Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 1

Question:

Find the value of x for which
$2\tanh x - 1 = 0$,
giving your answer in terms of a natural logarithm. **[E]**

Solution:

$$2\tanh x - 1 = 0$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$2\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1$$

$$2e^x - 2e^{-x} = e^x + e^{-x}$$

$$e^x = 3e^{-x}$$

You multiply both sides of this equation by e^x.

$$e^{2x} = 3$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

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Review Exercise 1
Exercise A, Question 2

Question:

Starting from the definition of $\cosh x$ in terms of exponentials, find, in terms of natural logarithms, the values of x for which $5 = 3\cosh x$. **[E]**

Solution:

$$5 = 3\cosh x$$

$$5 = 3\left(\frac{e^x + e^{-x}}{2}\right)$$

The wording of the question requires you to use the definition $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$10 = 3e^x + 3e^{-x}$$

$$3e^x - 10 + 3e^{-x} = 0$$

You multiply this equation throughout by e^x.

$$3e^{2x} - 10e^x + 3 = 0$$

$$(3e^x - 1)(e^x - 3) = 0$$

You may find it helpful to substitute $y = e^x$ and then, factorise, $3y^2 - 10y + 3 = (3y - 1)(y - 3) = 0$. This gives $y = \frac{1}{3}$ and $y = 3$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 3$. With practice, the substitution can be omitted.

$$e^x = \frac{1}{3}, e^x = 3$$

$$x = \ln\left(\frac{1}{3}\right), \ln 3$$

The answer $x = -\ln 3$ would also be acceptable.

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Review Exercise 1
Exercise A, Question 3

Question:

The curves with equations $y = 5\sinh x$ and $y = 4\cosh x$ meet at the point $A(\ln p, q)$.
Find the exact values of p and q. **[E]**

Solution:

The curves intersect when

$$5\sinh x = 4\cosh x$$

$$5\left(\frac{e^x - e^{-x}}{2}\right) = 4\left(\frac{e^x + e^{-x}}{2}\right)$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$5e^x - 5e^{-x} = 4e^x + 4e^{-x}$$

$$e^x = 9e^{-x}$$

$$e^{2x} = 9$$

$$2x = \ln 9$$

$$x = \frac{1}{2}\ln 9 = \ln\sqrt{9} = \ln 3$$

Using the law of logarithms $n\ln a = \ln a^n$ with $n = \frac{1}{2}$ and $a = 9$.

$$y = 5\sinh(\ln 3) = 5\left(\frac{e^{\ln 3} - e^{-\ln 3}}{2}\right) = \frac{5}{2}\times\left(3 - \frac{1}{3}\right)$$

$$= \frac{5}{2}\times\frac{8}{3} = \frac{20}{3}$$

$$p = 3, q = \frac{20}{3}$$

$e^{\ln 3} = 3$ and $e^{-\ln 3} = e^{\ln 1 - \ln 3} = e^{\ln\frac{1}{3}} = \frac{1}{3}$, using $\ln 1 = 0$ and the law of logarithms $\ln a - \ln b = \ln\frac{a}{b}$.

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Review Exercise 1
Exercise A, Question 4

Question:

Find the values of x for which
$5\cosh x - 2\sinh x = 11$,
giving your answers as natural logarithms. **[E]**

Solution:

$5\cosh x - 2\sinh x = 11$

$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$

$5e^x + 5e^{-x} - 2e^x + 2e^{-x} = 22$

$3e^x - 22 + 7e^{-x} = 0$

$3e^{2x} - 22e^x + 7 = 0$

$(3e^x - 1)(e^x - 7) = 0$

$e^x = \frac{1}{3}, 7$

$x = \ln\frac{1}{3}, \ln 7$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

You multiply this equation throughout by e^x.

You may find it helpful to substitute $y = e^x$ and then, factorising $3y^2 - 22y + 7 = (3y - 1)(y - 7) = 0$. This gives $y = \frac{1}{3}$ and $y = 7$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 7$. With practice, the substitution can be omitted.

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Review Exercise 1
Exercise A, Question 5

Question:

By expressing $\sinh 2x$ and $\cosh 2x$ in terms of exponentials, find the exact values of x for which
$6\sinh 2x + 9\cosh 2x = 7$,
giving each answer in the form $\frac{1}{2}\ln p$, where p is a rational number. **[E]**

Solution:

$$6\sinh 2x + 9\cosh 2x = 7$$
$$6\left(\frac{e^{2x}-e^{-2x}}{2}\right)+9\left(\frac{e^{2x}+e^{-2x}}{2}\right) = 7$$
$$6e^{2x}-6e^{-2x}+9e^{2x}+9e^{-2x} = 14$$
$$15e^{2x}-14+3e^{-2x} = 0$$
$$15e^{4x}-14e^{2x}+3 = 0$$
$$(3e^{2x}-1)(5e^{2x}-3) = 0$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ replacing x by $2x$.

You multiply this equation throughout by e^{2x}.

$$e^{2x} = \frac{1}{3}, \frac{3}{5}$$
$$2x = \ln\frac{1}{3}, \ln\frac{3}{5}$$
$$x = \frac{1}{2}\ln\frac{1}{3}, \frac{1}{2}\ln\frac{3}{5}$$
$$p = \frac{1}{3}, \frac{3}{5}$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

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Review Exercise 1
Exercise A, Question 6

Question:

Given that
$\sinh x + 2\cosh x = k$,
where k is a positive constant,

a find the set of values of k for which at least one real solution of this equation exists,
b solve the equation when $k = 2$. **[E]**

Solution:

a $\sinh x + 2\cosh x = k$

$$\frac{e^x - e^{-x}}{2} + 2\left(\frac{e^x + e^{-x}}{2}\right) = k$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$e^x - e^{-x} + 2e^x + 2e^{-x} = 2k$$
$$3e^x - 2k + e^{-x} = 0$$
$$3e^{2x} - 2ke^x + 1 = 0$$

Let $y = e^x$

$3y^2 - 2ky + 1 = 0$

$$y = \frac{2k \pm \sqrt{(4k^2 - 12)}}{6}$$

Using the quadratic formula $y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

$$= \frac{k \pm \sqrt{(k^2 - 3)}}{3} \quad *$$

For real y

$k^2 - 3 \geq 0 \Rightarrow k \geq \sqrt{3}, k \leq -\sqrt{3}$

If $x \leq -\sqrt{3}$, then both $\frac{k + \sqrt{(k^2 - 3)}}{3}$ and $\frac{k - \sqrt{(k^2 - 3)}}{3}$ are negative.

As $y = e^x > 0$ for all real x, $k \leq -\sqrt{3}$ is rejected.

$k \geq \sqrt{3}$.

b Using $*$ above with $k = 2$

You could solve the equation in part **b** without using part **a** but it is efficient to use the work you have already done.

$$y = e^x = \frac{2 \pm \sqrt{(4-3)}}{3} = \frac{2 \pm 1}{3}$$

$$e^x = 1, \frac{1}{3} \Rightarrow x = \ln 1, \ln\frac{1}{3} = 0, -\ln 3$$

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Review Exercise 1
Exercise A, Question 7

Question:

Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials,

a prove that $\cosh^2 x - \sinh^2 x = 1$,

b solve the equation $\operatorname{cosech} x - 2\coth x = 2$, giving your answer in the form $k \ln a$, where k and a are integers. **[E]**

Solution:

a $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4}$

$= \frac{4}{4} = 1$, as required.

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b $\operatorname{cosech} x - 2\coth x = 2$

$\frac{1}{\sinh x} - \frac{2\cosh x}{\sinh x} = 2$

You use $\operatorname{cosech} x = \frac{1}{\sinh x}$ and $\coth x = \frac{\cosh x}{\sinh x}$.

$\times \sinh x$

$1 - 2\cosh x = 2\sinh x$

$2\sinh x + 2\cosh x = 1$

$2\left(\frac{e^x - e^{-x}}{2}\right) + 2\left(\frac{e^x - e^{-x}}{2}\right) = 1$

You use the definitions $\sinh x$ and $\cosh x$ in terms of exponentials to obtain an equation in exponentials which you solve using logarithms.

$e^x - e^{-x} + e^x + e^{-x} = 1$

$2e^x = 1 \Rightarrow e^x = \frac{1}{2}$

$x = \ln \frac{1}{2} = -\ln 2$

$\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, as $\ln 1 = 0$.

$k = -1, a = 2$

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Review Exercise 1
Exercise A, Question 8

Question:

a From the definition of $\cosh x$ in terms of exponentials, show that $\cosh 2x = 2\cosh^2 x - 1$.

b Solve the equation $\cosh 2x - 5\cosh x = 2$, giving the answers in terms of natural logarithms. **[E]**

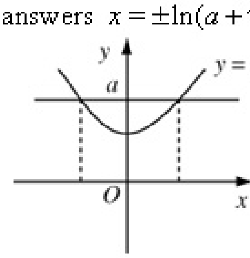
Solution:

a
$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$
$$= 2 \times \frac{e^{2x} + 2 + e^{-2x}}{4} - 1$$
$$= \frac{2e^{2x}}{4} + \frac{4}{4} + \frac{2e^{-2x}}{4} - 1$$
$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x, \text{ as required}$$

$$\left(e^x + e^{-x}\right)^2 = \left(e^x\right)^2 + 2e^x \cdot e^{-x} + \left(e^{-x}\right)^2 = e^{2x} + 2 + e^{-2x}$$

b Using the result in part a
$$\cosh 2x - 5\cosh x = 2$$
$$2\cosh^2 x - 1 - 5\cosh x = 2$$
$$2\cosh^2 x - 5\cosh x - 3 = 0$$
$$(2\cosh x + 1)(\cosh x - 3) = 0$$
$$\cosh x = -\frac{1}{2}, \cosh x = 3$$
$\cosh x = -\frac{1}{2}$ is impossible
$$x = \pm \operatorname{arcosh} x = \pm\ln\left(3 + \sqrt{8}\right)$$

If $\operatorname{arcosh} x > 0$ then $\operatorname{arcosh} x = \ln(x + \sqrt{(x^2 - 1)})$. However if $\cosh x = a$, where $a > 0$, then there are two answers $x = \pm\ln(a + \sqrt{(a^2 - 1)})$



These answers can also be written as $x = \ln\left(3 \pm \sqrt{8}\right)$.

