# FP3 Second Order Differential Equation Questions

**1** (a) Find the roots of the equation $m^2 + 2m + 2 = 0$ in the form $a + \mathrm{i}b$. *(2 marks)*

(b) (i) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x$$

*(6 marks)*

(ii) Hence express $y$ in terms of $x$, given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$. *(4 marks)*

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**1** It is given that $y$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x - 10 - 10\cos 2x$$

(a) Show that $y = 2x + \sin 2x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Find the general solution of the differential equation. *(4 marks)*

(c) Hence express $y$ in terms of $x$, given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. *(4 marks)*

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6 (a) Show that the substitution

$$u = \frac{dy}{dx} + 2y$$

transforms the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

into

$$\frac{du}{dx} + 2u = e^{-2x}$$ *(4 marks)*

(b) By using an integrating factor, or otherwise, find the general solution of

$$\frac{du}{dx} + 2u = e^{-2x}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

giving your answer in the form $y = g(x)$. *(5 marks)*

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5 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 + 5\sin x$$ *(12 marks)*

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1 (a) Find the value of the constant $k$ for which $kx^2e^{5x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$$ *(6 marks)*

(b) Hence find the general solution of this differential equation. *(4 marks)*

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**5** (a) A differential equation is given by

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

Show that the substitution

$$u = \frac{dy}{dx} + x$$

transforms this differential equation into

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

*(4 marks)*

(b) Find the general solution of

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

giving your answer in the form $y = g(x)$. *(3 marks)*

# FP3 Second Order Differential Equation Answers

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| **1(a)** | $(m+1)^2 = -1$ | M1 | | Completing sq or formula |
| | $m = -1 \pm i$ | A1 | 2 | |
| **(b)(i)** | CF is $e^{-x}(A\cos x + B\sin x)$ | M1 | | If $m$ is real give M0 |
| | {or $e^{-x}A\cos(x+B)$ **but not** $Ae^{(-1+i)x} + Be^{(-1-i)x}$} | A1√ | | On wrong $a$'s and $b$'s but roots must be complex. |
| | {P.Int.} try $y = px + q$ | M1 | | OE |
| | $2p + 2(px+q) = 4x$ | A1 | | |
| | $p = 2, \quad q = -2$ | A1√ | | On one slip |
| | GS $y = e^{-x}(A\cos x + B\sin x) + 2x - 2$ | B1√ | 6 | Their CF + their PI with two arbitrary constants. |
| **(ii)** | $x=0,\ y=1 \Rightarrow A = 3$ | B1√ | | Provided an M1gained in (b)(i) |
| | $y'(x) = -e^{-x}(A\cos x + B\sin x) +$ | M1 | | Product rule used |
| | $+ e^{-x}(-A\sin x + B\cos x) + 2$ | A1√ | | |
| | $y'(0) = 2 \Rightarrow 2 = -A+B+2 \Rightarrow B = 3$ | A1√ | | Slips |
| | $y = 3e^{-x}(\cos x + \sin x) + 2x - 2$ | | 4 | |
| | **Total** | | **12** | |

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|---|---|---|---|---|
| **1(a)** | $y = 2x + \sin 2x \Rightarrow y' = 2 + 2\cos 2x$ | | | |
| | $\Rightarrow y'' = -4\sin 2x$ | M1 A1 | | Need to attempt both $y'$ and $y''$ |
| | $-4\sin 2x - 5(2 + 2\cos 2x) + 4(2x + \sin 2x) = 8x - 10 - 10\cos 2x$ | A1 | 3 | CSO AG Substitute. and confirm correct |
| **(b)** | Auxiliary equation $m^2 - 5m + 4 = 0$ | M1 | | |
| | $m = 4$ and 1 | A1 | | |
| | CF: $Ae^{4x} + Be^{x}$ | M1 | | |
| | GS: $y = Ae^{4x} + Be^{x} + 2x + \sin 2x$ | B1√ | 4 | Their CF + $2x + \sin 2x$ |
| **(c)** | $x = 0, y = 2 \Rightarrow \quad 2 = A + B$ | B1√ | | Only ft if exponentials in GS |
| | $x = 0, y' = 0 \Rightarrow \quad 0 = 4A + B + 4$ | B1√ | | Only ft if exponentials in GS and differentiated four terms at least |
| | Solving the simultaneous equations | M1 | | |
| | gives $A = -2$ and $B = 4$ | A1 | 4 | |
| | $y = -2e^{4x} + 4e^{x} + 2x + \sin 2x$ | | | |
| | **Total** | | **11** | |

