Further Pure 3 Summary Notes

1. Limits

The function f(x) → l when x → a is written

$\lim_{x \to a} f(x) = l$ (l is called the **limiting value**)

Look for the highest power of x (Divide by x^2)

Example

Find $\lim_{x\to\infty} \frac{3x^2+2x+4}{1-2x^2-3x}$

$$\frac{3x^2/x^2 + 2x/x^2 + 4/x^2}{1/x^2 - 2x^2/x^2 - 3x/x^2} = \frac{3 + 2/x + 4/x^2}{1/x^2 - 2 - x/3}$$

State clearly $\lim_{x\to\infty} \frac{1}{x} = 0$ *and* $\lim_{x\to\infty} \frac{1}{x^2} = 0$

$$\lim_{x\to\infty} \frac{3x^2+2x+4}{1-2x^2-3x} = -\frac{3}{2}$$

Important limits – may be quoted

$$\lim_{x\to\infty} x^k e^{-x} = 0$$

$$\lim_{x\to 0} x^k lnx = 0$$

2. Maclaurin's Series

$$\boldsymbol{f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \ldots\ldots\ldots}$$

Assumption

- f(x) can be expressed as f(x) = $a_0 + a_1x + a_2x^2 + a_3x^3$.......
- The series can be differentiated term by term
- The function f(x) and all of its derivatives exist at x = 0

The following are given in the formula book

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

You may need to use the BINOMIAL SERIES too

Use brackets when substituting e.g (-3x) to avoid errors when using 'powers'

Example
Find the Maclaurin expansion of f(x) = ln(1 + 2x) up to the x^3 term.
State the range of values for which it is valid

ln(1 + x) = $x - \frac{x^2}{2} + \frac{x^3}{3}$

ln(1 + 2x) = $(2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3}$

= $2x - 2x^2 + \frac{8x^3}{3}$

ln(1 + x) valid for -1 < x < 1

ln(1 + 2x) valid for -1 < 2x < 1 so -½ < x < ½

3. Improper Integrals

An integral $\int_a^b f(x)dx$ is improper if

- the interval of integration is infinite
- f(x) is undefined at one or both of a/b
- *f(x) is not defined at one or more of the interior points (not considered in FP3)*

Example

$\int_0^e xlnx\ dx$ (Improper – not defined at lower boundary x = 0 - cannot have ln 0)

1. Replace undefined limit with c

$I = \int_0^c xlnx\ dx$

Integration by parts / substitution / partial fractions may be needed

2. Integrate and substitute in the boundary values

$I = \left[\frac{1}{2}x^2 lnx\right] - \int_0^c \frac{1}{2}x^2 \times \frac{1}{x}\ dx$

Remember Integration by parts
u = ln x $\frac{dv}{dx} = x$

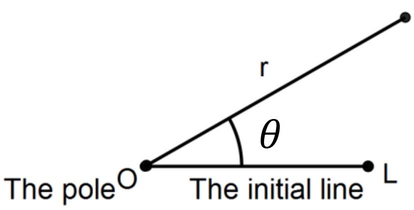
$= \frac{1}{2}e^2 - \frac{1}{2}c^2 lnc - \frac{1}{4}e^2 + \frac{1}{4}c^2$

$\lim_{x \to 0} c^2 lnc = 0 \qquad \lim_{x \to 0} c^2 = 0$

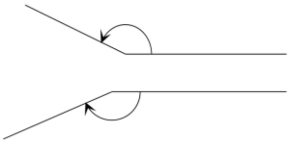
Make sure you show this clearly with the correct notation

$\int_0^e xlnx\ dx = \frac{1}{4}e^2$

4. Polar Coordinates and curves



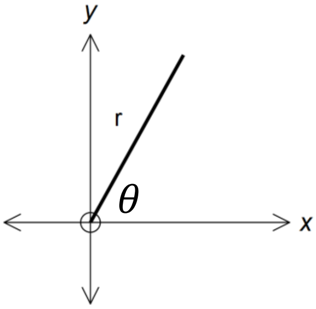
P has polar coordinate (r,θ)