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Review Exercise 1
Exercise A, Question 9

Question:

a Using the definition of $\cosh x$ in terms of exponentials, prove that
$4\cosh^3 x - 3\cosh x = \cosh 3x$.

b Hence, or otherwise, solve the equation $\cosh 3x = 5\cosh x$, giving your answer as natural logarithms. **[E]**

Solution:

a
$$4\cosh^3 x - 3\cosh x = 4\left(\frac{e^x + e^{-x}}{2}\right)^3 - 3\left(\frac{e^x + e^{-x}}{2}\right)$$
$$= \frac{e^{3x} + 3e^x + 3e^{-x} + e^{-3x}}{2} - \frac{3e^x + 3e^{-x}}{2}$$
$$= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x\text{, as required.}$$

Using the binomial expansion $(e^x + e^{-x})^3$
$$= (e^x)^3 + 3(e^x)^2 \cdot e^{-x} + 3e^x(e^{-x})^2 + (e^{-x})^3$$
$$= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}.$$

b $\cosh 3x = 5\cosh x$

Using the result in part **a**
$$4\cosh^3 x - 3\cosh x = 5\cosh x$$
$$4\cosh^3 x - 8\cosh x = 0$$
$$4\cosh x(\cosh^2 x - 2) = 0$$
As for all x, $\cosh x \geq 1$,
$$\cosh x = \sqrt{2}$$

There are 3 possible answers to this cubic, $\cosh x = 0$, $\cosh x = -\sqrt{2}$ and $\cosh x = \sqrt{2}$. As for all real x, $\cosh x \geq 1$ only the last of the three gives real values of x.

$$x = \pm\ln(\sqrt{2} + 1)$$

Using $\operatorname{arcosh} x = \ln\left(x + \sqrt{(x^2 - 1)}\right)$ with

$$-\ln(\sqrt{2}+1) = \ln\left(\frac{1}{\sqrt{2}+1}\right) = \ln\left(\frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$
$$= \ln\left(\frac{\sqrt{2}-1}{1}\right) = \ln(\sqrt{2}-1)$$

The solutions of $\cosh 3x = 5\cosh x$, as natural logarithms, are $x = \ln(\sqrt{2} \pm 1)$.

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Review Exercise 1
Exercise A, Question 10

Question:

a Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that $\cosh(A-B) = \cosh A\cosh B - \sinh A\sinh B$.

b Hence, or otherwise, given that $\cosh(x-1) = \sinh x$, show that

$$\tanh x = \frac{e^2+1}{e^2+2e-1}.$$ **[E]**

Solution:

a $\cosh A\cosh B - \sinh A\sinh B$

$$= \left(\frac{e^A+e^{-A}}{2}\right)\left(\frac{e^B+e^{-B}}{2}\right) - \left(\frac{e^A-e^{-A}}{2}\right)\left(\frac{e^B-e^{-B}}{2}\right)$$

$$= \frac{1}{4}\left(e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}-e^{A+B}+e^{-A+B}+e^{A-B}-e^{-A-B}\right)$$

$$= \frac{1}{4}\left(2e^{-A+B}+2e^{A-B}\right) = \frac{e^{A-B}+e^{-(A-B)}}{2}$$

$= \cosh(A-B)$, as required.

When multiplying out the brackets you must be careful to obtain all eight terms with the correct signs.

You use the definition $\cosh x = \frac{e^x+e^{-x}}{2}$ with $x = A-B$.

b $\cosh x\cosh 1 - \sinh x\sinh 1 = \sinh x$

$$\cosh x\cosh 1 = \sinh x(1+\sinh 1)$$

$$\tanh x = \frac{\cosh 1}{1+\sinh 1}$$

You expand $\cosh(x-1)$ using the result of part **a**.

Divide both sides of this equation by $\cosh x(1+\sinh 1)$ and use $\tanh x = \frac{\sinh x}{\cosh x}$.

$$\tanh x = \frac{\frac{e+e^{-1}}{2}}{1+\frac{e-e^{-1}}{2}} = \frac{e+e^{-1}}{2+e-e^{-1}} = \frac{e^2+1}{e^2+2e-1}$$, as required.

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Review Exercise 1
Exercise A, Question 11

Question:

a Starting from the definition $\sinh y = \frac{e^y - e^{-y}}{2}$, prove that, for all real values of x,

$\text{arsinh}x = \ln[x + \sqrt{(1+x^2)}]$.

b Hence, or otherwise, prove that, for $0 < \theta < \pi$,

$\text{arsinh}(\cot\theta) = \ln\left(\cot\frac{\theta}{2}\right)$. **[E]**

Solution:

a Let $y = \text{arsinh}x$

then $x = \sinh y = \frac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x + \sqrt{(4x^2+4)}}{2}$

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

b $\text{arsinh}(\cot\theta) = \ln\left[\cot\theta + \sqrt{(1+\cot^2\theta)}\right]$

Using the result of part **a** with $x = \cot\theta$.

$= \ln(\cot\theta + \text{cosec}\theta)$

Using $\text{cosec}^2\theta = 1 + \cot^2\theta$.

$= \ln\left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right) = \ln\left(\frac{\cos\theta + 1}{\sin\theta}\right)$

$= \ln\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

You use both double angle formulae $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x\cos x$ with $2x = \theta$.

$= \ln\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) = \ln\left(\cot\frac{\theta}{2}\right)$, as required.

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Review Exercise 1
Exercise A, Question 12

Question:

Given that $n \in \mathbb{Z}^+, x \in \mathbb{R}$ and $\mathbf{M} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$, prove that $\mathbf{M}^n = \mathbf{M}$. **[E]**

Solution:

Let $n = 1$
The result $\mathbf{M}^n = \mathbf{M}$ becomes $\mathbf{M}^1 = \mathbf{M}$, which is true.
Assume the result is true for $n = k$.
That is

$$\mathbf{M}^k = \mathbf{M} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M}$$

$$= \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix} \cdot \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^4 x - \cosh^2 x \sinh^2 & x \cosh^4 x - \cosh^2 x \sinh^2 x \\ -\sinh^2 x \cosh^2 x + \sinh^4 x & -\sinh^2 x \cosh^2 + \sinh^4 x \end{pmatrix}$$

$$\cosh^4 x - \cosh^2 x \sinh^2 x = \cosh^2 x(\cosh^2 x - \sinh^2 x)$$
$$= \cosh^2 x$$
$$-\sinh^2 x \cosh^2 x + \sinh^4 x = \sinh^2 x(-\cosh^2 x + \sinh^2 x)$$
$$= -\sinh^2 x$$

You can prove this result using mathematical induction, a method of proof you learnt in the FP1 module. The prerequisites in the FP3 specification state that a knowledge of FP1 is assumed and may be tested.

You use the identity $\cosh^2 x - \sinh^2 x = 1$ to simplify the terms in the matrix.

Hence $\mathbf{M}^{k+1} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$
and this is the result for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 13

Question:

Solve for real x and y, the simultaneous equations

$\cosh x = 3\sinh y$

$2\sinh x = 5 - 6\cosh y,$

expressing your answers in terms of natural logarithms. **[E]**

Solution:

$\cosh x = 3\sinh y$

Multiply by 2

$2\cosh x = 6\sinh y$

Squaring both sides

$4\cosh^2 x = 36\sinh^2 y$ ①

$2\sinh x = 5 - 6\cosh y$

Squaring both sides

$4\sinh^2 x = (5 - 6\cosh y)^2$ ②

① − ②

$4\cosh^2 x - 4\sinh^2 x = 36\sinh^2 y - (5 - 6\cosh y)^2$

$4 = 36\sinh^2 y - 25 + 60\cosh y - 36\cosh^2 y$

$4 = 60\cosh y - 25 - 36(\cosh^2 y - \sinh^2 y)$

$4 = 60\cosh y - 25 - 36$

$60\cosh y = 65 \Rightarrow \cosh y = \frac{13}{12}$

$y = \pm\ln\left(\frac{13}{12} + \sqrt{\left(\frac{169}{144} - 1\right)}\right) = \pm\ln\left(\frac{13}{12} + \sqrt{\left(\frac{25}{144}\right)}\right)$

$= \pm\ln\left(\frac{13}{12} + \frac{5}{12}\right) = \pm\ln\frac{3}{2}$

If $y = -\ln\frac{3}{2}$, then

$\sinh\left(-\ln\frac{3}{2}\right) = \frac{e^{-\ln\frac{3}{2}} - e^{\ln\frac{3}{2}}}{2} = \frac{\frac{2}{3} - \frac{3}{2}}{2} = -\frac{5}{12}$

As $\cosh x = 3\sinh y$, this gives

$\cosh x = 3 \times -\frac{5}{12} = -\frac{5}{4}$

As $\cosh x \geq 1$ for all real x, this is impossible and the solution $y = -\ln\frac{3}{2}$ is rejected.

$2\sinh x = 5 - 6\cosh y = 5 - 6 \times \frac{13}{12} = -\frac{3}{2}$

$\sinh x = -\frac{3}{4}$

$x = \ln\left(-\frac{3}{4} + \sqrt{\left(\frac{9}{16} + 1\right)}\right) = \ln\left(-\frac{3}{4} + \sqrt{\left(\frac{25}{16}\right)}\right)$

$= \ln\left(-\frac{3}{4} + \frac{5}{4}\right) = \ln\frac{1}{2} = -\ln 2$

$x = -\ln 2, y = \ln\frac{3}{2}$

When you square an equation, you may introduce false solutions. In this case equation ① will contain any solutions of $2\cosh x = -6\sinh y$ as well as $2\cosh x = 6\sinh y$, so you will need to check any solutions you obtain.

