in part (a) is

$$\frac{2.947266-2.853638}{2.853638}\times 100\% \approx 3.3\%.$$

**Exercise 4D**

1. The function $y(x)$ satisfies the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln(xy)$$
and the condition $y(2) = 1$. An estimate of the value of $y(3)$ to three decimal places is required.
   (a) Use the improved Euler method with a step length of 0.25 to obtain this estimate, setting out your working in a similar way to that of Example 4.4.1.
   (b) Repeat the calculations using the alternative method of working as shown in Example 4.4.2.

2. The function $y(x)$ satisfies the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x^2+y^2}$$
and the condition $y(0) = 0$.
Use the improved Euler method with a step length of 0.2 to obtain an approximate value of $y(0.4)$. Give your answer to three decimal places.

3. The function $y(x)$ satisfies the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y),$$
where $\mathrm{f}(x, y) = \cos^2(x + y)$.
When $x = 1$, $y = 1$.
   (a) Use the Euler formula with a step length of 0.1 to obtain an estimate of the value of $y(1.1)$.
   (b) Obtain a second estimate of the value of $y(1.1)$ by using the improved Euler formula
$$y_{r+1} = y_r + \frac{h}{2}\left[\mathrm{f}(x_r, y_r) + \mathrm{f}(x_{r+1}, y^*_{r+1})\right]$$
taking $y_1^*$ to be the value obtained for the first estimate of $y(1.1)$ in part (a).
   (c) Repeat the calculation in part (b) to obtain a third estimate of the value of $y(1.1)$, taking $y_1^*$ to be the value obtained for the second estimate.
   (d) Verify that the second and third estimates of $y(1.1)$ are in agreement to three decimal places.

## 4.5 Error analysis: some practical considerations

The three numerical formulae that were used for solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}\left(x, y\right)$ were derived by geometrical considerations. They can also be derived analytically. There is some advantage in this alternative method because it enables some insight to be gained into the errors involved in the approximations.

The Maclaurin series for $y\left(h\right)$ is

$$y\left(h\right) = y\left(0\right) + h\,y'\left(0\right) + \frac{h^2}{2!}y''\left(0\right) + \frac{h^3}{3!}y'''\left(0\right) + \ldots,$$

where, as usual, $y', y'', y''', \ldots$ denote derivatives of first order, second order, third order, … . If the origin is transferred to the point $x_r$, this becomes

$$y\left(x_r + h\right) = y\left(x_r\right) + h\,y'\left(x_r\right) + \frac{h^2}{2!}y''\left(x_r\right) + \frac{h^3}{3!}y'''\left(x_r\right) + \ldots,$$

which is **Taylor's series**. (See also exercise 1B, question 6).

Using the notation introduced in Section 4.1, and noting that $y'\left(x_r\right) = \mathrm{f}\left(x_r, y_r\right)$, the Taylor series may be expressed as

$$y_{r+1} = y_r + h\,\mathrm{f}\left(x_r, y_r\right) + \frac{h^2}{2!}y''\left(x_r\right) + \frac{h^3}{3!}y'''\left(x_r\right) + \ldots.$$

If $h$ is assumed to be sufficiently small for terms in $h^2, h^3, \cdots$ to be negligible, then the series reduces to

$$y_{r+1} = y_r + h\,\mathrm{f}\left(x_r, y_r\right),$$

which is **Euler's formula**.

Taylor's series can also be used to step backwards from $x_r$ to $x_{r-1}$ $\left(r \geq 1\right)$ by replacing $h$ by $-h$. This gives

$$y_{r-1} = y_r - h\,\mathrm{f}\left(x_r, y_r\right) + \frac{h^2}{2!}y''\left(x_r\right) - \frac{h^3}{3!}y'''\left(x_r\right) + \ldots.$$

Subtracting this from the series for $y_{r+1}$ and neglecting all terms in $h^3$ and higher powers, gives

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}\left(x_r, y_r\right), \qquad \left(r \geq 1\right).$$

This is the **mid-point formula** derived in Section 4.3.

To obtain the improved Euler formula, first replace $y$ by $y'$ in the Taylor series. This gives

$$y'\left(x_r + h\right) = y'\left(x_r\right) + h\,y''\left(x_r\right) + \frac{h^2}{2!}y'''\left(x_r\right) + \ldots,$$

which may be expressed as

$$\mathrm{f}\left(x_{r+1}, y_{r+1}\right) = \mathrm{f}\left(x_r, y_r\right) + h\,y''\left(x_r\right) + \frac{h^2}{2!}y'''\left(x_r\right) + \ldots.$$

Multiplying each side of this equation by $\frac{1}{2}h$ and subtracting the result from the above Taylor series for $y_{r+1}$ gives

$$y_{r+1} - \frac{h}{2}\mathrm{f}\left(x_{r+1}, y_{r+1}\right) = y_r + \frac{h}{2}\mathrm{f}\left(x_r, y_r\right) + \text{terms in } h^3 \text{ and higher powers.}$$

If the terms in $h^3$ and higher powers are neglected, this becomes, after rearrangement,

$$y_{r+1} = y_r + \frac{h}{2}\left[\mathrm{f}\left(x_r, y_r\right) + \mathrm{f}\left(x_{r+1}, y_{r+1}\right)\right].$$

This is the **improved Euler formula**.

The error incurred by neglecting terms in an expansion is called the **truncation error**. The derivations above show that the truncation error in Euler's formula is greater than that in the mid-point formula and the improved Euler formula because an extra term (that in $h^2$) is neglected. Hence, for a given step length *h*, Euler's method cannot be expected to achieve the same accuracy as the other two methods.

In addition to the truncation error, for all values of $r \geq 1$, another error will be incurred in the calculation of $y_{r+1}$ because an approximate value of $y_r$ has to be used in the evaluation of $\mathrm{f}\left(x_r, y_r\right)$. For example, whichever method is used, in order to calculate $y_2$, the value of $\mathrm{f}\left(x_1, y_1\right)$ is required and only an approximate value of $y_1$ is available. As the calculation of $y$-values progresses, errors of this type sometimes decay but they can also sometimes become increasingly large. In the latter case, it will usually become apparent by wildly erratic behaviour of the solution being generated – the method is then said to be **unstable**.

It is often difficult to determine with complete certainty the accuracy of $y$-values that have been calculated by a numerical method. However, improved accuracy is almost invariably achieved by using a smaller step length, though this is at the expense of having to take more steps to reach a given end point. To estimate the order of accuracy of a solution, the usual procedure is to reduce the step length to half its previous value and repeat the calculations. If the results obtained are the same to, say, three decimal places, then it is usually safe to assume that they are correct to this order of accuracy.

Of the three methods discussed, Euler's improved method is best as it achieves reasonable accuracy and has a good record for stability. It is, in fact, the simplest of a class of methods – called Runge-Kutta methods – which are amongst the most reliable.

In practice, nowadays all numerical methods for solving differential equations are programmed to be carried out on a computer. It is therefore relatively easy to experiment with different formulae and to see how changing the step length affects the solution generated.

**Exercise 4E**

1. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \frac{y}{x}$$

and the condition $y(1) = 0$.

(a) Apply Euler's formula twice with a step length of 0.2 to calculate the value of $y(1.4)$ to three decimal places.

(b) Repeat the calculations using four steps with a step length of 0.1.

(c) Use the mid-point formula with a step length of 0.2 to obtain the value of $y(1.4)$ to three decimal places. Take the value of $y(1.2)$ to be that obtained in part (a).

(d) Use the improved Euler method with a step length of 0.2 to obtain the value of $y(1.4)$ to three decimal places.

(e) Verify that the exact solution of this problem is $y = 2x \ln x$.

(f) Hence calculate, to one decimal place, the percentage errors in the values of $y(1.4)$ obtained in parts (a) – (d).

(g) Comment on the results of the error calculations in part (f).

2 Use Taylor's series to show that

$$y'(x_r - h) = y'(x_r) - h\,y''(x_r) + \frac{h^2}{2!}y'''(x_r) + \cdots.$$

Deduce that, when $y'(x) = \mathrm{f}(x, y)$,

$h\,y''(x_r) = \mathrm{f}(x_r, y_r) - \mathrm{f}(x_{r-1}, y_{r-1}) +$ terms in $h^2$ and higher powers.

Hence show that when terms in $h^3$ and higher powers are neglected,

$$y_{r+1} = y_r + \frac{h}{2}\left[3\mathrm{f}(x_r, y_r) - \mathrm{f}(x_{r-1}, y_{r-1})\right].$$

3. The expression for $y_{r+1}$ obtained in Question 2 is just one of several other formulae that may be used to obtain numerical solutions of differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$.

(a) Apply this formula to the problem in Question 1 above, using a step length of 0.2, to calculate the value of $y(1.4)$. Take the value of $y(1.2)$ to be that obtained using Euler's formula with a step length of 0.2.

(b) By continuing the calculation with a step length of 0.2, show that the formula gives $y(2.0) \approx 2.722$.

**Miscellaneous exercises 4**

1. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(x^3 + y^3\right)^{\frac{1}{2}}$$

and the condition $y(1) = 0.5$.

(a) Use the Euler formula,

$$y_{r+1} = y_r + h\,\mathrm{f}\left(x_r, y_r\right),$$

with $h = 0.1$, to show that $y(1.1) = 0.606$ to three decimal places.

(b) Use the mid-point formula,

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}\left(x_r, y_r\right),$$

with $h = 0.1$, to find an approximate value for $y(1.3)$ giving your answer to three decimal places.

[AQA, 2001]

2. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y - 4)\,y$$

and the condition $y(1) = 2$.

(a) Use the Euler formula,

$$y_{r+1} = y_r + h\,\mathrm{f}\left(x_r, y_r\right),$$

to show that

$$y(1+h) \approx 2(1-h).$$

(b) (i) Denoting the value for $y(1+h)$ obtained in part (a) by $y_1^*$, show that

$$\mathrm{f}\left(x_1, y_1^*\right) = -2\left(1-h^2\right).$$

(ii) Hence obtain an improved estimate for the value of $y(1+h)$ using the formula

$$y_{r+1} = y_r + \tfrac{1}{2}h\left[\mathrm{f}\left(x_r, y_r\right) + \mathrm{f}\left(x_{r+1}, y_{r+1}^*\right)\right]$$

giving your answer in the form

$$a(1-h) + bh^3,$$

where $a$ and $b$ are numbers to be found.

[AQA, 2000]

3. (a) (i) Show that the integrating factor of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}, \qquad |x| < 1,$$

is $\sqrt{1-x^2}$.

(ii) Hence, or otherwise, solve the differential equation given that $x = 0$ when $y = 0$.

(iii) Show that when $x = 0.5$, $y = \dfrac{\pi}{3\sqrt{3}}$.

(b) The above differential equation may be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y),$$

where $\mathrm{f}(x, y) = \dfrac{1+xy}{1-x^2}$.

The table below shows approximate values of $y$ obtained using the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2),$$

where $k_1 = h\,\mathrm{f}(x_r, y_r)$

and $k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1)$.

| $x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|
| $y$ | 0 | 0.1010101 | 0.2062236 | 0.3205873 | 0.4508417 |

Use the improved Euler formula, with $h = 0.1$, to calculate an approximate value of $y$ at $x = 0.5$ giving your answer to five decimal places.

(c) Use the answer to part (a)(iii) to find the percentage error in your answer to part (b).

[AQA, 2001]

4. The electric current $C(t)$ flowing in a circuit at time $t$ due to an applied voltage $V(t)$ is modelled by the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} + 2C = V(t).$$

The initial condition is $C(0) = 0$.

(a) In the case when $V(t) = \mathrm{e}^{-2t}$, find an explicit expression for $C(t)$.

(b) In the case when $V(t) = \mathrm{e}^{-t^2}$, an explicit expression for $C(t)$ cannot be found. Letting $\mathrm{f}(t, C) = \mathrm{e}^{-t^2} - 2C$,

(i) use the Euler formula

$$C_{r+1} = C_r + h\,\mathrm{f}(t_r, C_r),$$

with $h = 0.1$, to find an approximate value for $C(0.1)$;

(ii) use the mid-point formula,

$$C_{r+1} = C_{r-1} + 2h\,\mathrm{f}(t_r, C_r),$$

with $h = 0.1$ and your result from (i) to find approximate values for $C(0.2)$ and $C(0.3)$.