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|---|---|---|---|---|
| 6(a) | $u=\frac{dy}{dx}+2y \Rightarrow \frac{du}{dx}=\frac{d^2y}{dx^2}+2\frac{dy}{dx}$ | M1 A1 | | 2 terms correct |
| | LHS of DE$\Rightarrow \frac{du}{dx}-2\frac{dy}{dx}+4\frac{dy}{dx}+4y$ | | | |
| | LHS: $\frac{du}{dx}+2(u-2y)+4y$ | M1 | | Substitution into LHS of DE as far as no derivatives of $y$ |
| | $\Rightarrow \frac{du}{dx}+2u=e^{-2x}$ | A1 | 4 | CSO AG |
| (b) | IF is $e^{\int 2dx}=e^{2x}$ | B1 | | |
| | $\frac{d}{dx}\left[ue^{2x}\right]=1$ | M1 A1 | | |
| | $\Rightarrow ue^{2x}=x+A$ | A1 | | |
| | $\Rightarrow u=xe^{-2x}+Ae^{-2x}$ | A1 | 5 | |
| | **Alternative : Those using CF+PI** Auxiliary equation, | | | |
| | $m+2=0 \Rightarrow u_{CF}=Ae^{-2x}$ | B1 | | |
| | For $u_{PI}$ try $u_{PI}=kxe^{-2x} \Rightarrow$ | M1 | | |
| | $ke^{-2x}-2kxe^{-2x}+2kxe^{-2x}\{=e^{-2x}\}$ | A1 | | LHS |
| | $\Rightarrow k=1 \Rightarrow u_{PI}=xe^{-2x}$ | A1 | | |
| | $\Rightarrow u_{GS}=Ae^{-2x}+xe^{-2x}$ | A1 | | |
| (c) | $\Rightarrow \frac{dy}{dx}+2y=xe^{-2x}+Ae^{-2x}$ | M1 | | Use (b) to reach a 1st order DE in $y$ and $x$ |
| | IF is $e^{\int 2dx}=e^{2x}$ | B1 | | |
| | $\Rightarrow \frac{d}{dx}\left[ye^{2x}\right]=x+A$ | A1√ | | |
| | $\Rightarrow ye^{2x}=\frac{x^2}{2}+Ax+B$ | A1√ | | |
| | $\Rightarrow y=e^{-2x}\left(\frac{x^2}{2}+Ax+B\right)$ | A1 | 5 | |
| | **Total** | | **14** | |

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|---|---|---|---|---|
| 5 | Auxl. eqn $m^2-4m+3=0$ | M1 | | PI |
| | $m=3$ and 1 | A1 | | PI |
| | CF is $Ae^{3x}+Be^{x}$ | A1F | | |
| | PI Try $y=a+b\sin x+c\cos x$ | M1 | | Condone '$a$' missing here |
| | $y'(x)=b\cos x-c\sin x$ | A1 | | |
| | $y''(x)=-b\sin x-c\cos x$ | A1F | | ft can be consistent sign error(s) |
| | Substitute into DE gives | M1 | | |
| | $a=2$ | B1 | | |
| | $4c+2b=5$ and $2c-4b=0$ | A1 | | |
| | $b=0.5$, | A1F | | ft a slip |
| | $c=1$ | A1F | | ft a slip |
| | GS: $y=Ae^{3x}+Be^{x}+2+0.5\sin x+\cos x$ | B1F | 12 | $y=$ candidate's CF and candidate's PI (must have exactly two arbitrary constants) |
| | **Total** | | **12** | |

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|---|---|---|---|---|
| **1(a)** | $y_{PI} = kx^2e^{5x} \Rightarrow y' = 2kxe^{5x} + 5kx^2e^{5x}$ | M1<br>A1 | | Product rule to differentiate $x^2e^{5x}$ |
| | $\Rightarrow y'' = 2ke^{5x} + 10kxe^{5x} + 10kxe^{5x} + 25kx^2e^{5x}$ | A1ft | | |
| | $\Rightarrow 2ke^{5x} + 20kxe^{5x} + 25kx^2e^{5x}$<br>$-10\left(2kxe^{5x} + 5kx^2e^{5x}\right) + 25kx^2e^{5x} = 6e^{5x}$ | M1<br>A1 | | Substitution into differential equation |
| | $2k = 6 \Rightarrow k = 3$ | A1ft | 6 | Only ft if $xe^{5x}$ and $x^2e^{5x}$ terms all cancel out |
| **(b)** | Aux. eqn. $m^2 - 10m + 25 = 0 \Rightarrow m = 5$ | B1 | | PI |
| | CF is $(A + Bx)e^{5x}$ | M1 | | |
| | GS $y = (A + Bx)e^{5x} + 3x^2e^{5x}$ | M1<br>A1ft | 4 | Their CF + their/our PI<br>ft only on wrong value of $k$ |
| | **Total** | | **10** | |

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|---|---|---|---|---|
| **5(a)** | $u = \frac{dy}{dx} + x \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2} + 1$ | M1A1 | | |
| | $(x^2 - 1)\left(\frac{du}{dx} - 1\right) - 2x(u - x) = x^2 + 1$ | M1 | | Substitution into LHS of DE as far as no $y$s |
| | DE $\Rightarrow (x^2 - 1)\frac{du}{dx} - 2xu = 0$ | | | |
| | $\Rightarrow \frac{du}{dx} = \frac{2xu}{x^2 - 1}$ | A1 | 4 | CSO; AG |
| **(b)** | $\int \frac{1}{u}\, du = \int \frac{2x}{x^2 - 1}\, dx$ | M1<br>A1 | | Separate variables |
| | $\ln u = \ln \lvert x^2 - 1 \rvert + \ln A$ | A1A1 | | |
| | $u = A(x^2 - 1)$ | A1 | 5 | |
| **(c)** | $\frac{dy}{dx} + x = A(x^2 - 1)$ | M1 | | Use (b) $(\neq 0)$ to form DE in $y$ and $x$ |
| | $\frac{dy}{dx} = A(x^2 - 1) - x$ | | | |
| | $y = A\left(\frac{x^3}{3} - x\right) - \frac{x^2}{2} + B$ | M1<br>A1ft | 3 | Solution must have two different constants and correct method used to solve the DE |
| | **Total** | | **12** | |