FP3 Series & Limits Questions

2 (a) Find $\int_0^a xe^{-2x}\,dx$, where $a > 0$. *(5 marks)*

(b) Write down the value of $\lim_{a\to\infty} a^k e^{-2a}$, where k is a positive constant. *(1 mark)*

(c) Hence find $\int_0^\infty xe^{-2x}\,dx$. *(2 marks)*

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4 (a) Use the series expansion

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$

to write down the first four terms in the expansion, in ascending powers of x, of $\ln(1-x)$. *(1 mark)*

(b) The function f is defined by

$$f(x) = e^{\sin x}$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$1 + x + \tfrac{1}{2}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(c) Find

$$\lim_{x\to 0} \frac{e^{\sin x} - 1 + \ln(1-x)}{x^2 \sin x}$$ *(4 marks)*

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5 (a) Show that $\lim_{a\to\infty}\left(\frac{3a+2}{2a+3}\right) = \frac{3}{2}$. *(2 marks)*

(b) Evaluate $\int_1^\infty \left(\frac{3}{3x+2} - \frac{2}{2x+3}\right) dx$, giving your answer in the form $\ln k$, where k is a rational number. *(5 marks)*

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7 (a) (i) Write down the first three terms of the binomial expansion of $(1+y)^{-1}$, in ascending powers of y. *(1 mark)*

(ii) By using the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

and your answer to part (a)(i), or otherwise, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\sec x$ are

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}$$ *(5 marks)*

(b) By using Maclaurin's theorem, or otherwise, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\tan x$ are

$$x + \frac{x^3}{3}$$ *(3 marks)*

(c) Hence find $\lim\limits_{x \to 0}\left(\dfrac{x \tan 2x}{\sec x - 1}\right)$. *(4 marks)*

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4 (a) Explain why $\displaystyle\int_0^{\mathrm{e}} \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$ is an improper integral. *(1 mark)*

(b) Use integration by parts to find $\displaystyle\int x^{-\frac{1}{2}} \ln x \,\mathrm{d}x$. *(3 marks)*

(c) Show that $\displaystyle\int_0^{\mathrm{e}} \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$ exists and find its value. *(4 marks)*

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6 The function f is defined by $f(x) = (1+2x)^{\frac{1}{2}}$.

(a) (i) Find $f'''(x)$. *(4 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$f(x) \approx 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$ *(4 marks)*

(b) Use the expansion of e^x together with the result in part (a)(ii) to show that, for small values of x,

$$e^x(1+2x)^{\frac{1}{2}} \approx 1 + 2x + x^2 + kx^3$$

where k is a rational number to be found. *(3 marks)*

(c) Write down the first four terms in the expansion, in ascending powers of x, of e^{2x}. *(1 mark)*

(d) Find

$$\lim_{x \to 0} \frac{e^x(1+2x)^{\frac{1}{2}} - e^{2x}}{1 - \cos x}$$ *(4 marks)*

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6 (a) The function f is defined by

$$f(x) = \ln(1 + e^x)$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$\ln 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(b) Hence write down the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\frac{1+e^x}{2}\right)$. *(1 mark)*

(c) Use the series expansion

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots$$

to write down the first three terms in the expansion, in ascending powers of x, of $\ln\left(1-\frac{x}{2}\right)$. *(1 mark)*

(d) Use your answers to parts (b) and (c) to find

$$\lim_{x \to 0}\left[\frac{\ln\left(\frac{1+e^x}{2}\right)+\ln\left(1-\frac{x}{2}\right)}{x-\sin x}\right]$$ *(4 marks)*

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7 (a) Write down the value of

$$\lim_{x \to \infty} xe^{-x}$$ *(1 mark)*

(b) Use the substitution $u = xe^{-x}+1$ to find $\int \frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx$. *(2 marks)*

(c) Hence evaluate $\int_1^{\infty} \frac{1-x}{x+e^x}\,dx$, showing the limiting process used. *(4 marks)*

FP3 Series & Limits Answers

2(a)	$\int xe^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\mathrm{d}x$	M1 A1		Reasonable attempt at parts
	$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\ \{+c\}$	A1√		Condone absence of $+c$
	$\int_0^a xe^{-2x}\mathrm{d}x = -\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} - (0 - \frac{1}{4})$	M1		$F(a) - F(0)$
	$= \frac{1}{4} - \frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a}$	A1	5	
(b)	$\lim_{a\to\infty} a^k e^{-2a} = 0$	B1	1	
(c)	$\int_0^\infty xe^{-2x}\mathrm{d}x =$			
	$= \lim_{a\to\infty} \{\frac{1}{4} - \frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a}\}$	M1		If this line oe is missing then 0/2
	$= \frac{1}{4} - 0 - 0 = \frac{1}{4}$	A1√	2	On candidate's "1/4" in part (a). B1 must have been earned
	Total		**8**	