The identity $\cosh^2\theta - \sinh^2\theta = 1$ is used twice.

If $\cosh x = a$, then there are two possible values of x, $x = \pm\ln(a + \sqrt{(a^2 - 1)})$. You need to check that both answers are possible.

If $y = \ln\frac{3}{2}$, then $\sinh y = \frac{5}{12}$ and $\cosh x = \frac{5}{4}$ and this is the correct solution.

If $\sinh x = a$, then there is just one possible value of x, $x = \ln(a + \sqrt{(a^2 + 1)})$.

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Review Exercise 1
Exercise A, Question 14

Question:

a Starting from the definition of tanh x in terms of e^x, show that

$$\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

and sketch the graph of $y = \text{artanh}\, x$.

b Solve the equation $x = \tanh[\ln\sqrt{(6x)}]$ for $0 < x < 1$. **[E]**

Solution:

a Let $y = \text{artanh}\, x$

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y}$$

$$= \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$xe^{2y} + x = e^{2y} - 1$$

$$e^{2y}(1 - x) = 1 + x$$

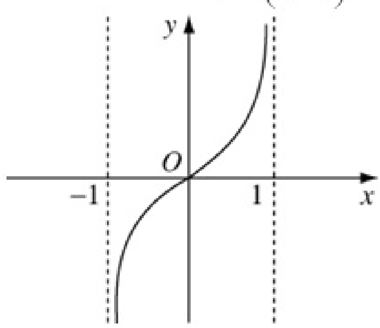
$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \ln\left(\frac{1+x}{1-x}\right)$$

You have been asked to prove a standard result in this question. You should learn the proof of this and other similar results as part of your preparation for the examination.

Make e^{2y} the subject of the formulas and then take logarithms.

$y = \text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as required.



You need to be able to sketch the graphs of the hyperbolic and inverse hyperbolic functions. When you sketch a graph you should show any important features of the curve. In this case, you should show the asymptotes $x = -1$ and $x = 1$ of the curve.

Graph of $y = \text{artanh}\, x$

b $x = \tanh\left[\ln\sqrt{(6x)}\right]$

$$\ln\sqrt{(6x)} = \text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\ln\sqrt{(6x)} = \ln\sqrt{\left(\frac{1+x}{1-x}\right)}$$

$$\sqrt{(6x)} = \sqrt{\left(\frac{1+x}{1-x}\right)}$$

You use the result in part **a**.

Squaring

$$6x = \frac{1+x}{1-x}$$

$$6x - 6x^2 = 1 + x$$

$$6x^2 - 5x + 1 = (3x-1)(2x-1) = 0$$

$$x = \frac{1}{2}, \frac{1}{3}$$

As you have squared this equation, you might have introduced an incorrect solution. It would be sensible to check on your calculator that $x = \frac{1}{2}, \frac{1}{3}$ are solutions of $x = \tanh\left[\ln\sqrt{(6x)}\right]$. In this case, both are correct.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 15

Question:

a Show that, for $0 < x \leq 1$,

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right) = -\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right).$$

b Using the definitions of cosh x and sinh x in terms of exponentials, show that, for $0 < x \leq 1$,

$$\text{arsech}\,x = \ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right).$$

c Solve the equation

$3\tanh^2 x - 4\text{sech}\ x + 1 = 0$,

giving exact answers in terms of natural logarithms. **[E]**

Solution:

a $\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)+\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$

$=\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$

$=\ln\frac{1-(1-x^2)}{x^2}=\ln\frac{x^2}{x^2}=\ln 1=0$

Hence

$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)=-\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$, as required.

There are a number of different ways of starting this question. The method used here begins by using the log rule $\log a+\log b=\log ab$.

This is the difference of two squares $a^2-b^2=(a-b)(a+b)$ with $a=1$ and $b=\sqrt{(1-x^2)}$.

b Let $y=\text{arsech}\, x$

$\text{sech}\, y=x$

$\frac{2}{e^y+e^{-y}}=x$

$2=xe^y+xe^{-y}$

$xe^{2y}-2e^y+x=0$

$e^y=\frac{2\pm\sqrt{(4-4x^2)}}{2x}=\frac{1\pm\sqrt{(1-x^2)}}{x}$

$y=\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right),\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)$

$=\pm\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$, using the result of **a**

$y=\text{arsech}\, x=\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$, as required

Multiply throughout by e^y and treat the result as a quadratic in e^y.

Either of the two answers is possible but it is conventional to take $0<\text{arsech}\, x\le 1$.

c Using $\text{sech}^2 x=1-\tanh^2 x$

$3\tanh^2 x-4\text{sech}\, x+1=0,$

$3-3\text{sech}^2 x-4\text{sech}\, x+1=0$

$3\text{sech}^2 x+4\text{sech}\, x-4=0$

$(3\text{sech} x-2)(\text{sech}\, x+2)=0$

$\text{sech}\, x=\frac{2}{3}$

$x=\pm\ln\left(\frac{1+\sqrt{\left(1-\frac{4}{9}\right)}}{\frac{2}{3}}\right)=\pm\ln\left(\frac{3+\sqrt{5}}{2}\right)$

$\text{sech}\, x=-2$ is impossible with real values of x.

$x=\ln\left(\frac{3\pm\sqrt{5}}{2}\right)$ is another correct form of the answer.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 16

Question:

a Express $\cosh 3\theta$ and $\cosh 5\theta$ in terms of $\cosh\theta$.
b Hence determine the real roots of the equation
$2\cosh 5x + 10\cosh 3x + 20\cosh x = 243$,
giving your answers to 2 decimal places. **[E]**

Solution:

a $\cosh 3\theta = \cosh(2\theta + \theta)$
$= \cosh 2\theta\cosh\theta + \sinh 2\theta\sinh\theta$
$\cosh\theta = c$ and $\sinh\theta = s$
$\cosh 3\theta = (2c^2 - 1)c + 2sc \times s$
$= 2c^3 - c + 2s^2c$
$= 2c^3 - c + 2(c^2 - 1)c$
$= 2c^3 - c + 2c^3 - 2c$
$= 4\cosh^3\theta - 3\cosh\theta$
$\cosh 5\theta = \cosh(3\theta + 2\theta) = \cosh 3\theta\cosh 2\theta + \sinh 3\theta\sinh 2\theta$
$\cosh 3\theta\cosh 2\theta = (4c^3 - 3c)(2c^2 - 1)$
$= 8c^5 - 10c^3 + 3c$
$\sinh 3\theta\sinh 2\theta = \sinh(2\theta + \theta)\sinh 2\theta$
$= (\sinh 2\theta\cosh\theta + \cosh 2\theta\sinh\theta)\sinh 2\theta$
$= (2sc \times c + (2c^2 - 1)s)2sc$
$= 2(4c^2 - 1)s^2c$
$= 2(4c^2 - 1)(c^2 - 1)c$
$= 8c^5 - 10c^3 + 2c$
Combining the results
$\cosh 5\theta = 8c^5 - 10c^3 + 3c + 8c^5 - 10c^3 + 2c$
$= 16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta$

In a complicated calculation like this, it is sensible to use the abbreviated notation suggested here but, if you intend to use a notation like this, you should state the notation in the solution so that the marker knows what you are doing.

You use the 'double angle' for hyperbolics $\cosh 2\theta = 2\cosh^2\theta - 1$ and $\sinh 2\theta = 2\sinh\theta\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$. The signs in these formulae can be worked out using **Osborn's rule**.

b $2\cosh 5x + 10\cosh 3x + 20\cosh x = 243$,
Letting $\cosh x = c$ and using the results in **a**
$32c^5 - 40c^3 + 10c + 40c^3 - 30c + 20c = 243$
$$c^5 = \frac{243}{32} \Rightarrow c = \frac{3}{2}$$
$$x = \pm\text{arcosh}\,\frac{3}{2} \approx \pm 0.96$$

You can use an inverse hyperbolic button on your calculator to find $\text{arcosh}\,\frac{3}{2}$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 17

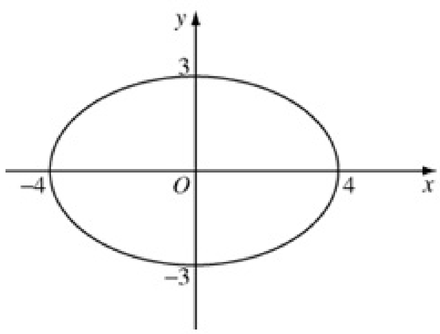
Question:

An ellipse has equation $\frac{x^2}{16}+\frac{y^2}{9}=1$.

a Sketch the ellipse.
b Find the value of the eccentricity e.
c State the coordinates of the foci of the ellipse. **[E]**

Solution:

a



When you draw a sketch, you should show the important features of the curve. When drawing an ellipse, you should show that it is a simple closed curve and indicate the coordinates of the points where the curve intersects the axes.

b $b^2 = a^2(1-e^2)$

$9 = 16(1-e^2) = 16-16e^2$

$e^2 = \frac{16-9}{16} = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

c The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 4\times\frac{\sqrt{7}}{4}, 0\right) = (\pm\sqrt{7}, 0)$

The formula you need for calculating the eccentricity and the coordinates of the foci are given in the Edexcel formula booklet you are allowed to use in the examination. You should be familiar with the formulae in that booklet. You should quote any formulae you use in your solution.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 18

Question:

The hyperbola H has equation $\frac{x^2}{16}-\frac{y^2}{4}=1$. Find

a the value of the eccentricity of H,
b the distance between the foci of H.

The ellipse E has equation $\frac{x^2}{16}+\frac{y^2}{4}=1$.

c Sketch H and E on the same diagram, showing the coordinates of the points where each curve crosses the axes. **[E]**

Solution:

a $b^2=a^2(e^2-1)$

$4=16(e^2-1)=16e^2-16$

$e^2=\frac{16+4}{16}=\frac{20}{16}=\frac{5}{4}$

$e=\frac{\sqrt{5}}{2}$

The formula for calculating the eccentricity is $b^2=a^2(e^2-1)$. It is important not to confuse this with the formula for calculating the eccentricity of an ellipse $b^2=a^2(1-e^2)$.