Positive θ

Negative θ

Polar – Cartesian



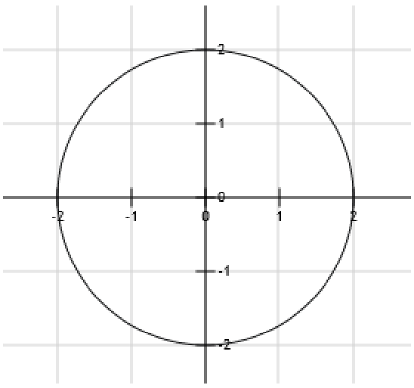
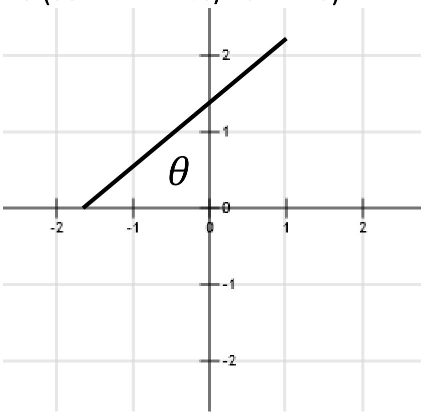
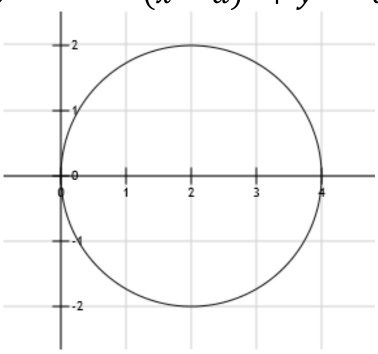
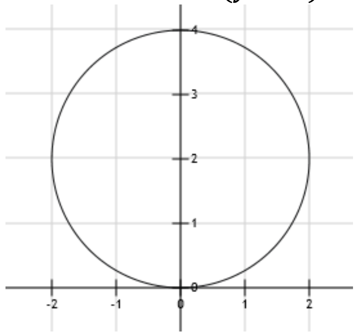
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

Use these when you need to change between polar and Cartesian equations

Curves and Graphs

r = a $x^2 + y^2 = a^2$ circle centre (0,0) radius a 	$\theta = \alpha$ Straight line (semi-infinite/half line) 
r = 2a cos θ $(x-a)^2 + y^2 = a^2$  **r=4cos θ**	**r = 2a sin θ** $x^2 + (y-a)^2 = a^2$  **r=4sin θ**

Hints for sketching curves

- if r is expressed as a function of cos only - **symmetrical about the initial line**
- if r is expressed as a function of sin only - **symmetrical about the line** $\boldsymbol{\theta = \frac{\pi}{2}}$
- if $r \to 0\ as\ \theta \to \infty$ then the line $\theta = \alpha$ is a tangent to the curve at the pole
- NEGATIVE VALUES of r are not allowed – if $r < 0$ in the interval $\alpha < \theta < \beta$ then there is no curve in this interval

Area bounded by a polar curve

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

r must be non-negative and defined for $\alpha < \theta < \beta$

If you need to find an area between curves use the values of θ at the point(s) of intersection for the limits

MAKE SURE YOU CAN INTEGRATE TRIG FUNCTIONS
You may need to use

$$\int tan^2\theta \, d\theta = \int (sec^2\theta - 1) d\theta = tan\theta - \theta$$

$$\int sin^2\theta = \int \frac{1}{2}(1 - cos2\theta) = \frac{1}{2}(\theta - \frac{1}{2}sin2\theta)$$

$$\int sin^2 2\theta \, d\theta = \int \frac{1}{2}(1 - cos4\theta) d\theta = \frac{1}{2}(\theta - \frac{1}{4}sin4\theta)$$

$$\int sin^n\theta cos\theta \, d\theta = \frac{1}{n+1} sin^{n+1}\theta$$

5. First order differential equations

Linear 1st order differential equation $\frac{dy}{dx} - \frac{y}{x} = 2x^2$ (dependent variable ‘y’ is linear)

Non-Linear 1st order differential equation $y\frac{dy}{dx} = 2x + y^2$ (dependent variable ‘y’ is quadratic)

A **General solution** has unknown variables and represents a family of solutions e.g $y = Ae^{x^2}$

To find a **Particular solution** boundary or initial conditions are used to evaluate unknown variables

e.g. when y = 2 x = 0 or y(0) = 2 both mean the same thing

METHOD 1 : Separating the variables (seen in Core 4)

$\frac{dy}{dx} = 3x^2 y$

$\frac{1}{y} dy = 3x^2\, dx$

$\int \frac{1}{y} dy = \int 3x^2\, dx$

$\ln|y| = x^3 + c$

$y = e^{x^3 + c}$

$y = Ae^{x^3}$

METHOD 2 : Using an Integrating Factor

Step 1 : Check that the equation is in the form $\frac{dy}{dx} + P(x)y = Q(x)$

$x\frac{dy}{dx} - 2y = x^3$

$\frac{dy}{dx} - \frac{2y}{x} = x^2$ P(x) = $-\frac{2}{x}$ Q(x) = x^2

Take care with the 'sign' of P(x)

