FP3 Groups

1. June 2010 qu.2

A multiplicative group with identity e contains distinct elements a and r, with the properties $r^6 = e$ and $ar = r^5 a$.

(i) Prove that $rar = a$. [2]

(ii) Prove, by induction or otherwise, that $r^n a r^n = a$ for all positive integers n. [4]

2. June 2010 qu.8

A set of matrices M is defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},\ C = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix},\ D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ E = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix},$$

where ω and ω^2 are the complex cube roots of 1. It is given that M is a group under matrix multiplication.

(i) Write down the elements of a subgroup of order 2. [1]

(ii) Explain why there is no element X of the group, other than A, which satisfies the equation $X^5 = A$. [2]

(iii) By finding BE and EB, verify the closure property for the pair of elements B and E. [4]

(iv) Find the inverses of B and E. [3]

(v) Determine whether the group M is isomorphic to the group N which is defined as the set of numbers {1, 2, 4, 8, 7, 5} under multiplication modulo 9. Justify your answer clearly. [3]

3. Jan 2010 qu. 2

H denotes the set of numbers of the form $a + b\sqrt{5}$, where a and b are rational. The numbers are combined under multiplication.

(i) Show that the product of any two members of H is a member of H. [2]

It is now given that, for a and b not both zero, H forms a group under multiplication.

(ii) State the identity element of the group. [1]

(iii) Find the inverse of $a + b\sqrt{5}$. [2]

(iv) With reference to your answer to part (iii), state a property of the number 5 which ensures that every number in the group has an inverse. [1]

4. Jan 2010 qu. 8

The function f is defined by $f : x \mapsto \dfrac{1}{2-2x}$ for $x \in \mathbb{R}, x \neq 0, x \neq \dfrac{1}{2}, x \neq 1$. The function g is defined by $g(x) = ff(x)$.

(i) Show that $g(x) = \dfrac{1-x}{1-2x}$ and that $gg(x) = x$. [4]

It is given that f and g are elements of a group K under the operation of composition of functions. The element e is the identity, where $e : x \mapsto x$ for $x \in \mathbb{R}, x \neq 0, x \neq \dfrac{1}{2}, x \neq 1$.

(ii) State the orders of the elements f and g. [2]

(iii) The inverse of the element f is denoted by h. Find $h(x)$. [2]

(iv) Construct the operation table for the elements e, f, g, h of the group K. [4]

5. June 2009 qu.2

It is given that the set of complex numbers of the form $re^{i\theta}$ for $-\pi < \theta \leq \pi$ and $r > 0$, under multiplication, forms a group.

(i) Write down the inverse of $5e^{\frac{1}{3}\pi i}$. [1]

(ii) Prove the closure property for the group. [2]

(iii) Z denotes the element $e^{i\gamma}$, where $\frac{1}{2}\pi < \gamma < \pi$. Express Z^2 in the form $e^{i\theta}$, where $-\pi < \theta < 0$. [2]

6. June 2009 qu.8

A multiplicative group Q of order 8 has elements $\{e, p, p^2, p^3, a, ap, ap^2, ap^3\}$, where e is the identity. The elements have the properties $p^4 = e$ and $a^2 = p^2 = (ap)^2$.

(i) Prove that $a = pap$ and that $p = apa$. [2]

(ii) Find the order of each of the elements p^2, a, ap, ap^2. [5]

(iii) Prove that $\{e, a, p^2, ap^2\}$ is a subgroup of Q. [4]

(iv) Determine whether Q is a commutative group. [4]

7. Jan 2009 qu. 1

In this question G is a group of order n, where $3 \leq n < 8$.

(i) In each case, write down the smallest possible value of n:

(a) if G is cyclic, [1]

(b) if G has a proper subgroup of order 3, [1]

(c) if G has at least two elements of order 2. [1]

(ii) Another group has the same order as G, but is not isomorphic to G. Write down the possible value(s) of n. [2]

8. Jan 2009 qu. 7

(i) The operation $*$ is defined by $x * y = x + y - a$, where x and y are real numbers and a is a real constant.