4(a)	$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \ldots$	B1	1	
(b)(i)	$f(x) = e^{\sin x} \Rightarrow f(0) = 1$	B1		
	$f'(x) = \cos x\, e^{\sin x}$ $\Rightarrow f'(0) = 1$	M1A1		
	$f''(x) = -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x}$ $f''(0) = 1$	M1A1		Product rule used
	Maclaurin $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0)$			
	so 1st three terms are $1 + x + \frac{1}{2}x^2$	A1	6	**CSO AG**
(ii)	$f'''(x) = \cos x(\cos^2 x - \sin x)e^{\sin x} +$ $+\{2\cos x(-\sin x) - \cos x\}\, e^{\sin x}$	M1A1		
	$f'''(0) = 0$ so the coefficient of x^3 in the series is zero	A1	3	**CSO AG** **SC** for (b): Use of series

				expansions….max of 4/9
(c)	$\sin x \approx x$.	B1		Ignore higher power terms in $\sin x$ expansion
	$\dfrac{e^{\sin x} - 1 + \ln(1-x)}{x^2 \sin x} = \dfrac{-\frac{1}{3}x^3 + o(x^4)}{x^3}$	M1 A1		Series from (a) & (b) used Numerator kx^3 (+…)
	$= \dfrac{-\frac{1}{3} + o(x)}{1 + o(x^2)}$			Condone if this step is missing
	$\lim_{x \to 0} \dfrac{e^{\sin x} - 1 + \ln(1-x)}{x^2 \sin x} = -\dfrac{1}{3}$	A1√	4	On candidate's x^3 coefficient in (a) provided lower powers cancel
	Total		**14**	

5(a)	$\Rightarrow \lim_{a \to \infty} \left(\dfrac{3 + \frac{2}{a}}{2 + \frac{3}{a}} \right) = \dfrac{3+0}{2+0} = \dfrac{3}{2}$	M1 A1	2	
(b)	$\int_1^\infty \dfrac{3}{(3x+2)} - \dfrac{2}{2x+3}\, dx$			
	$= [\ln(3x+2) - \ln(2x+3)]_1^\infty$	M1 A1		$a\ln(3x + 2) + b\ln(2x + 3)$
	$= \left[\ln\left(\dfrac{3x+2}{2x+3} \right) \right]_1^\infty$	m1		
	$= \ln\left\{ \lim_{a \to \infty} \left(\dfrac{3a+2}{2a+3} \right) \right\} - \ln 1$	M1		
	$= \ln\dfrac{3}{2} - \ln 1 = \ln\dfrac{3}{2}$	A1	5	CSO
	Total		**7**	

7(a)(i)	$(1+y)^{-1} = 1 - y + y^2 ......$	B1	1	
(ii)	$\sec x \approx \dfrac{1}{1-\dfrac{x^2}{2}+\dfrac{x^4}{24}....}$	B1		
	$= \left[1-\dfrac{x^2}{2}+\dfrac{x^4}{24}....\right]^{-1} =$	M1		
	$\left\{1-\left(-\dfrac{x^2}{2}+\dfrac{x^4}{24}\right)+\left(-\dfrac{x^2}{2}+\dfrac{x^4}{24}\right)^2\right\}$	M1		
	$= \left\{1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}+...\right\}$			
	$= 1+\dfrac{x^2}{2}; + \dfrac{5x^4}{24}$	A1;A1	5	AG be convinced
	Alternative: Those using Maclaurin			
	f(x) = sec x			
	f(0) = 1; f′(x) = sec$x$ tan$x$; {f′(0) = 0}	(B1)		
	f″(x) = sec$x$ $\tan^2 x$ + $\sec^3 x$; f ″(0) = 1	(M1)		Product rule oe
	f‴(x) = sec$x$ $\tan^3 x$ + 5tan$x$ $\sec^3 x$;	(m1)		Chain rule with product rule OE
	$f^{(iv)}(x)$ = sec$x$ $\tan^4 x$ +18$\tan^2 x$ $\sec^3 x$ … +5$\sec^5 x$ ⇒ $f^{(iv)}(0) = 5$			
	sec x ≈ printed result	(A2)		CSO AG
(b)	f(x) = tan x;			
	f(0) = 0; f′(x) = $\sec^2 x$; {f′(0) = 1}	B1		
	f″(x) = 2sec$x$(sec$x$ tan x); f ″(0) = 0			
	f‴(x) = 4sec$x$ tan$x$(sec$x$ tan$x$) + 2$\sec^4 x$	M1		Chain rule with product rule oe
	f‴(0) = 2			
	$\tan x = 0 + 1x + 0x^2 + \dfrac{2}{3!}x^3 ... = x + \dfrac{1}{3}x^3$	A1	3	CSO AG
	Alternative: Those using otherwise			
	$.. = \dfrac{\sin x}{\cos x} \approx \left(x-\dfrac{x^3}{6}...\right)\left(1+\dfrac{x^2}{2}...\right)$	(M1) (A1)		
	$= x + \dfrac{x^3}{2} - \dfrac{x^3}{6}... = x + \dfrac{1}{3}x^3....$	(A1)		
(c)	$\left(\dfrac{x\tan 2x}{\sec x - 1}\right) = \dfrac{x\left(2x+o(x^3)\right)}{\dfrac{x^2}{2}+o(x^4)}$	B1 M1		$\tan 2x = 2x + \dfrac{1}{3}(2x)^3$ Condone $o(x^k)$ missing
	$= \dfrac{2+o(x^2)}{\dfrac{1}{2}+o(x^2)}$	M1		
	$\lim_{x\to 0}\left(\dfrac{x\tan 2x}{\sec x - 1}\right) = 4$	A1√	4	ft on 2k after B0 for tan2x = kx+…
	Total		**13**	