Step 2 : Find the 'Integrating Factor' $\boldsymbol{I(x) = e^{\int P(x)dx}}$

$\int P(x)dx = \int \frac{-2}{x}\, dx = -2\ln|x|$

$I(x) = e^{-2\ln|x|} = e^{\ln x^{-2}} = \frac{1}{x^2}$

Step 3 : Multiply all terms by the Integrating Factor

$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$

The left hand side is the result of differentiating I(x)y using the product rule

$\frac{d}{dx}\left(\frac{y}{x^2}\right) = 1$

$\frac{d}{dx}(I(x)y) = I(x)Q(x)$
MAKE SURE YOU SHOW THIS CLEARLY

$\frac{y}{x^2} = \int 1\, dx$

$\frac{y}{x^2} = x + c$

$y = x^3 + cx^2$

Don't forget to include the 'c' in any rearrangement to isolate y e.g. cx^2

Step 4 : You have found the GENERAL solution – if necessary use values given to find a PARTICULAR solution

METHOD 3 : Complementary Functions and Particular Integrals

Step 1 : Check that the equation is in the form $a\frac{dy}{dx} + by = f(x)$

$\frac{dy}{dx} - 2y = e^{2x}$

Step 2 : Solve the REDUCED EQUATION $a\frac{dy}{dx} + by = 0$

$\frac{dy}{dx} - 2y = 0$

$\frac{1}{y}dy = 2\,dx$

$y = Ae^{2x}$ - this is the **COMPLEMENTARY FUNCTION**

Step 3 : Find a PARTICULAR Integral/solution of the complete equation $a\frac{dy}{dx} + by = f(x)$

Looking only at the f(x)

- If f(x) is of the form ce^{kx} try a particular solution of the form
  $y = ae^{kx}$ (or $y = axe^{kx}$ if the CF has the same exponential from as f(x)

- If f(x) is of the form $ccos\ kx\ \ or\ c\sin kx$ try a particular integral of the form
  $y = acoskx + bsin\ kx$

- If f(x) is a polynomial of degree n try a particular integral of the form
  $y = ax^n + bx^{n-1} + \cdots$

$\frac{dy}{dx} - 2y = e^{2x}$ ← Try $y = axe^{2x}$ $\frac{dy}{dx} = ae^{2x} + 2axe^{2x}$

$ae^{2x} + 2axe^{2x} - 2(axe^{2x}) = e^{2x}$ Substituting these into our original equation

$ae^{2x} = e^{2x}$
$ae^{2x} - e^{2x} = 0$
$e^{2x}(a-1) = 0$
$a = 1$ This gives a PARTICULAR integral/solution $y = xe^{2x}$

Step 4 : Write down the GENERAL SOLULTION y = CF + PI

$y = Ae^{2x} + xe^{2x}$

Step 5 : If boundary values are given to allow you to calculate A this must be done using y = CF + PI

6. Second order differential equations

Eulers Identity

$e^{ix} = cosx + isinx$ $e^{i\pi} = -1$

$e^{-ix} = cosx - isinx$

Solving equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

Step 1 : Solve the auxiliary equation $ak^2 + bk + c = 0$

Solve

Find the general solution to $2\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$

$$2k^2 + 6k + 5 = 0$$

$$k = \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$k = -\frac{3}{2} \pm \frac{1}{2}i$$

3 possibilities for values for

k_1 and k_2 real and unequal	$k_1 = k_2$ equal roots	$K_1 = p+qi$ $K_2 = p-qi$ (non real)
$y = Ae^{k_1x} + Be^{k_2x}$	$y = e^{k_1x}(A + Bx)$	$y = e^{px}(Acosqx + Bsinqx)$

Exceptional case if solving $\frac{d^2y}{dx^2} + n^2x = 0$ $\quad y = Acosnx + Bsinnx$

Step 2 : Use the chosen form of y for the general solution

General solution

$$y = e^{-\frac{3}{2}x}(Acos\frac{1}{2}x + Bsin\frac{1}{2}x)$$

Solving equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

Step 1 & 2 : Solve the auxiliary equation $ak^2 + bk + c = 0$ (as above)

This will find the COMPLEMENTARY Function , CF, y_c

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

Auxiliary equation

$$k^2 + 3k + 2 = 0$$

$$(k+1)(k+2) = 0$$

$K_1 = -1$ $K_2 = -2$

COMPLEMENTARY Function

$$y_c = Ae^{-x} + Be^{-2x}$$

Step 3 : **Finding a PARTICULAR solution of the complete equation (particular Integral) y_p**

You need to consider f(x) and y_c

$$y_c = Ae^{-x} + Be^{-2x} \qquad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

$$f(x) = 2e^{-2x}$$

$f(x) = ce^{\lambda x}$	y_c - contains $ae^{\lambda x}$ but **not** $axe^{\lambda x}$	$y = axe^{\lambda x}$
	y_c - contains $axe^{\lambda x}$	$y = ax^2e^{\lambda x}$
	y_c - does not contain $ae^{\lambda x}$ or $axe^{\lambda x}$	$y = ae^{\lambda x}$
$f(x) = c\cos\lambda x$ or $f(x) = c\sin\lambda x$	y_c - $A\cos\lambda x + B\sin\lambda x$	$y = ax\sin\lambda x$ if $f(x) = c\cos\lambda x$ $y = ax\cos\lambda x$ if $f(x) = c\sin\lambda x$
	y_c – does **not** contain $A\cos\lambda x + B\sin\lambda x$	$y = a\cos\lambda x + b\sin\lambda x$
$f(x)$ polynomial f degree n		$y = ax^n + bx^{n-1} + \cdots$