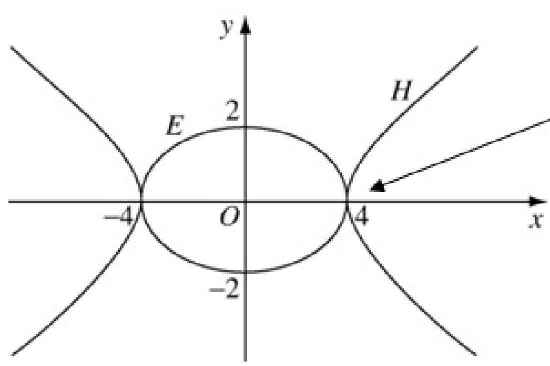
b The coordinates of the foci are given by

$(\pm ae,0)=\left(\pm 4\times\frac{\sqrt{5}}{2},0\right)=(\pm 2\sqrt{5},0)$

The distance between the foci is $4\sqrt{5}$.

The formulae for the foci of an ellipse and a hyperbola are the same $(\pm ae,0)$.

c



In this sketch, you should show where the curves cross the axes. Label which curve is H and which is E. These two curves touch each other on the x-axis.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 19

Question:

The ellipse D has equation $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the ellipse E has equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

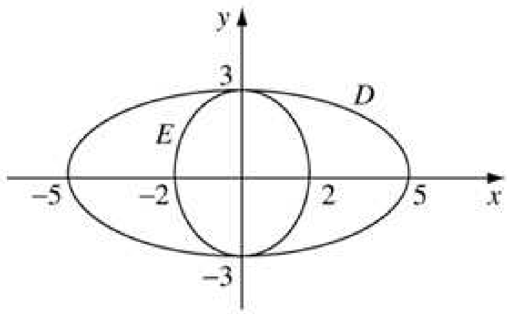
a Sketch D and E on the same diagram, showing the coordinates of the points where each curve crosses the axes.

The point S is a focus of D and the point T is a focus of E.

b Find the length of ST. **[E]**

Solution:

a



b For D

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2) = 25 - 25e^2$

$e^2 = \frac{25-9}{25} = \frac{16}{25}$

$e = \frac{4}{5}$

As the coordinates of a focus of D are $(ae, 0)$, you first need to find the eccentricity of the ellipse using $b^2 = a^2(1-e^2)$ with $a = 5$ and $b = 3$.

For S $(ae, 0) = \left(5 \times \frac{4}{5}, 0\right) = (4, 0)$

As an ellipse has two foci, you could choose either for S and there are also two possible choices for T. The symmetries of the diagram show that you would always get the same distance for ST whichever you choose. It does not matter which you choose but it is sensible to choose the positive coordinate.

For E

$b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) = 9 - 9e^2$

$e^2 = \frac{9-4}{9} = \frac{5}{9}$

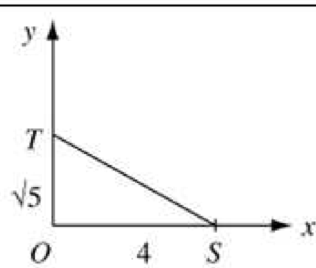
$e = \frac{\sqrt{5}}{3}$

The major axis for ellipse E is along the y-axis, so its foci have coordinates $(0, \pm ae)$. You find the eccentricity of E using $b^2 = a^2(1-e^2)$ with $a = 3$ and $b = 2$, a is always the semi-major axis and b the semi-minor axis, so $a > b$.

For T $(0, ae) = \left(0, 3 \times \frac{\sqrt{5}}{3}\right) = (0, \sqrt{5})$

$ST^2 = 4^2 + (\sqrt{5})^2 = 21$

$ST = \sqrt{21}$



As the focus of D is on the x-axis and the focus of E is on the y-axis, you find the distance between them using Pythagoras' Theorem.

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Review Exercise 1
Exercise A, Question 20

Question:

An ellipse, with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$, has foci S and S'.

a Find the coordinates of the foci of the ellipse.

b Using the focus-directrix property of the ellipse, show that, for any point P on the ellipse,
$SP+S'P=6$. **[E]**

Solution:

a $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) = 9 - 9e^2$

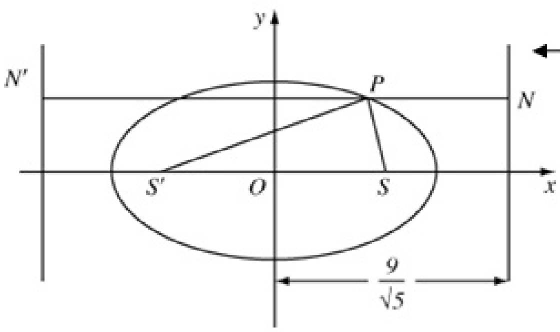
$e^2 = \frac{9-4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

As the coordinates of the foci of an ellipse are $(\pm ae, 0)$, you first need to find the eccentricity of the ellipse using $b^2 = a^2(1-e^2)$ with $a = 3$ and $b = 2$.

The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = (\pm\sqrt{5}, 0)$

b



In this question, you are not asked to draw a diagram but with questions on coordinate geometry it is usually a good idea to sketch a diagram so you can see what is going on.

The equations of the directrices are $x = \pm\frac{a}{e}$.

$x = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$

Let the line through P parallel to the x-axis intersect the directrices at N and N', as shown in the diagram

$N'N = 2 \times \frac{9}{\sqrt{5}} = \frac{18}{\sqrt{5}}$

If you introduce points, like N and N' here, you should define them in your solution and mark them on your diagram. This helps the examiner follow your solution.

The focus directrix property of the ellipse gives that

$SP = ePN$ and $S'P = ePN'$

$SP + S'P = ePN + ePN'$

$= e(PN + PN') = eN'N$

$= \frac{\sqrt{5}}{3} \times \frac{18}{\sqrt{5}} = 6$, as required.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 21

Question:

a Find the eccentricity of the ellipse with equation $3x^2 + 4y^2 = 12$.

b Find an equation of the tangent to the ellipse with equation $3x^2 + 4y^2 = 12$ at the point with coordinates $\left(1, \frac{3}{2}\right)$.

This tangent meets the y-axis at G. Given that S and S' are the foci of the ellipse,

c find the area of $\Delta SS'G$. **[E]**

Solution:

a $3x^2+4y^2=12$

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

$$b^2=a^2\left(1-e^2\right)$$

$$3=4\left(1-e^2\right)=4-4e^2$$

$$e^2=\frac{4-3}{4}=\frac{1}{4}\Rightarrow e=\frac{1}{2}$$

You divide this equation by 12. Comparing the result with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a^2=4$ and $b^2=3$ and you use $b^2=a^2\left(1-e^2\right)$ to calculate e.

b $3x^2+4y^2=12$

Differentiate implicitly with respect to x

$$6x+8y\frac{dy}{dx}=0$$

Differentiating implicitly using the chain rule, $\frac{d}{dx}\left(4y^2\right)=\frac{dy}{dx}\frac{d}{dy}\left(4y^2\right)=\frac{dy}{dx}\times 8y$.

$$\frac{dy}{dx}=-\frac{6x}{8y}=-\frac{3x}{4y}$$

At $\left(1,\frac{3}{2}\right)$

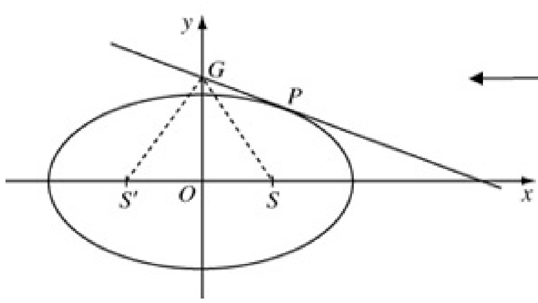
$$\frac{dy}{dx}=\frac{-3\times 1}{4\times\frac{3}{2}}=-\frac{1}{2}$$

Using $y-y_1=m\left(x-x_1\right)$, an equation of the tangent is

$$y-\frac{3}{2}=-\frac{1}{2}(x-1)=-\frac{1}{2}x+\frac{1}{2}$$

$$y=-\frac{1}{2}x+2$$

c



Sketching a diagram makes it clear that the area of the triangle is to be found using the standard expression $\frac{1}{2}$ base $\times$ height with the base $S'S$ and the height OG.

The coordinates of S are

$$(ae,0)=\left(2\times\frac{1}{2},0\right)=(1,0)$$

By symmetry, the coordinates of S' are $(-1,0)$. The y-coordinate of G is given by

$$y=0+2=2$$

You find the y-coordinate of G by substituting $x=0$ into the answer to part a.

$$\Delta SS'G=\frac{1}{2}\text{base}\times\text{height}$$

$$=\frac{1}{2}S'S\times OG$$

$$=\frac{1}{2}2\times 2=2$$

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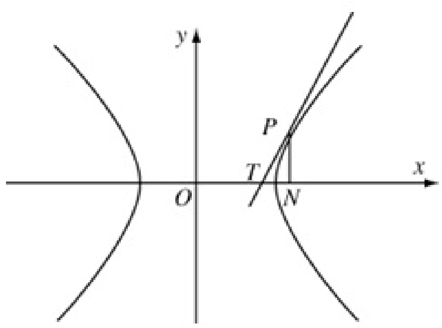
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 22

Question:

The point P lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, and N is the foot of the perpendicular from P onto the x-axis. The tangent to the hyperbola at P meets the x-axis at T.
Show that $OT \cdot ON = a^2$, where O is the origin. **[E]**

Solution:



To find the coordinates of T, it is easiest to carry out your calculation in terms of a parameter. As the question specifies no particular parametric form, you can choose your own. The hyperbolic form has been used here but (a sec t, b tan t) would work as well and there are other possible alternatives.