4(a)	Integrand is not defined at $x = 0$	E1	1	OE
(b)	$\int x^{-\frac{1}{2}} \ln x \, dx = 2x^{\frac{1}{2}} \ln x - \int 2x^{\frac{1}{2}} \left(\frac{1}{x}\right) dx$	M1		$... = kx^{\frac{1}{2}} \ln x \pm \int f(x)$, with $f(x)$ not involving the 'original' $\ln x$
		A1		
	$\ldots\ldots = 2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} (+c)$	A1	3	Condone absence of '$+ c$'
(c)	$\int_0^e \frac{\ln x}{\sqrt{x}} dx = \lim_{a \to 0} \int_a^e \frac{\ln x}{\sqrt{x}} dx$	M1		
	$= -2e^{\frac{1}{2}} - \lim_{a \to 0} \left[2a^{\frac{1}{2}} \ln a - 4a^{\frac{1}{2}} \right]$	M1		$F(b) - F(a)$
	But $\lim_{a \to 0} a^{\frac{1}{2}} \ln a = 0$	B1		Accept a general form e.g. $\lim_{x \to 0} x^k \ln x = 0$
	So $\int_0^e \frac{\ln x}{\sqrt{x}} dx$ exists and $= -2e^{\frac{1}{2}}$	A1	4	
	Total		**8**	

6(a)(i)	$f'(x) = \frac{1}{2}(1+2x)^{-\frac{1}{2}}(2) = (1+2x)^{-\frac{1}{2}}$	M1A1		
	$f''(x) = -(1+2x)^{-\frac{3}{2}}$	A1F		ft a slip
	$f'''(x) = 3(1+2x)^{-\frac{5}{2}}$	A1	4	
(ii)	$f(x) = (1+2x)^{\frac{1}{2}} \Rightarrow f(0) = 1;$	B1		
	$f'(0) = 1;\ f''(0) = -1;\ f'''(0) = 3$	M1		All three attempted
		A1F		ft on $k(1+2x)'''$
	$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0)$			
	$\ldots \approx 1 + x - \frac{x^2}{2} + \frac{x^3}{2}$	A1	4	CSO **AG**
(b)	$e^x (1+2x)^{\frac{1}{2}} \approx$			
	$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + x - \frac{x^2}{2} + \frac{x^3}{2}\right)$	M1		Attempt to expand needed
	$\approx 1 + x(1+1) + x^2(-0.5 + 1 + 0.5)$ $+ x^3\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{6}\right)$	A1		