[NEAB, 1996]

5. As part of a project, a student investigates the accuracy of numerical methods for solving differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y).$$

He chooses as a test function,

$$\mathrm{f}(x, y) = y^2\mathrm{e}^{-x}$$

and uses the boundary condition $y(0) = 1$.

Calculations are performed using two procedures, each with a step length of $h = 0.2$. The results are shown in the tables below, but two values have been omitted.

(a) **Table 1** shows the results obtained using the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r).$$

**Table 1**

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_r$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| $y_r$ | 1.000000 | 1.200000 | | 1.712168 | 2.033939 | 2.405705 |

Calculate the missing value.

(b) **Table 2** shows the results obtained using the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2),$$

where $\quad k_1 = h\,\mathrm{f}(x_r, y_r)$

and $\quad k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1).$

**Table 2**

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_r$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| $y_r$ | 1.000000 | | 1.482375 | 1.802968 | 2.190964 | 2.659637 |

Calculate the missing value.

(c) (i) Use the method of separation of variables to find the exact solution for $y(x)$.

(ii) Hence find, correct to one decimal place, the percentage error for each of the values of $y(1)$ given in parts (a) and (b).

[NEAB, 1997]

6. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y),$$

where $\mathrm{f}(x, y) = \sqrt{x + y}$. When $x = 1$, $y = 1$.

(a) Show that one application of Euler's formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r),$$

with a step length of 0.25, gives $y(1.25) \approx 1.35355$.

(b) Use the formula

$$y_{r+1} = y_r + \frac{h}{2}\left[3\mathrm{f}(x_r, y_r) - \mathrm{f}(x_{r-1}, y_{r-1})\right],$$

with a step length of 0.25, to estimate the value of $y$ when $x = 2$ giving your answer to three decimal places. Take $y_1$ to be the value obtained in part (a).

(c) Suggest how a more accurate estimate of the value of $y$ when $x = 2$ could be obtained using the above formulae.

# Chapter 5: Second Order Differential Equations

5.1 Introduction to complex numbers

5.2 Working with complex numbers

5.3 Euler's identity

5.4 Formation of second order differential equations

5.5 Differential equations of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\dfrac{\mathrm{d}y}{\mathrm{d}x}+cy=0$

5.6 Differential equations of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\dfrac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}(x)$

5.7 Solution of second order differential equations by reduction to simultaneous first order differential equations

This chapter deals with analytical methods for solving second order linear differential equations with constant coefficients. When you have completed it, you will:

- have been introduced to the concept of complex numbers, which prove useful in the analytical methods described;
- know sufficient about complex numbers, including Euler's identity, for the purposes of this chapter;
- be able to solve differential equations of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\dfrac{\mathrm{d}y}{\mathrm{d}x}+cy=0$ using the auxiliary equation $ak^2+bk+c=0$;
- be able to solve differential equations of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\dfrac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}(x)$ by finding a complementary function and a particular integral;
- know how second order linear differential equations can be solved by reduction to simultaneous first order linear differential equations.

## 5.1 Introduction to complex numbers

You will already have discovered that some algebraic equations have roots that cannot be expressed in terms of real numbers. A simple example is the quadratic equation $x^2+9=0$, which gives $x^2=-9$ and therefore $x=\pm\sqrt{-9}$. There is no real number with its square equal to $-9$ and so $\sqrt{-9}$ cannot be evaluated.

This problem can be overcome by introducing a new number i, called the imaginary unit, defined as $\mathrm{i}=\sqrt{-1}$. In terms of this unit, the solution of the equation $x^2+9=0$ can be expressed as

$$x=\pm\sqrt{9\times(-1)}=\pm3\sqrt{-1}=\pm3\mathrm{i}.$$

The number 3i is a special case of a complex number.

| |
|---|
| A complex number is defined as one which has the form $a+\mathrm{i}b$, where $a$ and $b$ are real and $\mathrm{i}=\sqrt{-1}$ |

Thus $1+\mathrm{i}$, $2-\mathrm{i}$, $3\mathrm{i}$ and $\sqrt{2}+\sqrt{3}\mathrm{i}$ are all examples of complex numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. When $b=0$, $a+\mathrm{i}b$ reduces to the real number $a$, so $\mathbb{C}$ includes the set $\mathbb{R}$ of all real numbers. Numbers of the form $b\mathrm{i}$, where $b\neq0$, are said to be **imaginary**. The number $a$ is called the **real part** of $a+\mathrm{i}b$, and $b$ is called the **imaginary part**.

Extending the number system to include new types has been a notable feature of the development of mathematics from earliest times and has often been controversial. Even as late as the seventeenth century, negative numbers were regarded with suspicion by some eminent mathematicians, though it was conceded that they were ‘a useful aid to calculation’. It is hardly surprising, therefore, to find that there was some resistance to the concept of complex numbers. Though the concept first appeared in 1572 in a book written by the Italian mathematician Raphael Bombelli, it was not until the nineteenth century that complex numbers can be said to have been completely accepted. The symbol i, to denote $\sqrt{-1}$, was introduced by the Swiss mathematician Leonhard Euler in the mid-eighteeenth century.

In this chapter you will see how complex numbers can be useful in solving certain types of differential equations. Only a little knowledge of the concept is needed; all that is required for the purpose of this chapter is covered in the two sections which follow.

## 5.2 Working with complex numbers

Complex numbers are added, subtracted and multiplied in much the same way as ordinary algebraic expressions, treating i as an algebraic quantity. However, since $\mathrm{i} = \sqrt{-1}$ and hence $\mathrm{i}^2 = -1$, $\mathrm{i}^2$ can be replaced by $-1$ wherever it occurs. The following examples show the processes involved.

**Example 5.2.1**

The complex numbers $z_1$ and $z_2$ are given by

$$z_1 = 1 + 2\mathrm{i}, \qquad z_2 = -2 + 3\mathrm{i}.$$

Find, in the form $a + \mathrm{i}b$,

(a) $z_1 + z_2$, (b) $z_1 - z_2$, (c) $3z_1 - 2z_2$, (d) $z_1 z_2$, (e) $z_1{}^2 + z_2{}^2$.

**Solution**

**(a)**
$$\begin{aligned} z_1 + z_2 &= 1 + 2\mathrm{i} - 2 + 3\mathrm{i} \\ &= -1 + 5\mathrm{i}. \end{aligned}$$

**(b)**
$$\begin{aligned} z_1 - z_2 &= 1 + 2\mathrm{i} - (-2 + 3\mathrm{i}) \\ &= 1 + 2\mathrm{i} + 2 - 3\mathrm{i} \\ &= 3 - \mathrm{i}. \end{aligned}$$

**(c)**
$$\begin{aligned} 3z_1 - 2z_2 &= 3(1 + 2\mathrm{i}) - 2(-2 + 3\mathrm{i}) \\ &= 3 + 6\mathrm{i} + 4 - 6\mathrm{i} \\ &= 7. \end{aligned}$$

**(d)**
$$\begin{aligned} z_1 z_2 &= (1 + 2\mathrm{i})(-2 + 3\mathrm{i}) \\ &= -2 + 3\mathrm{i} + 2\mathrm{i}(-2 + 3\mathrm{i}) \\ &= -2 + 3\mathrm{i} - 4\mathrm{i} + 6\mathrm{i}^2 \\ &= -2 - \mathrm{i} - 6 \\ &= -8 - \mathrm{i}. \end{aligned}$$

**(e)**
$$\begin{aligned} z_1{}^2 + z_2{}^2 &= (1 + 2\mathrm{i})^2 + (-2 + 3\mathrm{i})^2 \\ &= 1 + 4\mathrm{i} + 4\mathrm{i}^2 + 4 - 12\mathrm{i} + 9\mathrm{i}^2 \\ &= 1 + 4\mathrm{i} - 4 + 4 - 12\mathrm{i} - 9 \\ &= -8 - 8\mathrm{i}. \end{aligned}$$

**Example 5.2.2**

Find, in the form $a \pm \mathrm{i}b$, the values of $x$ satisfying the equation

$$(x+2)^2 + 4 = 0.$$

**Solution**

The equation can be rewritten as $(x+2)^2 = -4$.

Hence $\quad (x+2) = \pm\sqrt{-4}$

$\qquad\qquad = \pm 2\mathrm{i}$

giving $\quad x = -2 \pm 2\mathrm{i}.$

**Example 5.2.3**

Solve the quadratic equation $2x^2 + 2x + 3 = 0$ giving the roots in the form $a \pm \mathrm{i}b$.

**Solution**

Using the formula for the roots of a quadratic equation,

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4-(4\times 2\times 3)}}{4} \\ &= \frac{-2 \pm \sqrt{-20}}{4} \\ &= \frac{-2 \pm \sqrt{20}\,\mathrm{i}}{4} \\ &= \frac{-2 \pm 2\sqrt{5}\,\mathrm{i}}{4} \\ &= -\frac{1}{2} \pm \frac{\sqrt{5}}{2}\mathrm{i}. \end{aligned}$$

**Exercise 5A**

1. The complex numbers $z_1$ and $z_2$ are given by
$$z_1 = 2 + \mathrm{i}, \qquad z_2 = 1 - 3\mathrm{i}.$$
Find, in the form $a + \mathrm{i}b$:
(a) $z_1 - z_2$, (b) $3z_1 + z_2$, (c) $z_1 z_2$,
(d) ${z_1}^2 + {z_2}^2$, (e) ${z_1}^2 - {z_2}^2$, (f) ${z_1}^2 {z_2}^2$.

2. Show that
(a) $(1+\mathrm{i})^2$ is imaginary,
(b) $(1+\mathrm{i})^4$ is real,
(c) $\mathrm{i}\left(2\mathrm{i}^3 + 3\mathrm{i}^5\right)$ is real.

3. Find the real and imaginary parts of
(a) $\mathrm{i}(1-2\mathrm{i})(1+3\mathrm{i})$, (b) $\frac{1}{2}(1+\mathrm{i})^3$.

4. Solve the quadratic equation
$$(x-3)^2 + 4 = 0.$$

5. (a) Express each of the roots of the equation
$$x^2 + x + 1 = 0$$
in the form $a + \mathrm{i}b$.
(b) Denoting the two roots by $\alpha$ and $\beta$, use your answers to part (a) to verify that $\alpha + \beta = -1$ and $\alpha\beta = 1$.

## 5.3 Euler's identity

This identity, named after Euler but discovered by others, states that

$$\boxed{e^{ix} \equiv \cos x + i\sin x, \quad x \in \mathbb{R}}$$

The identity can be proved using series expansions. The left-hand side is

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots \end{aligned}$$

Now $i^2 = -1, \quad i^3 = i^2 \times i = -i, \quad i^4 = \left(i^2\right)^2 = 1, \quad i^5 = i^4 \times i = i,$ and the same sequence of results (–1, –i, 1 and i) is generated repeatedly when higher powers ($i^6$, $i^7$, …) are calculated. Substituting these expressions into the above expansion and collecting together real and imaginary parts gives

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

The two expressions in brackets are the series for $\cos x$ and $\sin x$, and hence

$$e^{ix} \equiv \cos x + i\sin x.$$

The special case when $x = \pi$ is of particular interest. As $\cos\pi = -1$ and $\sin\pi = 0$, substituting $x = \pi$ in Euler's identity gives

$$\boxed{e^{i\pi} = -1}$$

This is a famous equation connecting e, π and i, three of the most important numbers in mathematics, in an elegant and most unexpected way.

**Example 5.3.1**

Find the real and imaginary parts of

$$3e^{ix} + 2e^{-ix}.$$

**Solution**

Note first that

$$\begin{aligned} e^{-ix} &= \cos x + i\sin(-x) \\ &= \cos x - i\sin x. \end{aligned}$$

Hence

$$\begin{aligned} 3e^{ix} + 2e^{-ix} &= 3(\cos x + i\sin x) + 2(\cos x - i\sin x) \\ &= 5\cos x + i\sin x. \end{aligned}$$

The real and imaginary parts are therefore $5\cos x$ and $\sin x$, respectively.