Let the point P have coordinates (a cosh t, b sinh t)

To find an equation of the tangent PT,

$$\frac{dx}{dt}=a\sinh t,\ \frac{dy}{dt}=b\cosh t$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{b\cosh t}{a\sinh t}$$

Using $y-y_1=m(x-x_1)$

To find the x-coordinate of T, you substitute $y=0$ into a equation of the tangent at P, so first you must obtain an equation for the tangent.

$$y-b\sinh t=\frac{b\cosh t}{a\sinh t}(x-a\cosh t)$$

$$ay\sinh t-ab\sinh^2 t=bx\cosh t-ab\cosh^2 t$$

$$ay\sinh t=bx\cosh t-ab\left(\cosh^2 t-\sinh^2 t\right)$$

$$=bx\cosh t-ab$$

Using the identity $\cosh^2 t-\sinh^2 t=1$.

For T, $y=0$

$$bx\cosh t=ab \Rightarrow x=\frac{a}{\cosh t}$$

The coordinates of N are (a cosh t, 0)

$$OT\cdot ON=\frac{a}{\cosh t}\times a\cosh t=a^2, \text{ as required.}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 23

Question:

The hyperbola C has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

a Show that an equation of the normal to C at the point P $(a\sec t, b\tan t)$ is
$ax\sin t + by = (a^2 + b^2)\tan t$.

The normal to C at P cuts the x-axis at the point A and S is a focus of C. Given that the eccentricity of C is $\frac{3}{2}$, and that $OA = 3OS$, where O is the origin,

b determine the possible values of t, for $0 \le t \le 2\pi$. **[E]**

Solution:

a $\frac{dx}{dt} = a\sec t \tan t, \frac{dy}{dt} = b\sec^2 t$

$\frac{dy}{dx} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t} = \frac{b}{a\sin t}$

To find the gradient of the tangent, you use a version of the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Using $mm' = -1$, the gradient of the normal is given by $m' = -\frac{a\sin t}{b}$

An equation of the normal is

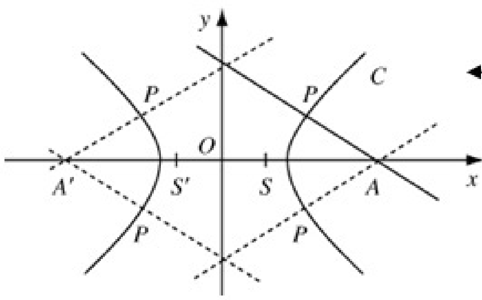
$$y - y_1 = m'(x - x_1)$$

$$y - b\tan t = -\frac{a\sin t}{b}(x - a\sec t)$$

$$by - b^2\tan t = -ax\sin t + a^2\tan t$$

$ax\sin t + by = (a^2 + b^2)\tan t$, as required

b



A diagram is essential here. Without it, you would be unlikely to see that there are four possible points where $OA = 3OS$. There are two to the right of the y-axis, corresponding to the focus S with coordinates $(ae, 0)$, and two to the left of the y-axis, corresponding to the focus, here marked S', with coordinates $(-ae, 0)$.

The x-coordinate of A is given by

$ax\sin t + 0 = (a^2 + b^2)\tan t$

$$x = \frac{a^2 + b^2}{a} \times \frac{\tan t}{\sin t} = \frac{a^2 + b^2}{a\cos t}$$

Hence $OA = \frac{a^2 + b^2}{a\cos t}$

Using $b^2 = a^2(e^2 - 1)$ with $e = \frac{3}{2}$

$b^2 = a^2\left(\frac{9}{4} - 1\right) = \frac{5a^2}{4}$

and $OA = \frac{a^2 + b^2}{a\cos t} = \frac{a^2 + \frac{5a^2}{4}}{a\cos t} = \frac{9a}{4\cos t}$

You need to eliminate b from the length OA to obtain a solvable equation in t from the condition $OA = 3AS$.

As $e = \frac{3}{2}$,

$OS = ae = \frac{3a}{2}$

$OA = 3OS$

$\frac{9a}{4\cos t} = \frac{9a}{2} \Rightarrow \cos t = \frac{1}{2}$

$t = \frac{\pi}{3}, \frac{5\pi}{3}$

These values give two points P, $(2a, \sqrt{3}b)$ and $(2a, -\sqrt{3}b)$.

These are the solutions in the first and fourth quadrants.
From the diagram, by symmetry, there are also solutions in the second and third quadrants giving

$t = \frac{2\pi}{3}, \frac{4\pi}{3}$

These correspond to the two points $(-2a, \sqrt{3}b)$ and $(-2a, -\sqrt{3}b)$ where $\cos t = -\frac{1}{2}$.

The possible values of t are

$t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 24

Question:

An ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are constants and $a > b$.

a Find an equation of the tangent at the point $P(a\cos t, b\sin t)$.

b Find an equation of the normal at the point $P(a\cos t, b\sin t)$.

The normal at P meets the x-axis at the point Q. The tangent at P meets the y-axis at the point R.

c Find, in terms of a, b and t, the coordinates of M, the mid-point of QR.

Given that $0 < t < \frac{\pi}{2}$,

d Show that, as t varies, the locus of M has equation $\left(\frac{2ax}{a^2-b^2}\right)^2+\left(\frac{b}{2y}\right)^2=1$. **[E]**

Solution:

a $x = a\cos t, \quad y = b\sin t$

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{b\cos t}{a\sin t}$$

For the tangent

$$y - y_1 = m(x - x_1)$$

$$y - b\sin t = -\frac{b\cos t}{a\sin t}(x - a\cos t)$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However the calculation in part c will be easier if you simplify the equation at this stage using $\sin^2 t + \cos^2 t = 1$.

$$ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$$

$$ay\sin t + bx\cos t = ab(\sin^2 t + \cos^2 t)$$

$$ay\sin t + bx\cos t = ab$$

b As $\frac{dy}{dx} = -\frac{b\cos t}{a\sin t}$, using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin t}{b\cos t}$$

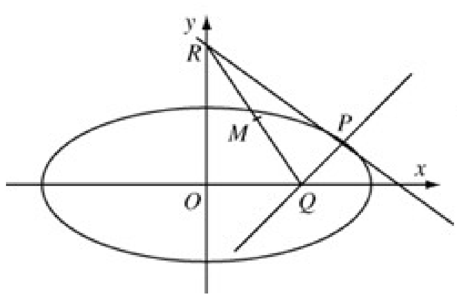
$$y - y_1 = m'(x - x_1)$$

$$y - b\sin t = \frac{a\sin t}{b\cos t}(x - a\cos t)$$

$$by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$$

$$ax\sin t - by\cos t = (a^2 - b^2)\sin t\cos t$$

c



The condition $0 < t < \frac{\pi}{2}$ implies that P is in the first quadrant.

Substituting $y = 0$ into the answer to part **b**

$$ax\sin t = (a^2 - b^2)\sin t\cos t \Rightarrow x = \frac{a^2 - b^2}{a}\cos t$$

You find the x-coordinate of Q by substituting $y = 0$ into the equation you found for the normal in part **b** and solving for x.

The coordinates of Q are $\left(\frac{a^2-b^2}{a}\cos t, 0\right)$

Substituting $x=0$ into the answer to part **a**

$ay\sin t = ab \Rightarrow y = \frac{b}{\sin t}$

You find the y-coordinate of R by substituting $x=0$ into the equation you found for the tangent in part **a** and solving for y.

The coordinates of R are $\left(0, \frac{b}{\sin t}\right)$

The coordinates of M are given by

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{a^2-b^2}{2a}\cos t, \frac{b}{2\sin t}\right)$

d If the coordinates of M are (x, y) then $x = \frac{a^2-b^2}{2a}\cos t \Rightarrow \cos t = \frac{2ax}{a^2-b^2}$ and

$y = \frac{b}{2\sin t} \Rightarrow \sin t = \frac{b}{2y}$

As $\cos^2 t + \sin^2 t = 1$, the locus of

M is $\left(\frac{2ax}{a^2-b^2}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$, as required

$x = \frac{a^2-b^2}{2a}\cos t$ and $y = \frac{b}{2\sin t}$ are the parametric equations of the locus of M. To find the Cartesian equation, you must eliminate t. The form of the answer given in the question gives you a hint that you can use the identity $\cos^2 t + \sin^2 t = 1$ to do this.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 25

Question:

The points S_1 and S_2 have Cartesian coordinates $\left(-\frac{a}{2}\sqrt{3}, 0\right)$ and $\left(\frac{a}{2}\sqrt{3}, 0\right)$ respectively.

a Find a Cartesian equation of the ellipse which has S_1 and S_2 as its two foci, and a semi-major axis of length a.

b Write down an equation of a directrix of this ellipse.

Given that parametric equations of this ellipse are
$x = a\cos\varphi, y = b\sin\varphi$,

c express b in terms of a.

The point P is given by $\varphi = \frac{\pi}{4}$ and the point Q by $\varphi = \frac{\pi}{2}$.

d Show that an equation of the chord PQ is
$(\sqrt{2} - 1)\,x + 2y - a = 0.$ **[E]**

Solution:

a S_2 has coordinates $\left(\frac{a}{2}\sqrt{3}, 0\right)$

Hence

$$e = \frac{\sqrt{3}}{2}$$

$$b^2 = a^2\left(1-e^2\right)$$

$$= a^2\left(1-\frac{3}{4}\right) = \frac{a^2}{4} \quad *$$

Comparing $\left(\frac{a}{2}\sqrt{3}, 0\right)$ with the formula for the focus $(ae, 0)$, $e = \frac{\sqrt{3}}{2}$.

An equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

You are given that a is the semi-major axis, so a can be left in the equation. The data in the question does not include b, so b must be replaced.