	$\approx 1+2x+x^2+\frac{2}{3}x^3$	A1	3	CSO
(c)	$e^{2x} = 1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{6}+\dots$	B1	1	
	$= 1+2x+2x^2+\frac{4}{3}x^3+\dots$			
(d)	$1-\cos x = \frac{1}{2}x^2+\{o(x^4)\}$	B1		
	$\frac{e^x(1+2x)^{\frac{1}{2}}-e^{2x}}{1-\cos x} =$			
	$\frac{1+2x+x^2+\frac{2}{3}x^3-\left[1+2x+2x^2+\frac{4}{3}x^3\right]}{\frac{1}{2}x^2+\{o(x^4)\}}$	M1		Series used
	$\lim_{x\to 0}\dots = \lim_{x\to 0}\frac{-x^2+\{o(x^3)\}}{\frac{1}{2}x^2+\{o(x^4)\}} =$	A1F		
	$\lim_{x\to 0}\frac{-1+o(x)}{\frac{1}{2}+o(x^2)} = -2$	A1F	4	ft a slip but must see the intermediate stage
	Total		**16**	

6(a)(i)	$f(x)=\ln(1+e^x)$:			
	$f(0)=\ln 2$	B1		
	$f'(x)=\frac{e^x}{1+e^x}\quad f'(0)=\frac{1}{2}$	M1 A1		Chain rule
	$f''(x)=\frac{(1+e^x)e^x-e^xe^x}{(1+e^x)^2}=\frac{e^x}{(1+e^x)^2}$	M1 A1		Quotient rule OE
	$f''(0)=\frac{1}{4}$			
	so first three terms are:			
	$f(x)=\ln 2+\frac{1}{2}x+\frac{1}{4}\frac{x^2}{2!}=\ln 2+\frac{1}{2}x+\frac{1}{8}x^2$	A1	6	CSO; AG
(ii)	$f'''(x)=\frac{(1+e^x)^2e^x-e^x\left[2(1+e^x)e^x\right]}{(1+e^x)^4}$	M1 A1ft		Chain rule with quotient/product rule ft on $f''(x)=ke^x(1+e^x)^n$ (integer $n<0$)
	$f'''(0)=\frac{4-4}{2^4}=0$ {so coefficient of x^3 is zero}	A1	3	CSO; AG; All previous differentiation correct

SC for those not using Maclaurin's theorem: **maximum** of 4/9

(b)	$\frac{1}{2}x+\frac{1}{8}x^2$	B1	1	
(c)	$\ln\left(1-\frac{x}{2}\right)=$			
	$\left(-\frac{x}{2}\right)-\frac{1}{2}\left(-\frac{x}{2}\right)^2+\frac{1}{3}\left(-\frac{x}{2}\right)^3-......$	B1	1	
(d)	$\ln\left(\frac{1+e^x}{2}\right)+\ln\left(1-\frac{x}{2}\right)=-\frac{x^3}{24}+...$	M1		Uses previous expansions to obtain first non-zero term of the form kx^3
	$x-\sin x\approx x-\left[x-\frac{x^3}{3!}+...\right]\approx\frac{x^3}{3!}+...$	B1		
	$\left[\frac{\ln\left(\frac{1+e^x}{2}\right)+\ln\left(1-\frac{x}{2}\right)}{x-\sin x}\right]=\frac{-\frac{1}{24}x^3+...}{\frac{1}{6}x^3+o\left(x^5\right)}$	M1		
	$=\frac{-\frac{1}{24}x^3+...}{x^3\left[\frac{1}{6}+o\left(x^2\right)\right]}=\frac{-\frac{1}{24}+...}{\frac{1}{6}+o\left(x^2\right)}$			
	$\lim_{x\to0}.....=-\frac{1}{4}$	A1	4	CSO
	Total		**15**	

7(a)	0	B1	1	
(b)	$u=xe^{-x}+1\Rightarrow du=\left(e^{-x}-xe^{-x}\right)dx$	M1		Attempts to find du
	$\int\frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx=\int\frac{1}{u}\,du=\ln u+c$			
	$=\ln\left(xe^{-x}+1\right)\{+c\}$	A1	2	Condone missing c
(c)	$\int\frac{1-x}{x+e^x}\,dx=\int\frac{e^{-x}(1-x)}{xe^{-x}+1}dx$	B1		
	$\int_1^\infty\frac{1-x}{x+e^x}\,dx=\lim_{a\to\infty}\left[\ln(xe^{-x}+1)\right]_1^a$			
	$=\lim_{a\to\infty}\left\{\ln\left(ae^{-a}+1\right)\right\}-\ln\left(e^{-1}+1\right)$	M1		For using part (b) and $F(B)-F(A)$
	$=\ln\left\{\lim_{a\to\infty}\left(ae^{-a}+1\right)\right\}-\ln\left(e^{-1}+1\right)$			
	$=\ln1-\ln\left(e^{-1}+1\right)=-\ln\left(e^{-1}+1\right)$	M1 A1	4	For using limiting process
	Total		**7**	