**Example 5.3.2**

The function $y(x)$ is given by

$$y(x) = Ae^{(2+3i)x} + Be^{(2-3i)x},$$

where $A$ and $B$ are constants. Show that this can be expressed as

$$y(x) = e^{2x}(C\cos 3x + D\sin 3x),$$

where $C$ and $D$ are constants.

**Solution**

$$\begin{aligned} y(x) &= A\left(e^{2x} \times e^{3ix}\right) + B\left(e^{2x} \times e^{-3ix}\right) \\ &= Ae^{2x}(\cos 3x + i\sin 3x) + Be^{2x}(\cos 3x - i\sin 3x) \\ &= e^{2x}\left[(A+B)\cos 3x + (A-B)i\sin 3x\right]. \end{aligned}$$

Writing $A + B = C$ and $(A - B)i = D$, this becomes

$$y(x) = e^{2x}(C\cos 3x + D\sin 3x).$$

**Exercise 5B**

1. (a) Use Euler's identity to show that $e^{\frac{i\pi}{2}} = i$.

   (b) Hence show that $i^i \approx 0.2079$.

2. Find the real and imaginary parts of $e^{(1+2i)x} + 3e^{(1-2i)x}$.

3. (a) The function $y(x)$ is given by

   $$y(x) = Ae^{-(1-i)x} + Be^{-(1+i)x},$$

   where $A$ and $B$ are constants. Show that this can be expressed as

   $$y(x) = e^{-x}(C\cos x + D\sin x),$$

   where $C$ and $D$ are constants to be found in terms of $A$ and $B$.

   (b) Given that $A = \alpha + i\beta$ and $B = \alpha - i\beta$, where $\alpha$ and $\beta$ are real, show that $C$ and $D$ are real.

## 5.4 Formation of second order differential equations

Consider the function $y = Ae^{x} + Be^{-2x} - x$, where $A$ and $B$ are arbitrary constants. Differentiating twice with respect to $x$ gives

$$\frac{dy}{dx} = Ae^{x} - 2Be^{-2x} - 1,$$

and

$$\frac{d^2y}{dx^2} = Ae^{x} + 4Be^{-2x}.$$

There are now three equations involving $A$ and $B$ and it is therefore possible to eliminate these arbitrary constants. From the first and second equations above,

$$y - \frac{dy}{dx} = 3Be^{-2x} - x + 1,$$

and from the second and third equations,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6Be^{-2x} + 1.$$

Hence,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2\left(y - \frac{dy}{dx}\right) = 1 - 2(-x+1)$$

giving

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2x - 1.$$

This shows that the function $y$ satisfies the differential equation above whatever the values of $A$ and $B$; $y$ is, in fact, the **general solution** as will be shown later.

The coefficients of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and $y$ in the differential equation above are constants (1, 1 and –2, respectively). Second order differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, where $a$, $b$ and $c$ are constants, have important applications in modelling real-world problems, especially in relation to studies of vibrations of mechanical systems. The example shows how such a differential equation can be derived from its general solution, but in practice it is the differential equation that is given and the problem is to obtain the general solution. Also, a particular solution satisfying specified boundary conditions is usually required.

The general solution, $y(x)$, of a second order differential equation must contain exactly two arbitrary constants. If the general solution contained just one arbitrary constant, $C$, it would be possible to eliminate $C$ using only the expressions for $y$ and $\frac{dy}{dx}$. The differential equation satisfied by $y$ would then be of first order. If the general solution contained three arbitrary constants, then the expressions for $y$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ would be needed to eliminate these constants so the differential equation for $y$ would be of third order. Similarly, the possibility that the general solution contains more than three arbitrary constants can be ruled out.

Sections 5.5, 5.6 and 5.7 show how second order linear differential equations with constant coefficients can be solved analytically.

**Example 5.4.1**

The function $y$ is given by $y=\left(Ax+B\right)\mathrm{e}^{-x}$, where $A$ and $B$ are arbitrary constants. Obtain the second order differential equation satisfied by $y$.

**Solution**

Using the product rule, the first and second derivatives of $y$ are

$$\frac{\mathrm{d}y}{\mathrm{d}x}=A\mathrm{e}^{-x}-\left(Ax+B\right)\mathrm{e}^{-x},$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-A\mathrm{e}^{-x}-A\mathrm{e}^{-x}+\left(Ax+B\right)\mathrm{e}^{-x}$$

$$=-2A\mathrm{e}^{-x}+\left(Ax+B\right)\mathrm{e}^{-x}.$$

Eliminating $Ax+B$ from the expressions for $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives

$$y+\frac{\mathrm{d}y}{\mathrm{d}x}=A\mathrm{e}^{-x}.$$

Similarly, eliminating $Ax+B$ from the expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-A\mathrm{e}^{-x}.$$

Hence, by addition of these expressions,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+y=0.$$

**Exercise 5C**

1. The function $y$ is given by

$$y=A\cos x+B\sin x+\mathrm{e}^{-2x},$$

where $A$ and $B$ are arbitrary constants. Show that $y$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=5\mathrm{e}^{-2x}.$$

2. The function $y$ is given by

$$y=A\mathrm{e}^{x}\cos x+B\mathrm{e}^{x}\sin x,$$

where $A$ and $B$ are arbitrary constants.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(b) Hence show that $y$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+2y=0.$$

3. The function $y$ is given by

$$y=\left(Ax+B\right)\mathrm{e}^{2x}+1,$$

where $A$ and $B$ are arbitrary constants. Find the differential equation satisfied by $y$.

## 5.5 Differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

This section deals with second order differential equations of the form $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0$, where $a$, $b$ and $c$ are given constants. Note that $a \neq 0$ because otherwise the differential equation would be of first order. The form of the general solution of differential equations of this type can be derived by a method which reduces the problem to one of solving first order linear differential equations.

Consider the quadratic equation

$$ak^2+bk+c=0,$$

the coefficients of which are the same as those in the differential equation above. This quadratic is called the **auxiliary equation**. Let the roots be $k_1$ and $k_2$. Then

$$k_1+k_2=-\frac{b}{a} \text{ and } k_1k_2=\frac{c}{a}.$$

The key step in the derivation of the general solution of the differential equation is to put

$$u=\frac{\mathrm{d}y}{\mathrm{d}x}-k_2y.$$

Then

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}-k_1u &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2}-k_2\frac{\mathrm{d}y}{\mathrm{d}x}-k_1\left(\frac{\mathrm{d}y}{\mathrm{d}x}-k_2y\right)\\
&=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\left(k_1+k_2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+k_1k_2y\\
&=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{b}{a}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{c}{a}y\\
&=\frac{1}{a}\left(a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy\right), \qquad a\neq 0,\\
&=0.\end{aligned}$$

Hence, the new variable $u$ satisfies the first order differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}=k_1u.$$

As shown in Chapter 3 (Exercise 3B), this has the general solution $u=C\mathrm{e}^{k_1x}$, where $C$ is an arbitrary constant.

Substituting this expression into that which defined $u$ in terms of $y$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}-k_2y=C\mathrm{e}^{k_1x}.$$

This is another first order differential equation of a type covered in Chapter 3. It is solved by using the integrating factor $\mathrm{e}^{\int -k_2\mathrm{d}x}=\mathrm{e}^{-k_2x}$.

Multiplying the differential equation through by this factor gives
$$\mathrm{e}^{-k_2 x}\frac{\mathrm{d}y}{\mathrm{d}x}-k_2\mathrm{e}^{-k_2 x}y=C\mathrm{e}^{(k_1-k_2)x},$$
which can be written as
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-k_2 x}y\right)=C\mathrm{e}^{(k_1-k_2)x}.$$

There are now two cases to consider.

$k_2 \neq k_1$

When $k_2 \neq k_1$, the integral of the differential equation above is
$$e^{-k_2 x}y=\frac{C}{k_1-k_2}e^{(k_1-k_2)x}+B,$$
where $B$ is an arbitrary constant. Writing $\dfrac{C}{k_1-k_2}$ as $A$, this may be expressed as
$$y=A\mathrm{e}^{k_1 x}+B\mathrm{e}^{k_2 x}.$$

This is, therefore, the form of the general solution when the auxiliary equation has unequal roots.

$k_2 = k_1$

In this case, the roots of the auxiliary equation are equal and the differential equation for $y$ reduces to
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-k_1 x}y\right)=C.$$

Hence, $$\mathrm{e}^{-k_1 x}y=Cx+D,$$

where $C$ and $D$ are arbitrary constants. The general solution in this case is therefore
$$y=(Cx+D)\mathrm{e}^{k_1 x}.$$

**Example 5.5.1**

Find the general solution of the differential equation
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-5\frac{\mathrm{d}y}{\mathrm{d}x}+6y=0.$$

**Solution**

The auxiliary equation is
$$k^2-5k+6=0.$$
Hence, $$(k-2)(k-3)=0,$$
so the roots are $k=2$ and $k=3$. The general solution of the differential equation is therefore
$$y=A\mathrm{e}^{2x}+B\mathrm{e}^{3x},$$

where $A$ and $B$ are arbitrary constants.

**Example 5.5.2**

Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+4y=0.$$

**Solution**

The auxiliary equation is

$$k^2+4k+4=0.$$

Hence, $$(k+2)^2=0,$$

so the roots are both equal to –2. The general solution of the differential equation is therefore

$$y=(Ax+B)\mathrm{e}^{-2x},$$

where $A$ and $B$ are arbitrary constants.

**Example 5.5.3**

Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+5y=0.$$

Express the answer in a form involving trigonometric functions.

**Solution**

The auxiliary equation is

$$k^2+2k+5=0.$$

Hence, $$\begin{aligned}k&=\frac{-2\pm\sqrt{4-20}}{2}\\&=\frac{-2\pm 4\mathrm{i}}{2}\\&=-1\pm 2\mathrm{i}.\end{aligned}$$

The general solution of the differential equation is therefore

$$\begin{aligned}y&=A\mathrm{e}^{(-1+2\mathrm{i})x}+B\mathrm{e}^{(-1-2\mathrm{i})x}\\&=\mathrm{e}^{-x}\left(A\mathrm{e}^{2\mathrm{i}x}+B\mathrm{e}^{-2\mathrm{i}x}\right)\\&=\mathrm{e}^{-x}\left[A(\cos 2x+\mathrm{i}\sin 2x)+B(\cos 2x-\mathrm{i}\sin 2x)\right].\end{aligned}$$

Hence, writing $A+B$ as $C$ and $A\mathrm{i}-B\mathrm{i}$ as $D$,

$$y=\mathrm{e}^{-x}(C\cos 2x+D\sin 2x),$$

where $C$ and $D$ are arbitrary constants.

The example above shows that when the roots of the auxiliary equation are not real, the general solution can still be expressed in a form not involving complex numbers.

In general, suppose that the roots of the auxiliary equation are

$$k_1 = p + \mathrm{i}q, \quad k_2 = p - \mathrm{i}q,$$

where $p$ and $q$ are real and $q \neq 0$. Then the general solution is

$$\begin{aligned} y &= A\mathrm{e}^{(p+\mathrm{i}q)x} + B\mathrm{e}^{(p-\mathrm{i}q)x} \\ &= \mathrm{e}^{px}\left(A\mathrm{e}^{\mathrm{i}qx} + B\mathrm{e}^{-\mathrm{i}qx}\right) \\ &= \mathrm{e}^{px}\left[A\left(\cos qx + \mathrm{i}\sin qx\right) + B\left(\cos qx - \mathrm{i}\sin qx\right)\right] \\ &= \mathrm{e}^{px}\left(C\cos qx + D\sin qx\right), \end{aligned}$$

where $C$ and $D$ are arbitrary constants.

The method for solving differential equations of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$ can now be summarised as follows.

1. Obtain the roots of the auxiliary equation
$$ak^2 + bk + c = 0.$$
2. If the roots of the auxiliary equation are $k_1$ and $k_2$ and these are real and unequal, then the general solution is
$$y = A\mathrm{e}^{k_1x} + B\mathrm{e}^{k_2x}.$$
3. If the roots of the auxiliary equation are equal, having the value $k_1$, then the general solution is
$$y = \mathrm{e}^{k_1x}\left(A + Bx\right).$$
4. If the roots of the auxiliary equation are non-real and have the values $p + \mathrm{i}q$ and $p - \mathrm{i}q$, then the general solution is
$$y = \mathrm{e}^{px}\left(A\cos qx + B\sin qx\right).$$

   In each of the solutions above, $A$ and $B$ are arbitrary constants.