(a) Prove that the set of real numbers, together with the operation $*$, forms a group. [6]

(b) State, with a reason, whether the group is commutative. [1]

(c) Prove that there are no elements of order 2. [2]

(ii) The operation $\circ$ is defined by $x \circ y = x + y - 5$, where x and y are **positive** real numbers. By giving a numerical example in each case, show that two of the basic group properties are not necessarily satisfied. [4]

9. June 2008 qu.1

(a) A cyclic multiplicative group G has order 12. The identity element of G is e and another element is r, with order 12.

(i) Write down, in terms of e and r, the elements of the subgroup of G which is of order 4. [2]

(ii) Explain briefly why there is no proper subgroup of G in which two of the elements are e and r. [1]

(b) A group H has order mnp, where m, n and p are prime. State the possible orders of proper subgroups of H. [2]

10. June 2008 qu.6

The operation $\circ$ on real numbers is defined by $a \circ b = a|b|$.

(i) Show that $\circ$ is not commutative. [2]

(ii) Prove that $\circ$ is associative. [4]

(iii) Determine whether the set of real numbers, under the operation $\circ$, forms a group. [4]

11. Jan 2008 qu. 1

(a) A group G of order 6 has the combination table shown below.

	e	a	b	p	q	r
e	e	a	b	p	q	r
a	a	b	e	r	p	q
b	b	e	a	q	r	p
p	p	q	r	e	a	b
q	q	r	p	b	e	a
r	r	p	q	a	b	e

(i) State, with a reason, whether or not G is commutative. [1]

(ii) State the number of subgroups of G which are of order 2. [1]

(iii) List the elements of the subgroup of G which is of order 3. [1]

(b) A multiplicative group H of order 6 has elements e, c, c^2, c^3, c^4, c^5, where e is the identity. Write down the order of each of the elements c^3, c^4 and c^5. [3]

12. Jan 2008 qu. 8

Groups A, B, C and D are defined as follows:

A: the set of numbers {2, 4, 6, 8} under multiplication modulo 10,

B: the set of numbers {1, 5, 7, 11} under multiplication modulo 12,

C: the set of numbers $\{2^0, 2^1, 2^2, 2^3\}$ under multiplication modulo 15,

D: the set of numbers $\left\{\frac{1+2m}{1+2n}, \text{where } m \text{ and } n \text{ are integers}\right\}$ under multiplication.

(i) Write down the identity element for each of groups A, B, C and D. [2]

(ii) Determine in each case whether the groups

A and B,

B and C,

A and C

are isomorphic or non-isomorphic. Give sufficient reasons for your answers. [5]

(iii) Prove the closure property for group D. [4]

(iv) Elements of the set $\left\{\frac{1+2m}{1+2n}, \text{where } m \text{ and } n \text{ are integers}\right\}$ are combined under **addition**.

State which of the four basic group properties are **not** satisfied. (Justification is not required.) [2]

13. June 2007 qu.4

Elements of the set {p, q, r, s, t} are combined according to the operation table shown below.

	p	q	r	s	t
p	t	s	p	r	q
q	s	p	q	t	r
r	p	q	r	s	t
s	r	t	s	q	p
t	q	r	t	p	s

(i) Verify that $q(st) = (qs)t$. [2]

(ii) Assuming that the associative property holds for all elements, prove that the set {p, q, r, s, t}, with the operation table shown, forms a group G. [4]

(iii) A multiplicative group H is isomorphic to the group G. The identity element of H is e and another element is d. Write down the elements of H in terms of e and d. [2]

14. June 2007 qu.9

The set S consists of the numbers 3^n, where $n \in \mathbb{Z}$. ($\mathbb{Z}$ denotes the set of integers $\{0, \pm 1, \pm 2, \ldots\}$.)

(i) Prove that the elements of S, under multiplication, form a commutative group G. (You may assume that **addition** of integers is associative and commutative.) [6]

(ii) Determine whether or not each of the following subsets of S, under multiplication, forms a subgroup of G, justifying your answers.