Particular solution / integral

$$y_p = axe^{-2x}$$

Step 4 : **Finding a PARTICULAR solution of the complete equation (particular Integral) y_p**

Differentiate y_p to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and substitute these into the original equation

$$y_p = axe^{-2x} \qquad \frac{dy}{dx} = ae^{-2x} - 2axe^{-2x} \qquad \frac{d^2y}{dx^2} = -2ae^{-2x} - 2(ae^{-2x} - 2axe^{-2x})$$
$$= -4ae^{-2x} + 4axe^{-2x}$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

$$-4ae^{-2x} + 4axe^{-2x} + 3(ae^{-2x} - 2axe^{-2x}) + 2axe^{-2x} = 2e^{-2x}$$
$$-ae^{-2x} = 2e^{-2x}$$
$$a = -2$$

$$y_p = -2xe^{-2x}$$

Step 5 : **General solution $y = y_c + y_p$**

$$y = Ae^{-x} + Be^{-2x} - 2xe^{-2x}$$

Step 6 : **Use boundary values and General solution $y = y_c + y_p$ to find the value of A and B**

7. Using substitution to reduce the order of differential equations

Show that the substitution $x = e^t$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$

We need to find $\frac{dy}{dx}$ *in terms of* $\frac{dy}{dt}$

Use the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dx}{dt} = e^t = x$$

$$\frac{dy}{dt} = x\frac{dy}{dx}$$

We need to find $\frac{d^2y}{dx^2}$ *in terms of* $\frac{d^2y}{dt^2}$ (*and* $\frac{dy}{dt}$)

$$\frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{d^2y}{dt^2}$$

Using the product rule

$$\frac{dx}{dt}\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d^2y}{dt^2}$$

$$\frac{dx}{dt}\left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dt^2}$$

Substitute $\frac{dx}{dt} = x$

$$x\left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dt^2}$$

Substitute $\frac{dy}{dt} = x\frac{dy}{dx}$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substituting into the original

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 4y = 2t$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$

These are usually SHOW THAT questions with the substitution given

8. Numerical solution of first order differential equations

- Problem given in the form $\frac{dy}{dx} = f(x, y)$
- Solution $y(x)$
- Boundary conditions $y(x_0) = y_0$
- Step length = h and $x_{n+1} = x_n + h$

It is essential that the correct notation is used !!

EULER'S FORMULA

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(x, y) \qquad y_{r+1} = y_r + hf(x_r, y_r)$$

Always work to at least 2 more decimal places than the number required by the final solution

Example

$\frac{dy}{dx} = f(x, y)$ $\quad$ *where* $f(x, y) = x + 3 + \sin y$ $\quad$ *and* $\quad$ $y(1) = 1$

Use the Euler formula with h = 0.1 to obtain an approximation to y(1.1) correct to 4 decimal places

$$y(x_0) = y_0 \quad x_0 = 1 \qquad y_0 = 1$$

$$y(1.1) = y_1 = 1 + 0.1(1 + 3 + \sin 1)$$
$$y_1 = 1.4841$$

Formulae are usually give in the exam paper

MIDPOINT FORMULA

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

$$y(x_0) = y_0 \text{ given but } y_1 \text{ calculated used Euler's formula}$$

IMPROVED EULER FORMULA

$$y_{r+1} = y_r + \frac{h}{2}(f(x_r, y_r) + f(x_{r+1}, y_{r+1}))$$

or can be given in the form

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

y_{r+1} calculated using the original Euler formula

Where $k_1 = hf(x_r, y_r)$ $\quad$ *and* $\quad$ $k_2 = hf(x_{r+1}, y_{r+1})$

Example

$\frac{dy}{dx} = f(x, y)$ $\quad$ *where* $f(x, y) = x + \sqrt{y}$ $\quad$ *and* $\quad$ $y(3) = 4$

Use the improved Euler formula with h = 0.1 to obtain an approximation for y(3.1) to 3 decimal places

$y_0 = 4 \qquad x_0 = 3$

$$k_1 = 0.1(3 + \sqrt{4}) = 0.5$$

$y_1 = 4 + 0.1(3 + \sqrt{4})$

$$k_2 = 0.1(3.1 + \sqrt{4.5}) = 0.522132$$

$$y(3.1) = 3 + 0.5(0.5 + 0.522132)$$
$$= 4.511$$