**You should memorise the form of the general solution in each of these cases**

The special case when the differential equation is of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + n^2x = 0,$$

where $n$ is a real number, is of particular interest. You may recognise this as the differential equation governing simple harmonic motion, and it occurs in other circumstances too. The auxiliary equation is $k^2 + n^2 = 0$, which has the roots $k = \mathrm{i}n$ and $k = -\mathrm{i}n$. In this case therefore, $p = 0$ and $q = n$ so the general solution is

$$\boxed{y = A\cos nx + B\sin nx}$$

This is well worth remembering.

Finally, it should be noted that the auxiliary equation can be derived by substituting $y = \mathrm{e}^{kx}$ into the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

This gives $\qquad ak^2\mathrm{e}^{kx} + bk\mathrm{e}^{kx} + c\mathrm{e}^{kx} = 0,$

and hence $\qquad ak^2 + bk + c = 0.$

This shows that $\mathrm{e}^{kx}$ will be a solution of the differential equation provided that $k$ is a root of the auxiliary equation. When there are two distinct roots, $k_1$ and $k_2$, each of the functions $\mathrm{e}^{k_1x}$ and $\mathrm{e}^{k_2x}$ will therefore be solutions.

It can be verified by direct substitution into the differential equation that the linear combination

$$y = A\mathrm{e}^{k_1x} + B\mathrm{e}^{k_2x},$$

where $A$ and $B$ are arbitrary constants, is also a solution in this case. This is, of course, identical to the general solution derived earlier.

When the roots of the auxiliary equation are equal, having the value $k_1$, $\mathrm{e}^{k_1x}$ is the only solution of exponential form. However, in this case it can be verified that $x\mathrm{e}^{k_1x}$ will also be a solution (see Exercise 5D, question 6). The linear combination

$$y = Cx\mathrm{e}^{k_1x} + D\mathrm{e}^{k_1x} = (Cx + D)\,e^{k_1x},$$

where $C$ and $D$ are arbitrary constants, then provides the general solution.

The method used earlier proves that the above are the only possible forms for the general solution.

**Example 5.5.4**

Find the solution of the differential equation

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0$$

satisfying the conditions that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

**Solution**

The auxiliary equation is $4k^2 - 4k + 5 = 0$. Hence

$$\begin{aligned} k &= \frac{4 \pm \sqrt{16 - 80}}{8} \\ &= \frac{4 \pm 8\mathrm{i}}{8} \\ &= \frac{1}{2} \pm \mathrm{i}. \end{aligned}$$

The general solution is therefore

$$y = e^{\frac{x}{2}}\left(A\cos x + B\sin x\right).$$

This follows because the roots of the auxiliary equation are of the form $p \pm iq$, with $p = \frac{1}{2}$ and $q = 1$.

Using the product rule to differentiate $y$ with respect to $x$,

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}}\left(A\cos x + B\sin x\right) + e^{\frac{x}{2}}\left(-A\sin x + B\cos x\right).$$

The condition $y = 1$ when $x = 0$ gives $1 = A$, and the condition $\frac{dy}{dx} = 0$ when $x = 0$ gives $0 = \frac{1}{2}A + B$. Hence $B = -\frac{1}{2}$. The required solution is therefore

$$y = e^{\frac{x}{2}}\left(\cos x - \frac{1}{2}\sin x\right).$$

**Exercise 5D**

1. (a) Write down the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 0, \qquad 0 \le x \le \frac{\pi}{2}.$$

(b) Hence find the particular solution satisfying the boundary conditions $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = 2$.

2. Find the general solution of each of the following differential equations.

(a) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$ (b) $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0.$

(c) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0.$ (d) $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0.$

(e) $2\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0.$ (f) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0.$

3. (a) Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2} - y = 0.$$

(b) Hence find the particular solution which is such that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

4. Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$$

subject to the conditions that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

5. Find the function $y(x)$ satisfying the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0$$

and the conditions that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

6. (a) Show that when $b^2 = 4ac$, the roots of the auxiliary equation for the differential equation

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

are both equal to $-\frac{b}{2a}$.

(b) Writing $-\frac{b}{2a}$ as $k_1$, verify that, in this case, $\mathrm{e}^{k_1 x}$ and $x\mathrm{e}^{k_1 x}$ are solutions of the differential equation.

7. Show that if $y_1$ and $y_2$ are solutions of the differential equation

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

then so also is the linear combination $y = Ay_1 + By_2$, where $A$ and $B$ are arbitrary constants.

## 5.6 Differential equations of the form $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=f(x)$

In Chapter 3 it was shown how first order differential equations of the form

$$a\frac{dy}{dx}+by=f(x)$$

can be solved by finding a complementary function (CF) and a particular integral (PI). The same method can be used to solve second order differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x),$$

where $a$, $b$ and $c$ are given constants and f is a given function of $x$. The procedure is as follows.

1. Find the general solution (GS) of the reduced equation
$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$
This solution is the CF, $y_c$.

2. Find a particular solution of the complete equation
$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x).$$
This solution is a PI, $y_p$.

3. The GS of the complete equation is then
$$y=y_c+y_p.$$

The reduced equation is of the form dealt with in Section 5.5 – the CF is therefore found by applying the methods explained there.

As with first order differential equations, a PI is found by substituting an appropriate trial function into the differential equation. The trial function depends on the form of the function $f(x)$ on the right-hand side of the differential equation, and contains at least one constant to be chosen so that the differential equation is satisfied.

Only certain forms of $\mathrm{f}(x)$ are included in the Further Pure 3 unit. For these cases, a PI can be found as follows.

- $\mathrm{f}(x) = C\,\mathrm{e}^{\lambda x}$, where $C$ and $\lambda$ are given constants.
  First note whether the CF contains a term of the form
  (i) *constant* $\times\,\mathrm{e}^{\lambda x}$ or (ii) *constant* $\times\, x\,\mathrm{e}^{\lambda x}$.
  - If it does not, then the PI will be of the form $y = a\,\mathrm{e}^{\lambda x}$, where $a$ is a constant to be found.
  - If there is a term of type (i) in the CF but not one of type (ii), then the PI will be of the form $y = ax\,\mathrm{e}^{\lambda x}$.
  - If there is a term of type (ii) in the CF, then the PI will be of the form $y = ax^2\mathrm{e}^{\lambda x}$.

- $\mathrm{f}(x) = C\cos\lambda x$ or $C\sin\lambda x$.
  - Provided that the CF is not of the form $A\cos\lambda x + B\sin\lambda x$, the PI will be of the form $y = a\cos\lambda x + b\sin\lambda x$, where $a$ and $b$ are constants to be found.
  - If the CF is of the form $A\cos\lambda x + B\sin\lambda x$, the PI will be of the form $y = ax\sin\lambda x$ when $\mathrm{f}(x) = C\cos\lambda x$, and $y = ax\cos\lambda x$ when $\mathrm{f}(x) = C\sin\lambda x$.

- $\mathrm{f}(x)$ is a polynomial of degree $n$.
  In this case try a PI of the form $y = ax^n + bx^{n-1} + \ldots$, where $a$, $b$, $\ldots$ are constants to be found. Thus, for example, if $\mathrm{f}(x) = x^3 + 3$ then the appropriate trial function is $y = ax^3 + bx^2 + cx + d$.
  This fails when the differential equation has no $y$ term. In this exceptional case, the appropriate trial function is a polynomial of degree $n+1$.

**Example 5.6.1**

The complementary function of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 2e^{2x}$$

is $Ae^{3x} + Be^{-2x}$.

(a) Find a particular integral.

(b) Hence write down the general solution.

**Solution**

**(a)** Since the CF does not contain a term of the form *constant* $\times e^{2x}$, the appropriate trial function for a PI is $y = ae^{2x}$. Differentiating this twice gives

$$\frac{dy}{dx} = 2ae^{2x}, \qquad \frac{d^2y}{dx^2} = 4ae^{2x}.$$

Substituting the expressions for $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the differential equation gives

$$4ae^{2x} - 2ae^{2x} - 6ae^{2x} = 2e^{2x}.$$

Cancelling the common factor $e^{2x}$ and collecting terms gives

$$-4a = 2$$

and therefore $a = -\frac{1}{2}$. Hence the PI is $y = -\frac{1}{2}e^{2x}$.

**(b)** The GS is obtained by adding the CF and the PI giving

$$y = Ae^{3x} + Be^{-2x} - \frac{1}{2}e^{2x}.$$

**Example 5.6.2**

The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2e^{-x}.$$

(a) Find the complementary function.

(b) Find a particular integral.

(c) Hence solve the differential equation given that $y = 3$ when $x = 0$ and that $y \to 0$ as $x \to \infty$.

**Solution**

**(a)** To obtain the CF, the GS of the reduced equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0$$

is required. The auxiliary equation is

$$k^2-k-2=0.$$

Hence $$(k-2)(k+1)=0,$$

giving $k=2$ or $k=-1$. The CF is therefore

$$y=Ae^{2x}+Be^{-x},$$

where $A$ and $B$ are arbitrary constants.

**(b)** The right-hand side of the differential equation is $2e^{-x}$ so the PI will be of the form $ae^{-x}$, $axe^{-x}$ or $ax^2e^{-x}$. Since there is a term of the form *constant* $\times e^{-x}$ in the CF, but not one of the form *constant* $\times xe^{-x}$, the rules stated earlier indicate that the appropriate choice is $y=axe^{-x}$. Differentiating this function using the product rule,

$$\frac{dy}{dx}=ae^{-x}-axe^{-x}$$

and $$\frac{d^2y}{dx^2}=-ae^{-x}-ae^{-x}+axe^{-x}$$

$$=-2ae^{-x}+axe^{-x}.$$

Substituting these expressions into the differential equation gives

$$-2ae^{-x}+axe^{-x}-\left(ae^{-x}-axe^{-x}\right)-2axe^{-x}=2e^{-x}.$$

The terms involving $xe^{-x}$ cancel and hence

$$-3ae^{-x}=2e^{-x},$$

giving $a=-\frac{2}{3}$. The PI is therefore

$$y=-\frac{2}{3}xe^{-x}.$$

It is worth noting here that the method should always give a **constant** value for $a$. If it does not then either the wrong form has been used for the PI or a mistake has been made in the working.

**(c)** From the results of parts (a) and (b), the GS of the differential equation is

$$y = Ae^{2x} + Be^{-x} - \frac{2}{3}xe^{-x}.$$

The condition $y = 3$ when $x = 0$ gives

$$3 = A + B.$$

When $x \to \infty$, $e^{2x} \to \infty$, $e^{-x} \to 0$ and $xe^{-x} \to 0$. Hence, to satisfy the condition that $y \to 0$ as $x \to \infty$, $A$ must be zero. It follows that $B = 3$ and therefore

$$\begin{aligned} y &= 3e^{-x} - \frac{2}{3}xe^{-x} \\ &= \left(3 - \frac{2}{3}x\right)e^{-x}. \end{aligned}$$

**Example 5.6.3**

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5\cos x$$

subject to the conditions that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

**Solution**

The auxiliary equation is

$$k^2 + 2k + 2 = 0,$$

which has roots $k = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \mathrm{i}.$

The CF is therefore $y = \mathrm{e}^{-x}\left(A\cos x + B\sin x\right).$

To find a PI, note first that in this case $\mathrm{f}\left(x\right) = 5\cos x.$ Hence, using the rules given earlier, the appropriate trial function is $y = a\cos x + b\sin x.$ The derivatives of this function are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a\sin x + b\cos x$$

and $$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -a\cos x - b\sin x.$$

Substituting these expressions into the differential equation gives

$$-a\cos x - b\sin x - 2a\sin x + 2b\cos x + 2a\cos x + 2b\sin x = 5\cos x$$

and hence $\left(a + 2b\right)\cos x + \left(b - 2a\right)\sin x = 5\cos x.$

The constants $a$ and $b$ must be chosen so that this equation is satisfied for all values of $x$.