Using *

$$\frac{x^2}{a^2} + \frac{y^2}{\frac{a^2}{4}} = 1$$

The required equation is

$$\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$$

$$x^2 + 4y^2 = a^2$$

b Equations of the directrices are

$$x = \pm\frac{a}{e} = \pm\frac{a}{\frac{\sqrt{3}}{2}} = \pm\frac{2a}{\sqrt{3}}$$

c From * above, $b = \frac{a}{2}$

d For Q

$$\left(a\cos\phi, \frac{1}{2}a\sin\varphi\right) = \left(a\cos\frac{\pi}{4}, \frac{1}{2}a\sin\frac{\pi}{4}\right)$$

$$= \left(\frac{a}{\sqrt{2}}, \frac{a}{2\sqrt{2}}\right) = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}\right)$$

For P

$$\left(a\cos\phi, \frac{1}{2}a\sin\varphi\right) = \left(a\cos\frac{\pi}{2}, \frac{1}{2}a\sin\frac{\pi}{2}\right)$$

$$= \left(0, \frac{a}{2}\right)$$

For PQ

$$\frac{y-\frac{a}{2}}{\frac{a\sqrt{2}}{4}-\frac{a}{2}}=\frac{x-0}{\frac{a\sqrt{2}}{2}-0}$$

Using the formula from module C1 for a line, $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$.

$$\frac{4y-2a}{\sqrt{2}-2}=\frac{2x}{\sqrt{2}}$$

The a cancels throughout the denominators of this equation. On the left-hand side $\frac{4\left(y-\frac{a}{2}\right)}{4\left(\frac{\sqrt{2}}{4}-\frac{1}{2}\right)}=\frac{4y-2a}{\sqrt{2}-2}$.

$$4\sqrt{2}y-2\sqrt{2}a=\left(2\sqrt{2}-4\right)x$$

$$\left(4-2\sqrt{2}\right)x+4\sqrt{2}y-2\sqrt{2}a=0$$

Dividing throughout by $2\sqrt{2}$

$\left(\sqrt{2}-1\right)x+2y-a=0$, as required.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 26

Question:

Show that the equations of the tangents with gradient m to the hyperbola with equation

$x^2 - 4y^2 = 4$

are

$y = mx \pm \sqrt{(4m^2 - 1)}$, where $|m| > \frac{1}{2}$. **[E]**

Solution:

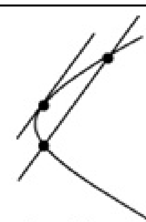
Let the equation of the tangent be $y = mx + c$

Eliminating y between $y = mx + c$ and $x^2 - 4y^2 = 4$

$$x^2 - 4(mx + c)^2 = 4$$

$$x^2 - 4m^2x^2 - 8mcx - 4c^2 = 4$$

$$(4m^2 - 1)x^2 + 8mcx + 4(c^2 + 1) = 0 \quad *$$



If the line was a chord, it would cut the curve in two distinct points and this equation would have a positive discriminant. As the line is a tangent it touches the curve at just one point and this equation has a repeated root. The discriminant is zero.

As the line is a tangent, equation * has repeated roots

'$b^2 - 4ac = 0$'

$$64m^2c^2 - 16(4m^2 - 1)(c^2 + 1) = 0$$

$$64m^2c^2 - 64m^2c^2 - 64m^2 + 16c^2 + 16 = 0$$

$$16c^2 = 64m^2 - 16$$

$$c^2 = 4m^2 - 1 \Rightarrow c = \pm\sqrt{(4m^2 - 1)}$$

The equation of the tangent is

$y = mx \pm \sqrt{(4m^2 - 1)}$, where $|m| > \frac{1}{2}$, as required.

If $|m| < \frac{1}{2}$, then $\sqrt{(4m^2 - 1)}$ would be the square root of a negative number and there would be no real answer. The cases $m = \pm\frac{1}{2}$ are interesting. For these values the equations are $y = \pm\frac{1}{2}x$. These are the asymptotes of the hyperbola and do not touch it at any point with finite coordinates. Asymptotes can be thought of as tangents to the curve 'at infinity'.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 27

Question:

The line with equation $y = mx + c$ is a tangent to the ellipse with equation
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

a Show that $c^2 = a^2m^2 + b^2$.

b Hence, or otherwise, find the equations of the tangents from the point (3, 4) to the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$. **[E]**

Solution:

a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

Multiply this equation throughout by a^2b^2. Then multiply out the bracket and collect the terms together as a quadratic in x.

As the line is a tangent this equation has repeated roots

'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(a^2m^2 + b^2)a^2(c^2 - b^2) = 0$$

$$a^2m^2c^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$$

$$\cancel{a^2m^2c^2} - \cancel{a^2m^2c^2} + a^2m^2b^2 - b^2c^2 + b^4 = 0$$

$$c^2 = a^2m^2 + b^2\text{, as required.}$$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

b $(3, 4) \in y = mx + c$

Hence $4 = 3m + c \Rightarrow c = 4 - 3m$ ①

The tangents have equations of the form $y = mx + c$ and $x = 3, y = 4$ must satisfy this relation.

For this ellipse, $a = 4$ and $b = 5$ and the result in part a becomes

$c^2 = 16m^2 + 25$ ②

Substituting ① into ②

$$(4 - 3m)^2 = 16m^2 + 25$$

$$16 - 24m + 9m^2 = 16m^2 + 25$$

$$7m^2 + 24m + 9 = (m+3)(7m+3) = 0$$

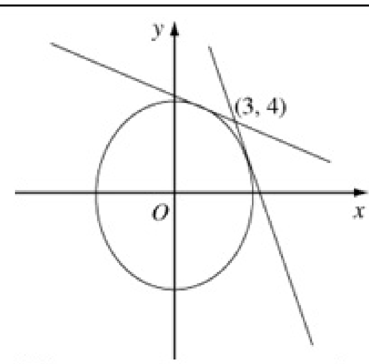
$$m = -3, -\frac{3}{7}$$

If $m = -3$, $c = 4 - 3m = 4 + 9 = 13$

If $m = -\frac{3}{7}$, $c = 4 - 3m = 4 + \frac{9}{7} = \frac{37}{7}$

The equations of the tangents are

$y = -3x + 13$ and $y = -\frac{3}{7}x + \frac{37}{7}$



There are two tangents to the ellipse which pass through (3, 4). Both have negative gradients.

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Review Exercise 1
Exercise A, Question 28

Question:

The ellipse E has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line L has equation $y = mx + c$, where $m > 0$ and $c > 0$.

a Show that, if L and E have any points of intersection, the x-coordinates of these points are the roots of the equation $(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$.

Hence, given that L is a tangent to E,

b show that $c^2 = b^2 + a^2m^2$.

The tangent L meets the negative x-axis at the point A and the positive y-axis at the point B, and O is the origin.

c Find, in terms of m, a and b, the area of the triangle OAB.

d Prove that, as m varies, the minimum area of the triangle OAB is ab.

e Find, in terms of a, the x-coordinate of the point of contact of L and E when the area of the triangle is a minimum. **[E]**

Solution:

a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0, \text{ as required}$$

Multiply this equation throughout by a^2b^2. Then multiply out the bracket and collect the terms together as a quadratic in x.

b As the line is a tangent the result of part **a** has repeated roots

'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(b^2 + a^2m^2)a^2(c^2 - b^2) = 0$$

Divide this equation throughout by $4a^2$.

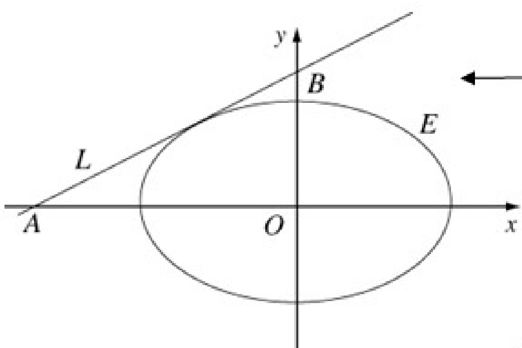
$$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0$$

$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$$

$$c^2 = a^2m^2 + b^2, \text{ as required.}$$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

c



As $c^2 = a^2m^2 + b^2$, $y = mx + c$ could have the forms $y = mx \pm \sqrt{(b^2 + a^2m^2)}$. However, the question specifies that the tangent crosses the positive y-axis. As the line has a positive y intercept, you can reject the negative possibility.

An equation of L is $y = mx + \sqrt{(b^2 + a^2m^2)}$

For A, $y = 0$

$$0 = mx + \sqrt{(b^2 + a^2m^2)} \Rightarrow x = -\frac{\sqrt{(b^2 + a^2m^2)}}{m}$$

Hence $OA = \frac{\sqrt{(b^2 + a^2m^2)}}{m}$

For B, $x = 0$

$y = \sqrt{(b^2 + a^2m^2)}$

Hence $OB = \sqrt{(b^2 + a^2m^2)}$

The area of triangle OAB, T say, is given by

$$T = \frac{1}{2}OA \times OB = \frac{1}{2}\frac{\sqrt{(b^2 + a^2m^2)}}{m}\sqrt{(b^2 + a^2m^2)}$$

$$= \frac{b^2 + a^2m^2}{2m}$$

d $T = \frac{b^2 + a^2m^2}{2m} = \frac{1}{2}b^2m^{-1} + \frac{1}{2}a^2m$

For a minimum

$$\frac{dT}{dm} = -\frac{1}{2}b^2m^{-2} + \frac{1}{2}a^2 = 0$$

$$\frac{b^2}{m^2} = a^2 \Rightarrow m^2 = \frac{b^2}{a^2}$$

As L has a positive gradient

$$m = \frac{b}{a}$$

The diagram shows that the tangent has a positive gradient and so the possible value $-\frac{b}{a}$ can be ignored.

$$\frac{d^2T}{dm^2} = b^2m^{-3} = \frac{b^2}{m^3}$$

At $m = \frac{b}{a}, \frac{d^2T}{dm^2} = \frac{b^2}{m^3} = \frac{a^3}{b} > 0$ and so this gives a minimum value of

$$T = \frac{b^2 + a^2\left(\frac{b}{a}\right)^2}{2\left(\frac{b}{a}\right)} = \frac{2b^2}{2\left(\frac{b}{a}\right)} = ab \text{, as required.}$$

e At $m = \frac{b}{a}, c^2 = a^2m^2 + b^2 = a^2\left(\frac{b}{a}\right)^2 + b^2 = 2b^2$

