FP3 Numerical Methods for the Solution of First Order Differential Equations Questions

5 (a) The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x,y) = x \ln x + \frac{y}{x}$$

and $$y(1) = 1$$

(i) Use the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$. *(3 marks)*

(ii) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r)$$

with your answer to part (a)(i) to obtain an approximation to $y(1.2)$, giving your answer to three decimal places. *(4 marks)*

(b) (i) Show that $\frac{1}{x}$ is an integrating factor for the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = x \ln x$$ *(3 marks)*

(ii) Solve this differential equation, given that $y = 1$ when $x = 1$. *(6 marks)*

(iii) Calculate the value of y when $x = 1.2$, giving your answer to three decimal places. *(1 mark)*

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2 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \frac{x^2 + y^2}{xy}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

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1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \ln(1 + x^2 + y)$$

and $$y(1) = 0.6$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.05$, to obtain an approximation to $y(1.05)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,f(x_r, y_r)$ and $k_2 = h\,f(x_r + h, y_r + k_1)$ and $h = 0.05$, to obtain an approximation to $y(1.05)$, giving your answer to four decimal places. *(6 marks)*

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2 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \sqrt{x^2 + y^2 + 3}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

FP3 Numerical Methods for the Solution of First Order Differential Equations Answers

5(a)(i)	$y(1.1) = y(1) + 0.1[1\ln 1 + 1/1]$	M1A1		
	$= 1 + 0.1 = 1.1$	A1	3	
(ii)	$y(1.2) = y(1) + 2(0.1)[f(1.1, y(1.1)]$	M1A1		
	$\ldots. = 1 + 2(0.1)[1.1\ln 1.1 + (1.1)/1.1]$	A1√		On answer to (a)(i)
	$\ldots. = 1 + 0.2\times 1.104841198\ldots.$			
	$\ldots. = 1.22096824 = 1.221$ to 3dp	A1	4	CAO
(b)(i)	IF is $e^{\int -\frac{1}{x}\,dx}$	M1		Condone $e^{\int \frac{1}{x}\,dx}$ for M mark
	$= e^{-\ln x}$	A1		
	$= e^{\ln x^{-1}} = x^{-1} = \dfrac{1}{x}$	A1	3	**AG** (be convinced) (b)(i) Solutions using the printed answer must be convincing before any marks are awarded
(ii)	$\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = \ln x$	M1A1		
	$\dfrac{y}{x} = \int \ln x\, dx = x\ln x - \int x\left(\dfrac{1}{x}\right)dx$	M1		Integration by parts for $x^k \ln x$
	$\dfrac{y}{x} = x\ln x - x + c$	A1		Condone missing c.
	$y(1) = 1 \Rightarrow 1 = \ln 1 - 1 + c$	m1		Dependent on at least one of the two previous M marks
	$\Rightarrow c = 2 \Rightarrow y = x^2\ln x - x^2 + 2x$	A1	6	OE eg $\dfrac{y}{x} = x\ln x - x + 2$
(iii)	$y(1.2) = 1.222543\ldots = 1.223$ to 3dp	B1	1	
	Total		**17**	

2(a)	$y_1 = 2 + 0.1 \times \left[\dfrac{1^2+2^2}{1\times 2}\right]$	M1 A1		
	$= 2 + 0.1 \times 2.5 = 2.25$	A1	3	
(b)	$k_1 = 0.1 \times 2.5 = 0.25$	M1		
		A1√		PI ft from (a)
	$k_2 = 0.1 \times \text{f}(1.1,\ 2.25)$	M1		
	$\ldots = 0.1 \times 2.53434\ldots = 0.2534(34\ldots)$	A1√		PI
	$y(1.1) = y(1) + \frac{1}{2}[0.25 + 0.253434\ldots]$	m1		
	$= 2.2517$ to 4dp	A1√	6	If answer not to 4dp withhold this mark
	Total		**9**	

1(a)	$y\,(1.05) = 0.6 + 0.05 \times [\ln(1+1+0.6)]$	M1A1		
	$= 0.6477(7557..) = 0.6478$ to 4dp	A1	3	Condone >4 dp
(b)	$k_1 = 0.05 \times \ln(1+1+0.6) = 0.0477(75\ldots)$	M1		PI
		A1F		ft candidate's evaluation in (a)
	$k_2 = 0.05 \times \text{f}\,(1.05,\ 0.6477\ldots)$ $\ldots = 0.05 \times \ln(1 + 1.05^2 + 0.6477\ldots)]$	M1		
	$\ldots = 0.0505(85\ldots)$	A1F		PI
	$y(1.05) = y(1) + \frac{1}{2}[k_1 + k_2]$ $= 0.6 + 0.5 \times 0.09836\ldots$	m1		Dep on previous two Ms and numerical values for k's
	$= 0.6492$ to 4dp	A1F	6	Must be 4 dp… ft one slip
	Total		**9**	

2(a)	$y_1 = 2 + 0.1 \times \sqrt{1^2 + 2^2 + 3}$	M1		
	$y\,(1.1) = 2 + 0.1 \times \sqrt{8}$	A1		
	$y\,(1.1) = 2.28284\ldots = 2.2828$ to 4dp	A1	3	
(b)	$k_1 = 0.1 \times \sqrt{8} = 0.2828$	M1		PI
		A1ft		
	$k_2 = 0.1 \times \text{f}\,(1.1,\ 2.2828\ldots)$	M1		
	$= 0.1 \times \sqrt{9.42137\ldots} = 0.3069(425\ldots)$	A1		PI
	$y\,(1.1) = y\,(1) + \frac{1}{2}[0.28284\ldots + 0.30694\ldots]$	m1		
	$2.29489\ldots = 2.2949$ to 4dp	A1	6	
	Total		**9**	