Hence $a + 2b = 5$ and $b - 2a = 0,$

giving $a = 1$ and $b = 2.$ The PI is therefore $y = \cos x + 2\sin x.$

Adding the CF and the PI, the GS is

$$y = \mathrm{e}^{-x}\left(A\cos x + B\sin x\right) + \cos x + 2\sin x.$$

To complete the solution it is necessary to find the constants $A$ and $B$ using the given conditions at $x = 0.$

The condition $y = 1$ when $x = 0$ gives $1 = A + 1.$ Therefore $A = 0$ and

$$y = B\mathrm{e}^{-x}\sin x + \cos x + 2\sin x.$$

Also $$\frac{\mathrm{d}y}{\mathrm{d}x} = -B\mathrm{e}^{-x}\sin x + B\mathrm{e}^{-x}\cos x - \sin x + 2\cos x.$$

Applying the condition that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ gives $0 = B + 2.$

Hence $B = -2$ and the required solution is

$$\begin{aligned} y &= -2\mathrm{e}^{-x}\sin x + \cos x + 2\sin x \\ &= 2\left(1 - \mathrm{e}^{-x}\right)\sin x + \cos x. \end{aligned}$$

**Exercise 5E**

1. The complementary function of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \mathrm{f}(x)$$

is $(A + Bx)\mathrm{e}^{2x}$. In each of the following cases find a particular integral, and hence write down the general solution of the differential equation in these cases.

(a) $\mathrm{f}(x) = \mathrm{e}^{-x}$, (b) $\mathrm{f}(x) = 4x^2 + 6$, (c) $\mathrm{f}(x) = 25\sin x$.

2. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{3x}.$$

(a) Find the complementary function.

(b) Show that there is a particular integral of the form $y = ax^2\mathrm{e}^{3x}$, and find the appropriate value of $a$.

(c) Hence write down the general solution for $y(x)$.

3. Find a particular integral for the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = x^4.$$

4. Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 4$$

subject to the conditions that $y = 0$ when $x = 0$ and $y \to -2$ as $x \to \infty$.

5. Find the solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\sin x$$

satisfying the conditions that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

6. The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 16y = 16\sin 4x.$$

(a) Find the complementary function.

(b) Show that the differential equation has a particular integral of the form $y = ax\cos 4x$, and find the appropriate value of $a$.

(c) Find the solution for $y(x)$ which is such that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

## 5.7 Second order linear differential equations with variable coefficients

This concluding section deals with differential equations which can be expressed in the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \mathrm{P}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{Q}y = \mathrm{R}$$

where P, Q, and R are, in general, functions of $x$. A particular example, which is solved as a worked example later in this chapter, is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x^2} = 5 \ .$$

In the special case when P and Q are constants and R is such that a particular integral of the differential equation can be found, the methods of the previous two sections of this chapter can, of course, be used. However, when one or both of P and Q are variable, these methods are no longer applicable. Usually, when P and/or Q are variable, finding an exact analytical solution of the differential equation becomes much more difficult and may well prove impossible. There are, however, some instances where a second order differential equation with variable coefficients can be transformed by a suitable substitution into a simpler one which can be solved by using either one of the analytical techniques described in chapter 3 or by the methods used in the previous two sections of this chapter. In such cases, solving the transformed differential equation readily enables the solution of the original differential equation to be deduced.

The examples and exercises which follow show how this technique is used. You should note that for the purposes of this module, the substitution required to simplify a differential equation with variable coefficients will always be given.

**Example 5.7.1**

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \ , \ x > 0.$$

(a) Show that the substitution $\frac{\mathrm{d}y}{\mathrm{d}x} = u$ reduces the differential equation to

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2u}{x} = 0.$$

(b) Hence find the general solution for $u$ in terms of $x$.

(c) Deduce the general solution for $y(x)$.

(d) Find the particular solution for $y(x)$ which is such that $y\,(1) = 0$ and $y\,(x) \to 1$ as $x \to \infty$.

**Solution**

**(a)** Since $\frac{dy}{dx} = u$, $\frac{d^2y}{dx^2} = \frac{du}{dx}$. Hence the given differential can be expressed as

$$\frac{du}{dx} + \frac{2u}{x} = 0.$$

**(b)** The differential equation above can be written as

$$\frac{du}{dx} = -\frac{2u}{x}.$$

The variables in this case can be separated giving

$$\int \frac{du}{u} = -2\int \frac{dx}{x}.$$

Hence $$\ln|u| = -2\ln|x| + C,$$

where $C$ is an arbitrary constant. Noting that $2\ln|x| = \ln|x|^2 = \ln x^2$, this equation can be expressed as

$$\ln|u| + \ln x^2 = C,$$

which gives $$\ln|u|x^2 = C$$

and hence $$|u|x^2 = e^C.$$

It follows that $$u = \pm\frac{e^C}{x^2},$$

which can be written as $$u = \frac{A}{x^2},$$

where $A$ is now the arbitrary constant.

**(c)** Since $u = \frac{dy}{dx}$, we now have

$$\frac{dy}{dx} = \frac{A}{x^2}$$

and hence $$y(x) = -\frac{A}{x} + B,$$

where $B$ is another arbitrary constant. This is the general solution for $y(x)$. Observe that it contains two arbitrary constants, which is as should be expected because the given differential equation for $y(x)$ is of second order.

**(d)** Using the condition $y(1)=0$ gives $0=-A+B$ and hence $A=B$. It is also given that $y(x)\to 1$ as $x\to\infty$ which implies that $B=1$. The particular solution required is therefore

$$y(x)=-\frac{1}{x}+1\ .$$

**Example 5.7.2**

(a) Use the substitution $\frac{\mathrm{d}y}{\mathrm{d}x}=u$ to transform the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}=4,$$

where $x>0$, into a first order differential equation in $u$. Show that the general solution of the transformed equation is

$$u=2x+\frac{A}{x},$$

where $A$ is an arbitrary constant.

(b) Find the particular solution for $y$ which satisfies the boundary conditions $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=1$.

**Solution**

**(a)** Since $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}u}{\mathrm{d}x}$. Hence, in terms of $u$, the given differential equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{u}{x} = 4 .$$

This type of differential was dealt with in section 3.3 and can be solved by finding an integrating factor. The integrating factor is

$$\begin{aligned} I &= \mathrm{e}^{\int \frac{1}{x}\mathrm{d}x} \\ &= \mathrm{e}^{\ln|x|} \\ &= |x| \\ &= x, \ \text{since } x > 0 . \end{aligned}$$

Multiplying each side of the differential equation by $x$ gives

$$x\frac{\mathrm{d}u}{\mathrm{d}x} + u = 4x ,$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(ux) = 4x .$$

Hence $$ux = 2x^2 + A ,$$

where $A$ is an arbitrary constant. This can be expressed as

$$u = 2x + \frac{A}{x} ,$$

as required.

**(b)** In order to find the particular solution for $y$ satisfying the given boundary conditions, the general solution for $y$ is needed. Since $u = \frac{dy}{dx}$, we have from part (a)

$$\frac{dy}{dx} = 2x + \frac{A}{x}.$$

Hence $$y = x^2 + A \ln x + B,$$

where $B$ is an arbitrary constant. This is the general solution for $y$.

Applying the conditions that $\frac{dy}{dx} = 0$ and $y = 0$ when $x = 1$ gives $0 = 2 + A$ and $0 = 1 + B$. Hence $A = -2$, $B = -1$ and therefore the solution required is

$$y = x^2 - 2\ln x - 1.$$

**Example 5.7.3**

(a) Show that the substitution $x = e^t$ transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{y}{x^2} = 5$$

into $$\frac{d^2y}{dt^2} + y = 5e^{2t}.$$

(b) Find the general solution for $y$ in terms of $t$.

(c) Given that $y = 3$ and $\frac{dy}{dx} = 3$ when $x = 1$, find $y$ in terms of $x$.

**Solution**

**(a)** Using the chain rule for derivatives,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

Now $$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{e^t} = e^{-t}.$$ Hence

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt}.$$

Differentiating each side of this equation with respect to $x$ and using the chain rule again gives

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left[e^{-t}\frac{dy}{dt}\right] \times \frac{dt}{dx}\\ &= \left[e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right] \times e^{-t}\\ &= e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}.\end{aligned}$$

Substituting the above expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the given differential equation and putting $x = e^t$ gives

$$e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt} + \frac{1}{e^t}\left(e^{-t}\frac{dy}{dt}\right) + \frac{y}{e^{2t}} = 5.$$

Multiplying this throughout by $e^{2t}$ it becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + y = 5e^{2t}.$$

Hence $$\frac{d^2y}{dt^2} + y = 5e^{2t}.$$

**(b)** The GS of the differential equation above is obtained by finding the CF and a PI. The reduced equation is

$$\frac{d^2y}{dt^2} + y = 0 .$$

This is a standard form met before (see page 108) with GS

$$y = A\cos t + B\sin t$$

where $A$ and $B$ are arbitrary constants. This is the required CF.

Since the CF does not contain a term of the form *constant* $\times\, e^{2t}$, the appropriate trial function for a PI is $y = ae^{2t}$. Differentiating this twice gives

$$\frac{dy}{dt} = 2ae^{2t} , \quad \frac{d^2y}{dt^2} = 4ae^{2t} .$$

Substituting the above expressions for $\frac{d^2y}{dt^2}$ and $y$ into the complete differential equation for $y$ in terms of $t$ gives

$$4ae^{2t} + ae^{2t} = 5e^{2t}.$$

It follows that $a = 1$ and the PI is therefore $y = e^{2t}$. The general solution for $y$ is obtained by adding the CF and PI giving

$$y = A\cos t + B\sin t + e^{2t} .$$

**(c)** Before the given boundary conditions can be applied, $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be expressed in terms of $x$. Since $x = \mathrm{e}^t$, $t = \ln x$. Hence

$$y = A\cos(\ln x) + B\sin(\ln x) + x^2 .$$

Also $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{A}{x}\sin(\ln x) + \frac{B}{x}\cos(\ln x) + 2x .$$

Applying the condition $y = 3$ when $x = 1$ gives

$$3 = A\cos 0 + B\sin 0 + 1 = A + 1 ;$$

and applying the condition $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 1$ gives

$$3 = -A\sin 0 + B\cos 0 + 2 = B + 2 .$$

Hence $A = 2$ and $B = 1$. The required solution is therefore

$$y = 2\cos(\ln x) + \sin(\ln x) + x^2 .$$

**Exercise 5F**

1. The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x-2}\frac{dy}{dx} = 0,$$

where $x > 2$.

(a) Show that the substitution $\frac{dy}{dx} = u$ reduces the differential equation to

$$\frac{du}{dx} - \frac{u}{x-2} = 0.$$

(b) Find the general solution for $u(x)$.

(c) Hence find the general solution for $y(x)$.

2. The function $y(x)$ satisfies the differential equation

$$(\sin x)\frac{d^2y}{dx^2} - 2(\cos x)\frac{dy}{dx} = 0, \ 0 < x < \pi.$$

(a) Use the substitution $\frac{dy}{dx} = u$ to transform the differential equation to one of first order in $u$. Find the general solution for $u$ and show that it can be expressed as

$$u = C(1 - \cos 2x)$$

where $C$ is an arbitrary constant.

(b) Given that $y = \pi$ and $\frac{dy}{dx} = 1$ when $x = \frac{\pi}{2}$, find $y(x)$.

3. (a) Use the substitution $\frac{dy}{dx} = u$ to transform the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{3x}$$

into a first order linear differential equation in $u$.

(b) Obtain an integrating factor for the differential equation in $u$, and hence show that the general solution is

$$u = e^{3x} + Ae^{2x},$$

where $A$ is an arbitrary constant.

(c) Given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$, find $y$ in terms of $x$.

4. The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - (\cot x)\frac{dy}{dx} = 2\sin^2 x,$$

where $0 < x < \pi$.

(a) Use the substitution $\frac{dy}{dx} = u$ to transform the differential equation into a first order linear differential equation in $u$.

(b) (i) Obtain an integrating factor of the differential equation in $u$.

(ii) Hence show that the general solution is

$$u = -2\sin x\cos x + A\sin x,$$

where $A$ is an arbitrary constant.

(c) Obtain the general solution for $y(x)$.