Substituting $m = \frac{b}{a}$ and $c = \sqrt{2}b$ into the result in part **a**

$$\left(b^2 + a^2 \times \frac{b^2}{a^2}\right)x^2 + 2a^2 \times \frac{b}{a} \times \sqrt{2}bx + a^2(2b^2 - b^2) = 0$$

$$2b^2x^2 + 2\sqrt{2}ab^2x + a^2b^2 = 0$$

Divide this equation throughout by b^2.

$$2x^2 + 2\sqrt{2}ax + a^2 = 0$$

$$(\sqrt{2}x + a)^2 = 0$$

$$x = -\frac{a}{\sqrt{2}}$$

As the line is a tangent, this quadratic must factorise to a complete square. If you cannot see the factors, you can use the quadratic formula.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 29

Question:

a Find the eccentricity of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

b Find also the coordinates of both foci and equations of both directrices of this ellipse.

c Show that an equation for the tangent to this ellipse at the point $P(3\cos\theta, 2\sin\theta)$ is $\frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1$.

d Show that, as θ varies, the foot of the perpendicular from the origin to the tangent at P lies on the curve $(x^2+y^2)^2=9x^2+4y^2$. **[E]**

Solution:

a $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) = 9 - 9e^2$

$e^2 = \frac{9-4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

The formulae you need for calculating the eccentricity, the coordinates of the foci, and the equations of the directrices are given in the Edexcel formula booklet you are allowed to use in the examination. However, it wastes time checking your textbook every time you need to use these formulae and it is worthwhile remembering them. **Remember** to quote any formulae you use in your solution.

b The coordinates of the foci are

$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = (\pm\sqrt{5}, 0)$

The equations of the directrices are

$x = \pm\frac{a}{e} = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$

c $x = 3\cos\theta, \quad y = 2\sin\theta$

$\frac{dx}{d\theta} = -3\sin\theta, \frac{dy}{d\theta} = 2\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{2\cos\theta}{3\sin\theta}$

$y - y_1 = m(x - x_1)$

$y - 2\sin\theta = -\frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

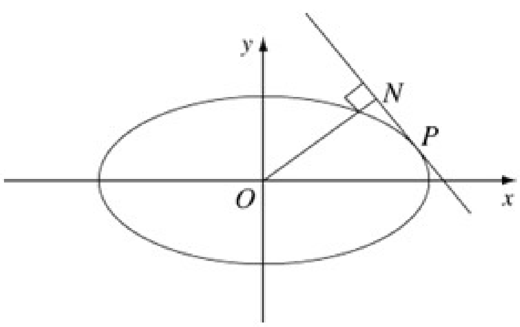
$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

$2x\cos\theta + 3y\sin\theta = 6(\cos^2\theta + \sin^2\theta) = 6$

Divide this line throughout by 6.

$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$, as required

d



Let the foot of the perpendicular from O to the tangent at P be N.
Using $mm' = -1$, the gradient of ON is given by

$m' = -\frac{1}{\frac{dy}{dx}} = \frac{3\sin\theta}{2\cos\theta}$

An equation of ON is $y = \frac{3\sin\theta}{2\cos\theta}x$ *

Eliminating y between equation * and the answer to part c

$$\frac{x\cos\theta}{3} + \frac{\sin\theta}{2}\left(\frac{3\sin\theta}{2\cos\theta}x\right) = 1$$

$$x\left(\frac{4\cos^2\theta + 9\sin^2\theta}{12\cos\theta}\right) = 1$$

$$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$ and $y = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$ are parametric equations of the locus. Eliminating θ between them to obtain a Cartesian equation is not easy and you will need to use the printed answer to help you.

Substituting this expression for x into equation *

$$y = \frac{3\sin\theta}{2\cos\theta} \times \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta} = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$$x^2 + y^2 = \left(\frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2 + \left(\frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2$$

$$= \frac{144\cos^2\theta + 324\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{36(4\cos^2\theta + 9\sin^2\theta)}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \frac{36}{4\cos^2\theta + 9\sin^2\theta}$$

$$9x^2 + 4y^2 = \frac{9\times144\cos^2\theta + 4\times324\sin^2\theta}{\left(4\cos^2\theta + 9\sin^2\theta\right)^2}$$

$$= \frac{1296\cos^2\theta + 1296\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{1296}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \left(\frac{36}{4\cos^2\theta + 9\sin^2\theta}\right)^2 = (x^2 + y^2)^2$$

The locus of N is $(x^2 + y^2)^2 = 9x^2 + 4y^2$, as required.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 30

Question:

a Show that the hyperbola $x^2 - y^2 = a^2, a > 0$, has eccentricity equal to $\sqrt{2}$.

b Hence state the coordinates of the focus S and an equation of the corresponding directrix L, where both S and L lie in the region $x > 0$.

The perpendicular from S to the line $y = x$ meets the line $y = x$ at P and the perpendicular from S to the line $y = -x$ meets the line $y = -x$ at Q.

c Show that both P and Q lie on the directrix L and give the coordinates of P and Q.

Given that the line SP meets the hyperbola at the point R,

d prove that the tangent at R passes through the point Q. **[E]**

Solution:

a $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

$b^2=a^2\left(e^2-1\right)$

For this hyperbola $b^2=a^2$

$a^2=a^2\left(e^2-1\right)\Rightarrow 1=e^2-1\Rightarrow e^2=2$

$e=\sqrt{2}$, as required.

> $x^2-y^2=a^2\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{a^2}=1$. This is an hyperbola in which $a=b$.

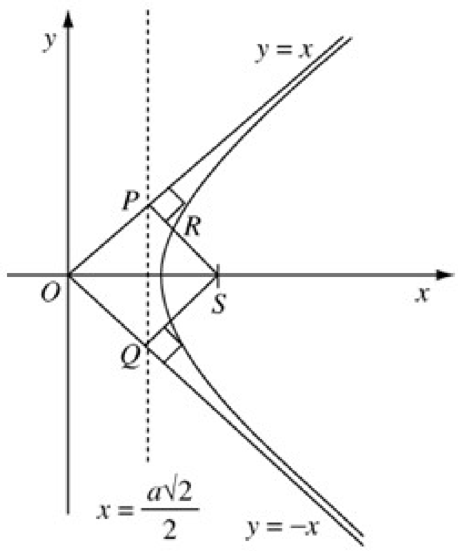
b The coordinates of S are

$(ae,0)=\left(a\sqrt{2},0\right)$

An equation of L is

$x=\frac{a}{e}=\frac{a}{\sqrt{2}}=\frac{a\sqrt{2}}{2}$

c



SP is perpendicular to $y=x$, so its gradient is -1. An equation of SP is

$y=-1\left(x-a\sqrt{2}\right)=-x+a\sqrt{2}$

$y+x=a\sqrt{2}$

SP meets $y=x$ where

$x+x=a\sqrt{2}\Rightarrow x=\frac{a\sqrt{2}}{2}$

Hence P is on the directrix L.

SQ is perpendicular to $y=-x$, so its gradient is 1.

> The asymptotes to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$. These formulae are given in the Edexcel formulae booklet. With this hyperbola $a=b$ and the asymptotes are $y=\pm x$. This question is about the intersection of line with the asymptotes. The lines $y=x$ and $y=-x$ are perpendicular to each other and a hyperbola with perpendicular asymptotes is called a rectangular hyperbola. In Module FP1, you studied another rectangular hyperbola, $xy=c^2$.

An equation of SQ is

$y = 1(x - a\sqrt{2}) = x - a\sqrt{2}$

$y = x - a\sqrt{2}$

SQ meets $y = -x$ where $-x = x - a\sqrt{2} \Rightarrow x = \frac{a\sqrt{2}}{2}$

Hence Q is on the directrix L.
Both P and Q lie on the directrix L.

The coordinates of P are $\left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}\right)$.

The coordinates of Q are $\left(\frac{a\sqrt{2}}{2}, -\frac{a\sqrt{2}}{2}\right)$.

d SP: $y + x = a\sqrt{2}$ ①

Hyperbola $x^2 - y^2 = a^2$ ②

From ① $y = a\sqrt{2} - x$ ③

Substitute ③ into ②

$$x^2 - (a\sqrt{2} - x)^2 = a^2$$

$$x^2 - 2a^2 + 2\sqrt{2}ax - x^2 = a^2$$

$$2\sqrt{2}ax = 3a^2 \Rightarrow x = \frac{3a}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}a$$

To find the coordinates of R, you solve equations ① and ② simultaneously.

Substituting for x in ③

$$y = a\sqrt{2} - \frac{3\sqrt{2}}{4}a = \frac{\sqrt{2}}{4}a$$

The coordinates of R are $\left(\frac{3\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a\right)$

To find the tangent to the hyperbola at R

$$x^2 - y^2 = a^2$$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Differentiating the equation of the hyperbola implicitly with respect to x.

At R

$$\frac{dy}{dx} = \frac{x}{y} = \frac{\frac{3\sqrt{2}}{4}a}{\frac{\sqrt{2}}{4}a} = 3$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{2}}{4}a = 3\left(x - \frac{3\sqrt{2}}{4}a\right) = 3x - \frac{9\sqrt{2}}{4}a$$

$$y = 3x - 2\sqrt{2}a$$

This is the equation of the tangent to the hyperbola at R. To establish that R passes through Q, you substitute the x-coordinate of Q into this equation and show that this gives the y-coordinate of Q.

At $x = \frac{a\sqrt{2}}{2}, y = 3\left(\frac{a\sqrt{2}}{2}\right) - 2\sqrt{2}a = -\frac{a\sqrt{2}}{2}$

This is the y-coordinate of Q.
Hence the tangent at R passes through Q.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 31

Question:

a Show that an equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$.

The normal at P cuts the x-axis at G.

b Show that the coordinates of M, the mid-point of PG, are

$$\left[\left(\frac{2a^2-b^2}{2a}\right)\cos\theta, \left(\frac{b}{2}\right)\sin\theta\right]$$

c Show that, as θ varies, the locus of M is an ellipse and determine the equation of this locus.