(a) The numbers 3^{2n}, where $n \in \mathbb{Z}$ [2]

(b) The numbers 3^n, where $n \in \mathbb{Z}$ and $n \geq 0$. [2]

(c) The numbers $3^{(\pm n^2)}$, where $n \in \mathbb{Z}$ [2]

15. Jan 2007 qu. 1

(i) Show that the set of numbers $\{3, 5, 7\}$, under multiplication modulo 8, does not form a group. [2]

(ii) The set of numbers $\{3, 5, 7, a\}$, under multiplication modulo 8, forms a group. Write down the value of a. [1]

(iii) State, justifying your answer, whether or not the group in part (ii) is isomorphic to the multiplicative group $\{e, r, r^2, r^3\}$, where e is the identity and $r^4 = e$. [2]

16. Jan 2007 qu. 5

A multiplicative group G of order 9 has distinct elements p and q, both of which have order 3. The group is commutative, the identity element is e, and it is given that $q \neq p^2$.

(i) Write down the elements of a proper sub group of G

(a) which does not contain q, [1]

(b) which does not contain p. [1]

(ii) Find the order of each of the elements pq and pq^2, justifying your answers. [3]

(iii) State the possible order (s) of proper subgroups of G. [1]

(iv) Find two proper subgroups of G which are distinct from those in part (i), simplifying the elements. [4]

17. June 2006 qu.1

(a) For the infinite group of non-zero complex numbers under multiplication, state the identity element and the inverse of 1 +2i, giving your answers in the form $a + ib$. [3]

(b) For the group of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ under matrix addition, where $a \in \mathbb{R}$,

state the identity element and the inverse of $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$. [2]

18. June 2006 qu.8

A group D of order 10 is generated by the elements a and r, with the properties $a^2 = e$, $r^5 = e$ and $r^4a = ar$, where e is the identity. Part of the operation table is shown below.

	e	a	r	r^2	r^3	r^4	ar	ar^2	ar^3	ar^4
e	e	a	r	r^2	r^3	r^4	ar	ar^2	ar^3	ar^4
a	a	e	ar	ar^2	ar^3	ar^4				
r	r		r^2	r^3	r^4	e				
r^2	r^2		r^3	r^4	e	r				
r^3	r^3		r^4	e	r	r^2				
r^4	r^4	ar	e	r	r^2	r^3				
ar	ar		ar^2	ar^3	ar^4	a				
ar^2	ar^2		ar^3	ar^4	a	ar		**E**		
ar^3	ar^3		ar^4	a	ar	ar^2				
ar^4	ar^4		a	ar	ar^2	ar^3				

(i) Give a reason why ***D*** is not commutative. [1]

(ii) Write down the orders of any possible proper subgroups of D. [2]

(iii) List the elements of a proper subgroup which contains
(a) the element a, [1] (b) the element r. [1]

(iv) Determine the order of each of the elements r^3, ar and ar^2. [4]

(v) Copy and complete the section of the table marked **E**, showing the products of the elements ar, ar^2, ar^3 and ar^4. [5]

19. Jan 2006 qu. 2

The tables shown below are the operation tables for two isomorphic groups G and H.

G	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	d	a
d	c	d	a	b

H	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(i) For each group, state the identity element and list the elements of any proper subgroups.m [4]

(ii) Establish the isomorphism between G and H by showing which elements correspond. [3]

20. Jan 2006 qu. 7

A group G has an element a with order n, so that $a^n = e$, where e is the identity. It is given that x is any element of G distinct from a and e.

(i) Prove that the order of $x^{-1}ax$ is n, making it clear which group property is used at each stage of your proof. [6]

(ii) Express the inverse of $x^{-1}ax$ in terms of some or all of x, x^{-1}, a and a^{-1}, showing sufficient working to justify your answer. [3]

(iii) It is now given that a commutes with every element of G. Prove that a^{-1} also commutes with every element. [2]