5. (a) Show that the substitution $x = \frac{1}{t}$ transforms the differential equation

$$x^4 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x^3 \frac{\mathrm{d}y}{\mathrm{d}x} - y = \frac{1}{x}$$

into
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y = t\,.$$

(b) (i) Find the general solution for $y$ in terms of $t$.

(ii) Hence find the general solution for $y$ in terms of $x$.

6 (a) Show that the substitution $y = ux$ transforms the differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x(2x-1)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(5x^2 - 4x + 2\right)y = 0$$

into
$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 4\frac{\mathrm{d}u}{\mathrm{d}x} + 5u = 0\,.$$

(b) Find the general solution for $u$.

(c) Show that $y \to 0$ as $x \to \infty$.

**Miscellaneous exercises 5**

1. The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 6e^x.$$

(a) Find the complementary function.

(b) Find a particular integral.

(c) Hence find the solution for $y(x)$ which is such that $y = 0$ and $\frac{dy}{dx} = 3$ when $x = 0$.

2. Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

subject to the conditions that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

3. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{-x}.$$

4. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2e^{-2x},$$

given that $y = 0$ when $x = 0$ and that $y \to 2$ as $x \to \infty$.

[JMB, 1989]

5. (a) Show that the differential equation

$$\frac{d^2y}{dx^2} + 4y = 12\cos 2x$$

has a particular integral of the form $Cx\sin 2x$, where $C$ is a constant.

(b) Solve the differential equation given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

[NEAB, 1995]

6. The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - 4y = 4x^2.$$

(a) Find a particular integral.

(b) Find the general solution for $y(x)$.

7. The function $y(x)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} = 0,$$

where $x > 0$.

(a) Use the substitution $\frac{dy}{dx} = u$ to transform the differential equation to one of first order in $u$. Obtain the general solution of this differential equation and show that it can be expressed as

$$u = Axe^{-x},$$

where $A$ is an arbitrary constant.

(b) (i) Find the general solution for $y(x)$.

(ii) Obtain the particular solution for $y(x)$ which satisfies the conditions $y(0) = 0$ and $y(x) \to 1$ as $x \to \infty$.

8. The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 1,$$

where $x > 0$.

(a) Use the substitution $\frac{dy}{dx} = u$ to transform the differential equation to one of first order in $u$. Find the general solution for $u$.

(b) Deduce that the general solution for $y(x)$ is

$$y(x) = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{1}{2}Ax^2 + B,$$

where $A$ and $B$ are arbitrary constants.

(c) Find the solution for $y(x)$ which is such that $y(1) = 0$ and $y(x) \to 1$ as $x \to 0$.

9. (a) Show that the substitution $x = t^2$ transforms the differential equation

$$4x\frac{d^2y}{dx^2} + 2\left(1 - 2\sqrt{x}\right)\frac{dy}{dx} + y = 3\sqrt{x}$$

into

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 3t.$$

(b) Find the general solution for $y$ in terms of $x$.

# Answers to Exercises – Further Pure 3

## Chapter 1

**Exercise 1A**

1. (a) 1 (b) $-3$ (c) $0+$ (d) $0-$

2. $\ln x \to -\infty$ as $x \to 0$ and limits must be finite.

3. (a) $\frac{\pi}{4}$ (b) 0

4. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 0

**Exercise 1B**

1. $1+x+\frac{x^2}{2}+\frac{x^3}{6}$ ; general term $=\frac{x^r}{r!}$.

2. $1-x+\frac{x^2}{2}-\frac{x^3}{6}$

3. $(-1)^r\frac{x^{2r}}{(2r)!}$ , $r=0,1,2,\ldots.$

5. $2+x^2+\frac{x^4}{12}$ ; general term$=\frac{2x^{2r}}{(2r)!}$ , $r=0,1,2,\ldots$ .

**Exercise 1C**

1. 0.540302

2. (a) 0.4794

3. (a) $2-\frac{4}{3}x^2+\frac{4}{15}x^4$ (b) $1-\frac{9}{2}x^2+\frac{27}{8}x^4$ (c) $x-\frac{4}{3}x^2+\frac{20}{9}x^3$
   (d) $1-3x+\frac{9}{2}x^2$ (e) $x+\frac{1}{2}x^2-\frac{2}{3}x^3$

4. (a) $-x-\frac{1}{2}x^2-\frac{1}{3}x^3$ (b) $1-5x+\frac{13}{2}x^2+\frac{7}{6}x^3$
   (c) $1+2x^2-\frac{5}{3}x^3$ (d) $x-\frac{7}{2}x^2+\frac{65}{6}x^3$

5. (a) $-1\le x<1$ (b) all $x$ (c) all $x$ (d) $-\frac{1}{3}<x<\frac{1}{3}$

**Exercise 1D**

1. (a) $1$ (b) $3$ (c) $\frac{1}{2}$ (d) $2$

2. (b) $-\frac{1}{2}$

3. $\frac{1}{4}\sqrt{2}$

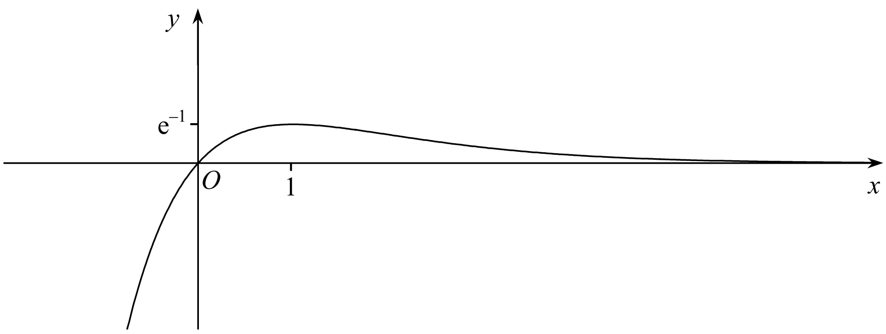
4. (a) $1+(\ln 2)x+\frac{1}{2}(\ln 2)^2 x^2$ (b) $\frac{\ln 2}{\ln 3}$

5. $\frac{3}{2}$

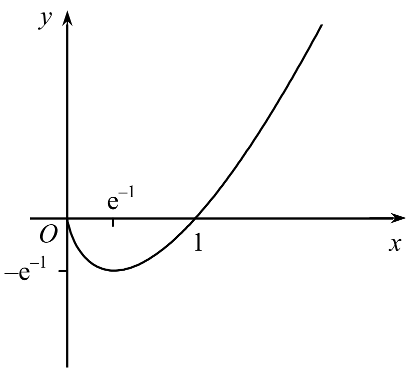
**Exercise 1E**

1. Each of these limits is zero

4. (b)




5. (b)




**Exercise 1F**

1. (a) The interval of integration is infinite (b) $\dfrac{\ln x}{x^{\frac{1}{2}}}$ is not defined at $x=0$

(c) $\dfrac{1}{\sqrt{1-x^2}}$ is not defined at $x=1$

**Exercise 1G**

1. (a) The first integral exists. The second does not. (b) $\frac{1}{2}$

2. (a) $\lim_{a\to 0}\left(\sin^{-1}1-\sin^{-1}a\right)=\frac{\pi}{2}$ (b) $\lim_{a\to\infty}\left(-\frac{1}{1+a}+1\right)=1$

   (c) $\lim_{a\to\infty}\left(-a\mathrm{e}^{-a}-\mathrm{e}^{-a}+1\right)=1$ (d) $\lim_{a\to-\infty}\left(1-\frac{2}{(4-a)^{\frac{1}{2}}}\right)=1$

   (e) $\lim_{a\to 0}\left(-\frac{1}{3}a^3\ln a-\frac{1}{9}+\frac{1}{9}a^3\right)=-\frac{1}{9}$ (f) $\lim_{a\to 0}\left(-a\ln a+a\right)=0$

3. (a) (i) The interval of integration is infinite
   (ii) $\frac{x}{1-x^2}$ is not defined at $x=1$

**Miscellaneous exercises 1**

4. (a) $x+x^2-\frac{1}{3}x^3$ (b) $-1<x\le\frac{1}{2}$

5. (a) $1-\frac{9}{2}x^3$ (b) $-\frac{1}{3}<x\le\frac{1}{3}$

6. (a)(i) $x^2+x^3+\frac{1}{2}x^4$ (ii) $1-2x^2+\frac{2}{3}x^4$ (b) $-\frac{1}{2}$

7. $-2$

8. (a) $0$ (b) $-1$

10. (a) $a-a\ln a-1$

(b) The integral is improper because $\ln x$ is not defined at $x=0$.
The integral exists because $(a-a\ln a-1)\to -1$ as $a\to 0$.

11. (a) $1-\frac{1}{2}x^2+\frac{1}{6}x^4$ (b) $\frac{1}{2}\mathrm{e}$

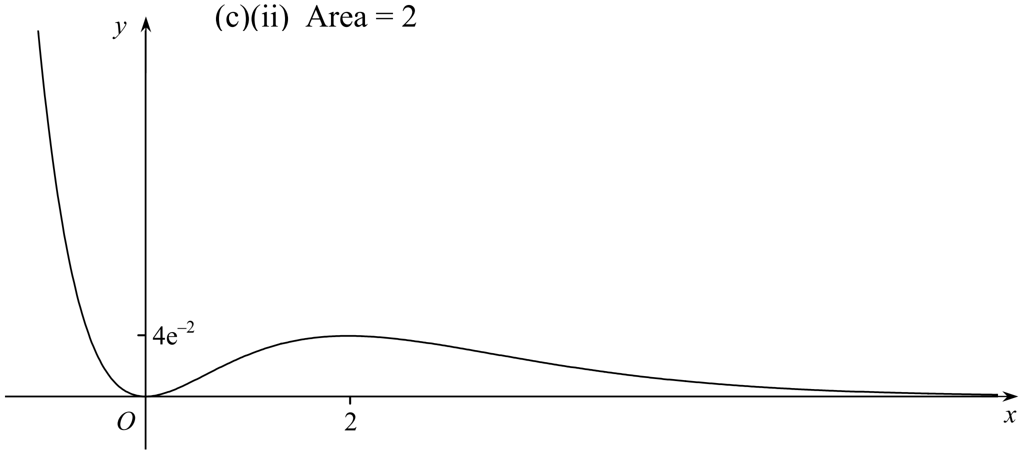
12 (a) $\frac{1}{2}$ (b) $\frac{3}{2}$; the integral is improper because the interval of integration is infinite

13. (a) $1+2x+4x^2$ (b) $2x+2x^2+\frac{14}{3}x^3$ (c) 1

14. $I=\int_a^1\frac{1}{x}\ln x\,\mathrm{d}x=-\frac{1}{2}(\ln a)^2\to-\infty$ as $a\to 0$. Hence I does not exist.

$$J=\lim_{a\to 0}\int_a^1\frac{1}{\sqrt{x}}\ln x\,\mathrm{d}x=\lim_{a\to 0}\left(-2a^{\frac{1}{2}}\ln a-4+4a^{\frac{1}{2}}\right)=-4$$

15. (b) (c)(ii) Area = 2




16. (a)(i) $1+\frac{1}{2}x+\frac{3}{8}x^2$ (b) $-1$ (c)(i) $p=1$ (ii) 1

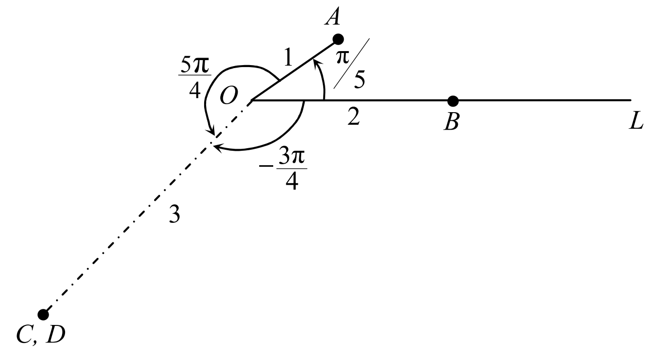
17. (a)(i) $2x-\frac{4}{3}x^3+\frac{4}{15}x^5$; $x-\frac{1}{2}x^2+\frac{1}{6}x^3$

(b)(ii) $\frac{1}{x}+\frac{1}{2}+\frac{1}{12}x$ (iii) $-\frac{1}{2}$

## Chapter 2

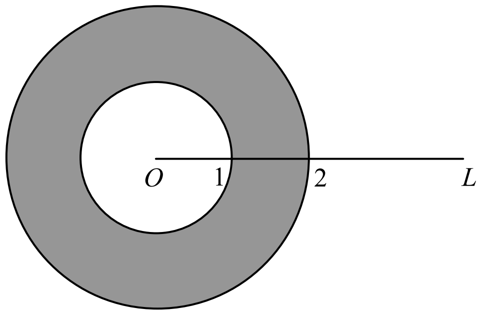
### Exercise 2A

1.

