FP3 Introduction to Differential Equations Questions

3 (a) Show that $y = x^3 - x$ is a particular integral of the differential equation

$$\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = 5x^2 - 1$$ *(3 marks)*

(b) By differentiating $(x^2 - 1)y = c$ implicitly, where y is a function of x and c is a constant, show that $y = \frac{c}{x^2 - 1}$ is a solution of the differential equation

$$\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = 0$$ *(3 marks)*

(c) Hence find the general solution of

$$\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = 5x^2 - 1$$ *(2 marks)*

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3 (a) Show that $\sin x$ is an integrating factor for the differential equation

$$\frac{dy}{dx} + (\cot x)y = 2\cos x$$ *(3 marks)*

(b) Solve this differential equation, given that $y = 2$ when $x = \frac{\pi}{2}$. *(6 marks)*

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3 (a) Show that x^2 is an integrating factor for the first-order differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 3(x^3 + 1)^{\frac{1}{2}}$$ *(3 marks)*

(b) Solve this differential equation, given that $y = 1$ when $x = 2$. *(6 marks)*

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3 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

given that $y = 3$ when $x = 0$. *(8 marks)*

FP3 Introduction to Differential Equations Answers

3(a)	$y = x^3 - x \Rightarrow y'(x) = 3x^2 - 1$	B1		Accept general cubic.
	$\frac{dy}{dx} + \frac{2xy}{x^2-1} = 3x^2 - 1 + \frac{2x(x^3-x)}{x^2-1}$	M1		Substitution into LHS of DE
	$= 3x^2 - 1 + \frac{2x^2(x^2-1)}{x^2-1} = 5x^2 - 1$	A1	3	Completion. If using general cubic all unknown constants must be found
(b)	$\frac{d}{dx}\left[(x^2-1)y\right] = 2xy + (x^2-1)\frac{dy}{dx}$	M1A1		
	Differentiating $(x^2-1)y = c$ wrt x leads to $2xy + (x^2-1)\frac{dy}{dx} = 0$			**SC** Differentiated but not implicitly give max of 1/3 for complete solution
	$\Rightarrow y = \frac{c}{x^2-1}$ is a soln. of $\frac{dy}{dx} + \frac{2xy}{x^2-1} = 0$	A1	3	Be generous
(c)	$\Rightarrow y = \frac{c}{x^2-1}$ is a soln with one arb. constant of $\frac{dy}{dx} + \frac{2xy}{x^2-1} = 0$			
	$\Rightarrow y = \frac{c}{x^2-1}$ is a CF of the DE			
	GS is CF + PI	M1		Must be using 'hence'; CF and PI functions of x only
	$y = \frac{c}{x^2-1} + x^3 - x$	A1	2	CSO Must have explicitly considered the link between one arbitrary constant and the GS of a first order differential equation.
	Total		**8**	

3(a)	IF is $e^{\int \cot x\,dx}$	M1		
	$= e^{\ln \sin x}$	A1		
	$= \sin x$	A1	3	AG
(b)	$\frac{d}{dx}(y\sin x) = 2\sin x\cos x$	M1 A1		
	$y\sin x = \int \sin 2x\,dx$	M1		Method to integrate 2sin*x*cos*x*
	$y\sin x = -\frac{1}{2}\cos 2x + c$	A1		OE
	$y = 2$ when $x = \frac{\pi}{2} \Rightarrow$			
	$2\sin\frac{\pi}{2} = -\frac{1}{2}\cos\pi + c$	m1		Depending on at least one M
	$c = \frac{3}{2} \Rightarrow y\sin x = \frac{1}{2}(3 - \cos 2x)$	A1	6	OE eg $y\sin x = \sin^2 x + 1$
	Total		**9**	

3(a)	IF is $\exp\left(\int \frac{2}{x}dx\right)$	M1		And with integration attempted
	$= e^{2\ln x}$	A1		
	$= x^2$	A1	3	CSO **AG** be convinced
(b)	$\frac{d}{dx}\left[yx^2\right] = 3x^2(x^3+1)^{\frac{1}{2}}$	M1A1		PI
	$\Rightarrow yx^2 = \frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} + A$	m1		$k\left(x^3+1\right)^{\frac{3}{2}}$
		A1		Condone missing 'A'
	$\Rightarrow 4 = \frac{2}{3}(9)^{\frac{3}{2}} + A$	m1		Use of boundary conditions to find constant
	$\Rightarrow A = -14$			
	$\Rightarrow y = x^{-2}\left\{\frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} - 14\right\}$	A1	6	Any correct form
	Total		**9**	

3	IF is $e^{\int \tan x\,dx}$	M1		
	$= e^{-\ln\cos x} = e^{\ln\sec x}$	A1		Accept either
	$= \sec x$	A1ft		ft on earlier sign error
	$\frac{d}{dx}(y\sec x) = \sec^2 x$	M1A1		
	$y\sec x = \int \sec^2 x\,dx$			
	$y\sec x = \tan x + c$	A1		Condone missing c
	$y = 3$ when $x = 0 \Rightarrow 3\sec 0 = 0 + c$	m1		
	$c = 3 \Rightarrow y\sec x = \tan x + 3$	A1	8	OE; condone solution finishing at $c = 3$ provided no errors
	Total		**8**	