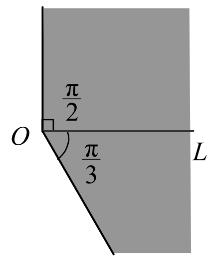



2. (a) $\frac{2\pi}{3}$ (b) $\sqrt{19}$

3. (a)




(b)




### Exercise 2B

1. (b) $A\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$, $B\left(2, -2\sqrt{3}\right)$

2. (a) $\left(2\sqrt{2}, \frac{\pi}{4}\right)$ (b) $\left(2, \frac{2\pi}{3}\right)$ (c) $r = 5,\quad \theta \approx 4.07$

3. 2

### Exercise 2C

1. $r = 2a\sin\theta,\quad 0 \le \theta \le \pi$

2. $r = 2a\cos\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

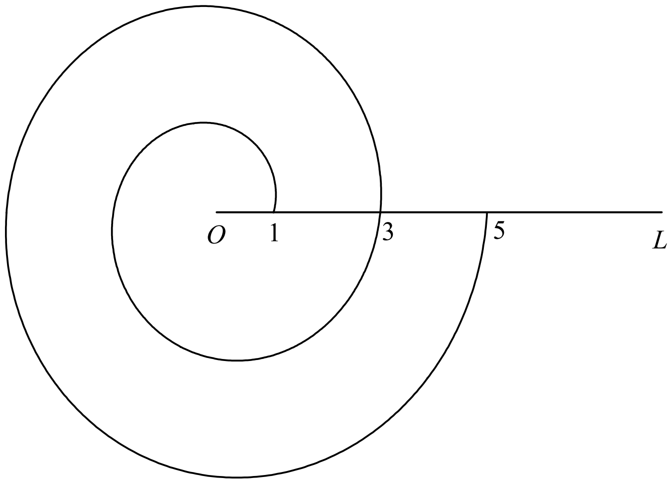
3. $r = a\sec\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

**Exercise 2D**

1. (a) A **rough** sketch should show a curve in the form of a spiral around the pole *O*.
   It should start at the point (1, 0), pass through the point (3, 0) and end at the point (5, 0).

   (b)

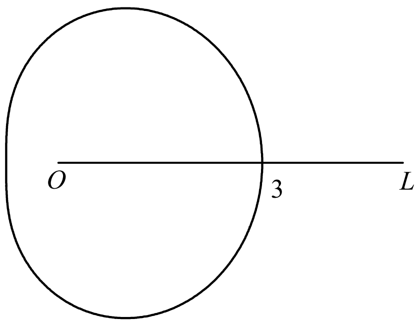
| $\theta$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ | $\frac{5\pi}{2}$ | $3\pi$ | $\frac{7\pi}{2}$ | $4\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |



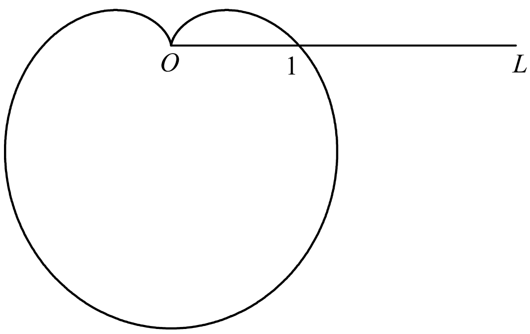

2. (a)

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | 3 | 2.87 | 2.5 | 2 | 1.5 | 1.13 | 1 |

   (b)

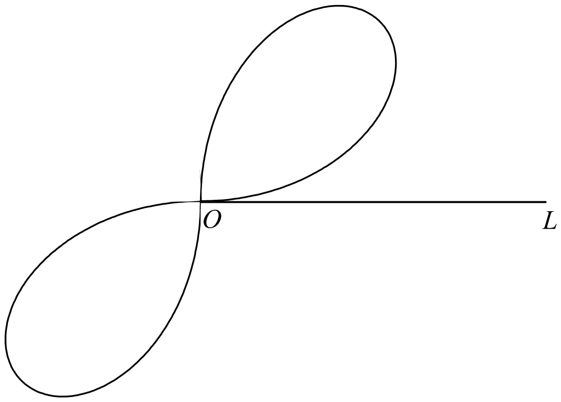



3. (a)

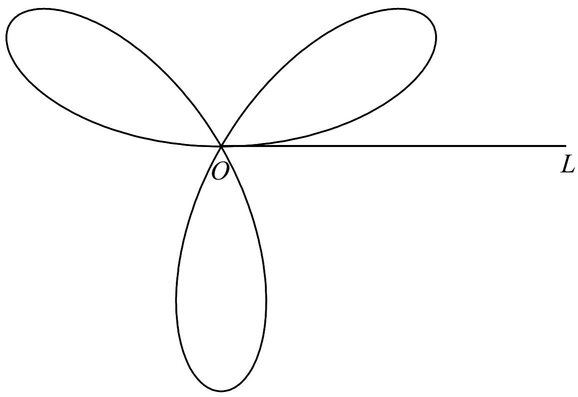



   (b) $\theta = \frac{\pi}{2}$

4. (b)

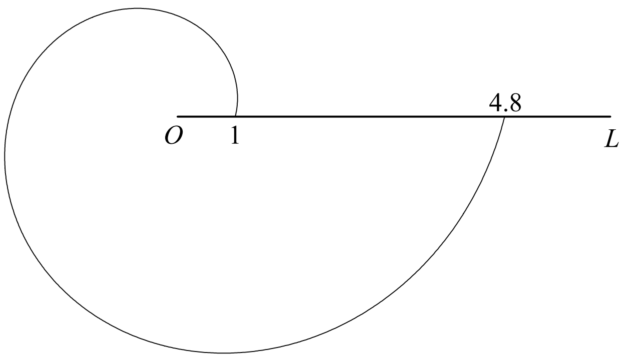



5. (a)




(b) $\theta = 0,\quad \frac{\pi}{3},\quad \frac{2\pi}{3},\quad \pi,\quad \frac{4\pi}{3},\quad \frac{5\pi}{3}$

6.



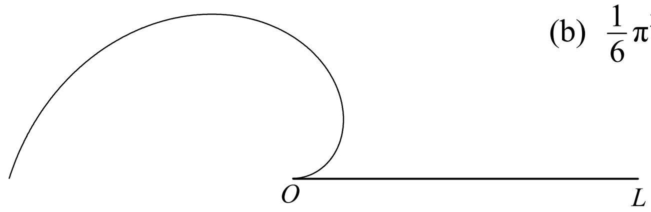

**Exercise 2E**

1. It is wrong because $r$ is negative in two parts of the interval $-\pi < \theta \leq \pi$.

2. (a) $r = a$

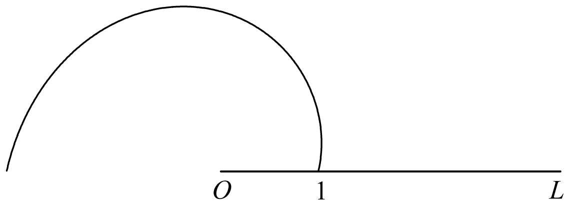
**Exercise 2F**

1. (a)




(b) $\frac{1}{6}\pi^3$

2. (a)

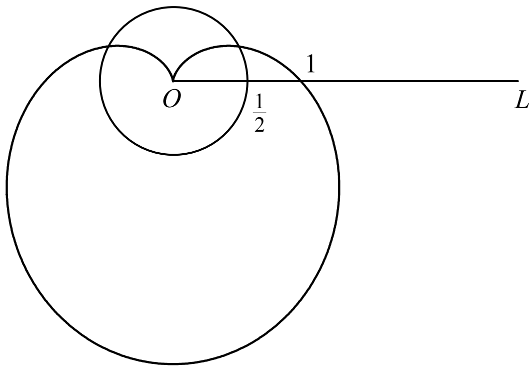



(b) $e^{\frac{\pi}{2}} - 1$

3. $\frac{9\pi}{2}$

4. $\frac{\pi a^2}{8}$

5. (a)



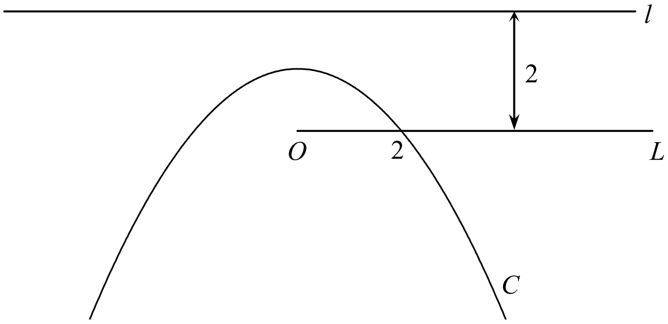

(b) $\left(\frac{1}{2}, \frac{\pi}{6}\right), \quad \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

(c) $\frac{2\pi}{3} - \frac{7\sqrt{3}}{8}$

**Miscellaneous exercises 2**

1. Area $= \dfrac{1}{2}\ln 2$

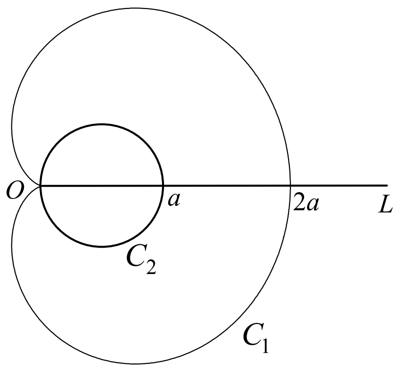
2. (a)




3. (a) $x = \tan 2\theta \cos\theta, \quad y = \tan 2\theta \sin\theta$ (b)(i) $\dfrac{3\sqrt{3}}{8}$

4. (b)(ii) $r = \dfrac{2}{1+\cos\theta}$

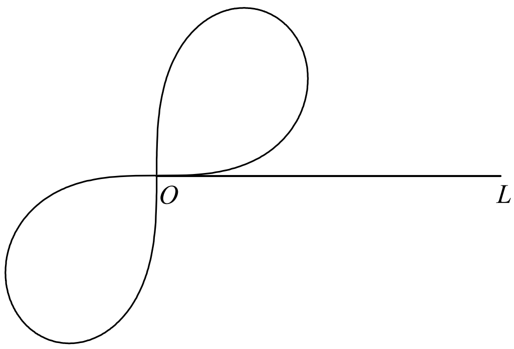
5. (a) and (b)(iii)




6. (b)(i) There is no curve when $\dfrac{\pi}{2} < \theta < \pi$ and $-\dfrac{\pi}{2} < \theta < 0$ because in these intervals $\sin 2\theta < 0$.

(ii) $\left(a, \dfrac{\pi}{4}\right)$, $\left(a, -\dfrac{3\pi}{4}\right)$

(iii)




(c) $2a^2$ (d) $\dfrac{\pi}{6}$

## Chapter 3

**Exercise 3A**

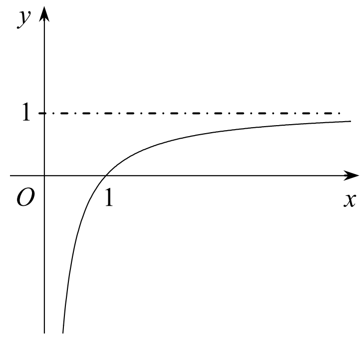
1. (a) 1 (b) 2 (c) 1 (d) 1

2. Equations (a), (b) and (c) are linear.

**Exercise 3B**

2. (a) $y=\dfrac{1}{\frac{1}{2}x^2+C}$ (b) $y=\dfrac{2}{1+x^2}$

3. (a) $y=-\dfrac{1}{x}+C$ (b) $y=1-\dfrac{1}{x}$ (c)




4. (a) $y=3\mathrm{e}^{-x}$

5. (a) $x=-\dfrac{250}{5+t}+C$ (b)(i) $x(0)=0$ (ii) $x=\dfrac{50t}{5+t}$

**Exercise 3C**

1. (a) $x^2$ (b) $y=x^2+\dfrac{C}{x^2}$

2. (a) $y=\dfrac{1}{4}\mathrm{e}^{2x}+C\mathrm{e}^{-2x}$ (b) $y=\dfrac{1}{4}\left(\mathrm{e}^{2x}-\mathrm{e}^{-2x}\right)$

3. $y=\dfrac{\ln x}{x}$

4. $y=x^2(x+C)$

5. (b)(i) $y=1-x\cot x+C\operatorname{cosec} x$ (ii) $y=1-x\cot x-\operatorname{cosec} x$

6. $y=\tan x+\sec x$

7. (a) $y=-\dfrac{1}{2}x\mathrm{e}^{-2x}+Cx$ (b) $y=-\dfrac{1}{2}x\mathrm{e}^{-2x}$

8. $y=x+1+\mathrm{e}^{-x}$

**Exercise 3D**

1. (a) $3x^2 - 2x + 1$ (b) $-x + 1$ (c) $\frac{1}{2}(\sin x + \cos x)$ (d) $\frac{4}{5}\sin x - \frac{2}{5}\cos x$
   (e) $\frac{1}{2}e^{3x}$ (f) $3xe^{3x}$ (g) $2x - 1 + \frac{1}{2}e^{-x}$

2. (a) $y_C = Ce^{3x}$, $y_P = -2$ (b) $y = 2\left(e^{3(x-1)} - 1\right)$

3. (a) $y_C = Ce^{2x}$, $y_P = xe^{2x}$ (b) $y = (2 + x)e^{2x}$

4. $y = \frac{1}{5}\left(7e^{-x} - 2\cos 2x + \sin 2x\right)$

**Exercise 3E**

1. (b) $u = \dfrac{x^2 + C}{x}$, $y = \dfrac{x}{x^2 + C}$

2. (b) $u = -\ln(x + C)$ (c) $y = x - \ln(x + 1)$ (d) $x > -1$

3. (b) $u = \dfrac{x + C}{x^2}$ (c) $y = \ln x$

**Miscellaneous exercises 3**

1. (b) $y = \left(3 + x^2\right)\cos x$

2. (a)(i) $Ae^{-x}$ (ii) $\frac{1}{2}(\sin x - \cos x)$ (b) $y = \frac{3}{2}e^{-x} + \frac{1}{2}(\sin x - \cos x)$

3. (a) $y = Ce^{-x} + xe^{-x}$ (b)(i) $y = (x - 1)e^{-x}$ (ii) $y = 0$ (because as $x \to \infty$, $y \to 0$)

4. (b) $y = (x - 1)^2 + (x - 1)\ln(x - 1)$

5. (a) $a = 4, b = 6, c = 6, d = -3$ (b) $Ce^{-2x}$ (c) $y = 3e^{-2x} + 4x^3 - 6x^2 + 6x - 3$

6. (b) $y = -2x - x\ln x$
   (c) when $x \to 0, x\ln x \to 0$ and therefore $y \to 0$. The curve will therefore approach the point $(0,0)$, i.e. the origin.

7. (b) $u = 1 + \ln x + Cx$ (c) $y = \dfrac{1}{1 + x + \ln x}$

8. (b) $y = \dfrac{x}{\left(x^6 + C\right)^{\frac{1}{3}}}$

## Chapter 4

**Exercise 4A**

(a) $x_0 = 1,\ y_0 = 2,\ h = 0.1,\ x_1 = 1.1,\ x_2 = 1.2$ (b) $y(1.1),\ y(2.0)$ (c) $\mathrm{f}(x_0, y_0) = 9$

**Exercise 4B**

1. 1.069, 1.147, 1.232, 1.325, 1.425

2. $y(1) = 2.230$

3. (a) $h = 0.0125$ (c) $y_3 = 0.160$

**Exercise 4C**

1. (a) 1.2, 1.5059 (b) 1.6118

2. (b) $y(2) = 1.204$

3. (b) $y(0.5) = 0.539$. (The intermediate values are $y_1 = 0.1,\ y_2 = 0.20199,\ y_3 = 0.30792$ and $y_4 = 0.41969$)

**Exercise 4D**

1. (a) $y(3) = 2.305$ (The intermediate values are $y_1^* = 1.7329,\ y_1 = 1.20799;\ y_2^* = 1.45796,$ $y_2 = 1.49464;\ y_3^* = 1.82418,\ y_3 = 1.86100;\ y_4^* = 2.26918$)

   (b) $y(3) = 2.305$. (The values of $k_1$ and $k_2$ in the working are as follows:
   first step: $k_1 = 0.17329,\ k_2 = 0.24269$
   second step: $k_1 = 0.24997,\ k_2 = 0.32333$
   third step: $k_1 = 0.32954,\ k_2 = 0.40318$
   fourth step: $k_1 = 0.40818,\ k_2 = 0.47951$)

2. $y(0.4) = 0.458$. (The value of $y_1$ is 0.20833)

3. (a) $y(1.1) = 1.01732$ (b) 1.02216 (c) 1.02238

**Exercise 4E**

1. (a) $y(1.4) = 0.867$ (b) 0.903 (c) 0.933 (d) 0.934

   (f) $-8.0\%$, $-4.1\%$, $-1.0\%$, $-0.8\%$

   (g) The advantage of reducing the step length is shown in parts (a) and (b). The error has been halved.
   Parts (c) and (d) show that the mid-point formula and the improved Euler method give better accuracy than the Euler method with the same step length (0.2). This is because the truncation error in Euler's formula is greater than in the formulae for the other two methods.

3. (a) $y(1.4) = 0.9$

**Miscellaneous exercises 4**

1. (b) $y(1.3) = 0.899$

2. (b)(ii) $a = 2,\ b = 1$

3. (a)(ii) $y = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$ (b) 0.60748 (c) 0.48% (approximately)

4. (a) $C(t) = t\,\mathrm{e}^{-2t}$ (The differential equation is of first order linear form and is solved using the methods given in Chapter 3)

   (b)(i) $C(0.1) = 0.1$ (ii) $C(0.2) \approx 0.158$, $C(0.3) \approx 0.229$

5. (a) 1.435794 (b) 1.217897 (c)(i) $y(x) = \mathrm{e}^{x}$ (ii) $-11.5\%$, $-2.2\%$

6. (b) 2.784 (c) Reduce the step length in both parts (a) and (b).

## Chapter 5

**Exercise 5A**

1. (a) $1+4\mathrm{i}$ (b) $7$ (c) $5-5\mathrm{i}$
   (d) $-5-2\mathrm{i}$ (e) $11+10\mathrm{i}$ (f) $-50\mathrm{i}$

3. (a) $-1, 7$ (b) $-1, 1$

4. $x = 3 \pm 2\mathrm{i}$

5. (a) $-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$ and $-\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}$

**Exercise 5B**

2. $4\mathrm{e}^{x}\cos 2x, \quad -2\mathrm{e}^{x}\sin 2x$

3. (a) $C = A+B, \quad D = (A-B)\mathrm{i}$ (b) $C = 2\alpha, \quad D = -2\beta$

**Exercise 5C**

2. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = A\mathrm{e}^{x}(\cos x - \sin x) + B\mathrm{e}^{x}(\cos x + \sin x)$,
   $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2A\mathrm{e}^{x}\sin x + 2B\mathrm{e}^{x}\cos x$

3. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4$

**Exercise 5D**

1. (a) $y = A\cos 3x + B\sin 3x$ (b) $y = \cos 3x - 2\sin 3x$

2. (a) $y = Ae^{3x} + Be^{-2x}$ (b) $y = (Ax+B)e^{\frac{x}{3}}$

   (c) $y = e^{3x}(A\cos x + B\sin x)$ (d) $y = (Ax+B)e^{-\frac{3x}{2}}$

   (e) $y = e^{-\frac{3x}{2}}\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right)$ (f) $y = A + Be^{6x}$

3. (a) $y = Ae^{\frac{x}{2}} + Be^{-\frac{x}{2}}$ (b) $y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$

4. $y = (1-x)e^{2x}$

5. $y = \frac{1}{2}e^{x}\sin 2x$

**Exercise 5E**

1. (a) PI: $y = \frac{1}{9}e^{-x}$; GS: $y = (A+Bx)e^{2x} + \frac{1}{9}e^{-x}$

   (b) PI: $y = x^2 + 2x + 3$; GS: $y = (A+Bx)e^{2x} + x^2 + 2x + 3$

   (c) PI: $y = 4\cos x + 3\sin x$; GS: $y = (A+Bx)e^{2x} + 4\cos x + 3\sin x$

2. (a) $y = (A+Bx)e^{3x}$ (b) $a = \frac{1}{2}$ (c) $y = (A+Bx)e^{3x} + \frac{1}{2}x^2e^{3x}$

3. $y = x^4 - 4x^3 + 24x - 24$

4. $y = 2\left(e^{-x} - 1\right)$

5. $y = e^{2x}(\sin x - \cos x) + \sin x + \cos x$

6. (a) $y = A\cos 4x + B\sin 4x$ (b) $a = -2$ (c) $y = \frac{3}{4}\sin 4x - 2x\cos 4x$

**Exercise 5F**

1. (b) $u(x) = A(x-2)$ (c) $y(x) = A(\frac{1}{2}x^2 - 2x) + B$

2. (a) $(\sin x)\frac{\mathrm{d}u}{\mathrm{d}x} - (2\cos x)u = 0$, GS is $u = A\sin^2 x$

   (b) $y(x) = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + \frac{3}{4}\pi$

3. (a) $\frac{\mathrm{d}u}{\mathrm{d}x} - 2u = \mathrm{e}^{3x}$ (b)(i) $\mathrm{e}^{-2x}$ (c) $y(x) = \frac{1}{3}\mathrm{e}^{3x} - \frac{1}{2}\mathrm{e}^{2x} + \frac{1}{6}$

4. (a) $\frac{\mathrm{d}u}{\mathrm{d}x} - (\cot x)u = 2\sin^2 x$ (b)(i) $\frac{1}{\sin x}$ (c) $\mathrm{y}(x) = -\sin^2 x - A\cos x + B$

5. (b)(i) $y = A\mathrm{e}^{t} + B\mathrm{e}^{-t} - t$ (ii) $y = A\mathrm{e}^{\frac{1}{x}} + B\mathrm{e}^{-\frac{1}{x}} - \frac{1}{x}$

6. (b) $u = \mathrm{e}^{-2x}(A\cos x + B\sin x)$

   (c) The GS is $y = x\mathrm{e}^{-2x}(A\cos x + B\sin x)$. For all $x$, $A\cos x + B\sin x$ is finite.

   As $x \to \infty$, $x\mathrm{e}^{-2x} \to 0$. Hence $y \to 0$ as $x \to \infty$.

**Miscellaneous exercise 5**

1. (a) $y_C = Ae^{4x} + Be^{-x}$ (b) $y_P = -e^x$ (c) $y = e^{4x} - e^x$

2. $y = e^{2x}(\cos x - 2\sin x)$

3. $y = e^{-x}(A\cos 3x + B\sin 3x) + \frac{1}{3}e^{-x}$

4. $y = 2 - 2e^{-2x} - xe^{-2x}$

5. (a) $C = 3$ (b) $y = \cos 2x + \sin 2x + 3x\sin 2x$

6. (a) $y_P = -x^2 - \frac{1}{2}$ (b) $y = Ae^{2x} + Be^{-2x} - x^2 - \frac{1}{2}$

7. (b)(i) $y(x) = A(x+1)e^{-x} + B$ (ii) $y(x) = 1 - (x+1)e^{-x}$

8. (a) $\frac{du}{dx} - \frac{1}{x}u = 1$, $u = x\ln x + Ax$ (c) $y(x) = \frac{1}{2}x^2\ln x - x^2 + 1$

9. (b) $y = (A + B\sqrt{x})e^{\sqrt{x}} + 3\sqrt{x} + 6$