Given that the normal at P meets the y-axis at H and that O is the origin,

d show that, if $a > b$, area ΔOMG : area$\Delta OGH = b^2 : 2(a^2-b^2)$. **[E]**

Solution:

a $x = a\cos\theta, \quad y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{b\cos\theta}{a\sin\theta}$$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin\theta}{b\cos\theta}$$

$$y - y_1 = m'(x - x_1)$$

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

Divide this equation throughout by $\sin\theta\cos\theta$.

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$, as required

b Substituting $y = 0$ in the result to part **a**

$$ax\sec\theta = a^2 - b^2$$

You find the x-coordinate of G by substituting $y = 0$ into the equation of the normal at P and solving the resulting equation for x.

$$x = \frac{a^2 - b^2}{a}\cos\theta$$

$$P: (a\cos\theta, b\sin\theta), G: \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right)$$

The coordinates (x_M, y_M) of M the mid-point of PG are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

$$x_M = \frac{a\cos\theta + \frac{a^2 - b^2}{a}\cos\theta}{2}$$

$$= \frac{\cos\theta}{2}\left(\frac{a^2 + a^2 - b^2}{a}\right) = \left(\frac{2a^2 - b^2}{2a}\right)\cos\theta$$

Hence, the coordinates of M are

$\left[\left(\frac{2a^2 - b^2}{2a}\right)\cos\theta, \left(\frac{b}{2}\right)\sin\theta\right]$, as required

c For M

$$x = \left(\frac{2a^2 - b^2}{2a}\right)\cos\theta, y = \left(\frac{b}{2}\right)\sin\theta$$

$$\cos\theta = \frac{x}{\left(\frac{2a^2-b^2}{2a}\right)}, \sin\theta = \frac{y}{\left(\frac{b}{2}\right)}$$

$$\cos^2\theta + \sin^2\theta = 1$$

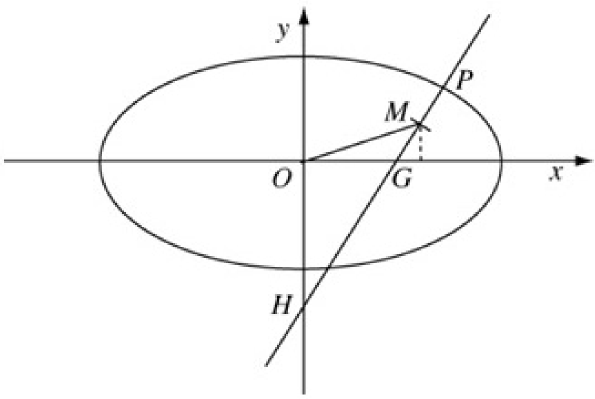
$$\frac{x^2}{\left(\frac{2a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{b}{2}\right)^2} = 1$$

This is an ellipse. A Cartesian equation of this ellipse is

$$\frac{4a^2x^2}{\left(2a^2-b^2\right)^2} + \frac{4y^2}{b^2} = 1$$

Any curve with an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse. If you are asked to show that a locus is an ellipse, it is sufficient to show that it has a Cartesian equation of this form.

d



Substituting $x = 0$ into the equation of the normal

$$-by\operatorname{cosec}\theta = a^2 - b^2 \Rightarrow y = -\frac{a^2-b^2}{b}\sin\theta$$

Hence $OH = \frac{a^2-b^2}{b}\sin\theta$.

$$\frac{\text{area}\Delta OMG}{\text{area}\Delta OGH} = \frac{y-\text{coordinate of } M}{OH}$$

$$= \frac{\left(\frac{b}{2}\right)\sin\theta}{\frac{a^2-b^2}{b}\sin\theta}$$

$$= \frac{b^2}{2\left(a^2-b^2\right)}, \text{ as required}$$

The triangles OMG and OGH can be looked at as having the same base OG. As the area of a triangle is $\frac{1}{2}\times\text{base}\times\text{height}$, triangles with the same base will have areas proportional to their heights. The height of the triangle OGM is shown by a dotted line in the diagram and is given by the y-coordinate of M.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 32

Question:

a Find equations for the tangent and normal to the rectangular hyperbola $x^2 - y^2 = 1$, at the point P with coordinates $(\cosh t, \sinh t), t > 0$.

The tangent and normal intersect the x-axis at T and G respectively. The perpendicular from P to the x-axis meets an asymptote in the first quadrant at Q.

b Show that GQ is perpendicular to this asymptote.

The normal intercepts the y-axis at R.

c Show that R lies on the circle with centre at T and radius TG. **[E]**

Solution:

a To find an equation of the tangent at P.

$x = \cosh t, \quad y = \sinh t$

$\frac{dx}{dt} = \sinh t, \quad \frac{dy}{dt} = \cosh t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cosh t}{\sinh t}$

Using $y - y_1 = m(x - x_1)$

$$y - \sinh t = \frac{\cosh t}{\sinh t}(x - \cosh t)$$

$$y \sinh t - \sinh^2 t = x \cosh t - \cosh^2 t$$

$$y \sinh t = x \cosh t - (\cosh^2 t - \sinh^2 t)$$

$$= x \cosh t - 1$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$.

$x \cosh t - y \sinh t = 1 \quad ①$

To find the equation of the normal at P

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{\sinh t}{\cosh t}$$

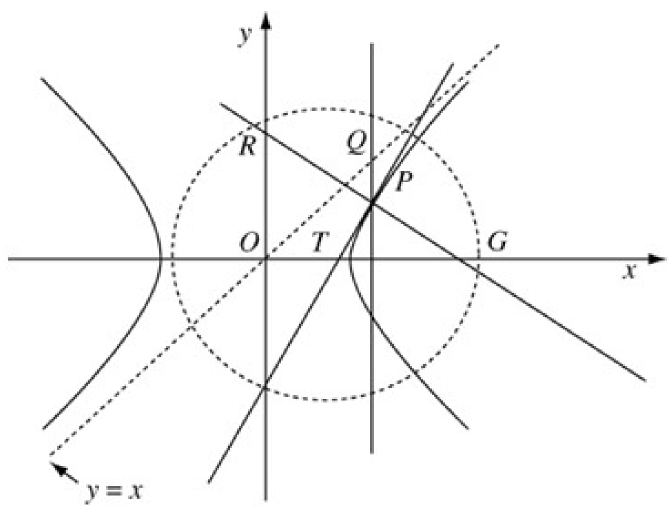
$$y - y_1 = m'(x - x_1)$$

$$y - \sinh t = -\frac{\sinh t}{\cosh t}(x - \cosh t)$$

$$y \cosh t - \sinh t \cosh t = -x \sinh t + \sinh t \cosh t$$

$$x \sinh t + y \cosh t = 2 \sinh t \cosh t \quad ②$$

b



Substitute $y=0$ into ②

$x\sinh t = 2\sinh t\cosh t$

$x = 2\cosh t$

The coordinates of G are $(2\cosh t, 0)$.

To find the coordinates of G, you substitute $y=0$ into the equation of the normal found in part **a**.

The x-coordinate of Q is $\cosh t$.

The asymptote in the first quadrant has equation $y=x$.

Hence the coordinates of Q are $(\cosh t, \cosh t)$.

The gradient of GQ is given by $\frac{y_1-y_2}{x_1-x_2}=\frac{0-\cosh t}{2\cosh t-\cosh t}=-1$

As the gradient of $y=x$ is 1 and $1\times -1=-1$, GQ is perpendicular to the asymptote.

The asymptotes to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$. These formulae are given in the Edexcel formulae booklet. With this hyperbola $a=b=1$ and the asymptotes are $y=\pm x$. The asymptote in the first quadrant has equation $y=x$.

c Substitute $y=0$ into ①

$x\cosh t = 1 \Rightarrow x=\frac{1}{\cosh t}$

The coordinates of T are $\left(\frac{1}{\cosh t}, 0\right)$.

To find the coordinates of T, you substitute $y=0$ into the equation of the tangent found in part **a**.

Substitute $x=0$ into ②

$y\cosh t = 2\sinh t\cosh t \Rightarrow y = 2\sinh t$

The coordinates of R are $(0, 2\sinh t)$

To find the coordinates of R, you substitute $x=0$ into the equation of the normal found in part **a**.

$$TG = 2\cosh t-\frac{1}{\cosh t}$$

$$TR^2 = OR^2+OT^2 = (2\sinh t)^2+\left(\frac{1}{\cosh t}\right)^2$$

$$=4\sinh^2 t+\frac{1}{\cosh^2 t}=4(\cosh^2 t-1)+\frac{1}{\cosh^2 t}$$

$$=4\cosh^2 t-4+\frac{1}{\cosh^2 t}$$

$$=\left(2\cosh t-\frac{1}{\cosh t}\right)^2=TG^2$$

If a circle can be drawn through R with centre T and radius TG then TR must also be a radius of the circle. So you can solve the problem by showing that TR and TG have the same length.

Hence $TR=TG$ and R lies on the circle with centre at T and radius TG.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 33

Question:

a Find the equations for the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.

b If these lines meet the y-axis at P and Q respectively, show that the circle described on PQ as diameter passes through the foci of the hyperbola. **[E]**

Solution:

a To find the equation of the tangent at $(a\sec\theta, b\tan\theta)$

$$x = a\sec\theta, \quad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \frac{dy}{dt} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a\sin\theta}$$

$$y - y_1 = m(x - x_1)$$

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However, the calculation in part **b** will be easier if you simplify the equation at this stage.

$$ay\sin\theta - \frac{ab\sin^2\theta}{\cos\theta} = bx - ab\sec\theta$$

$$bx - ay\sin\theta = ab\left(\frac{1-\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta}$$

$$bx - ay\sin\theta = ab\cos\theta \quad ①$$

To find the equation of the normal at $(a\sec\theta, b\tan\theta)$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{a\sin\theta}{b}$$

$$y - y_1 = m'(x - x_1)$$

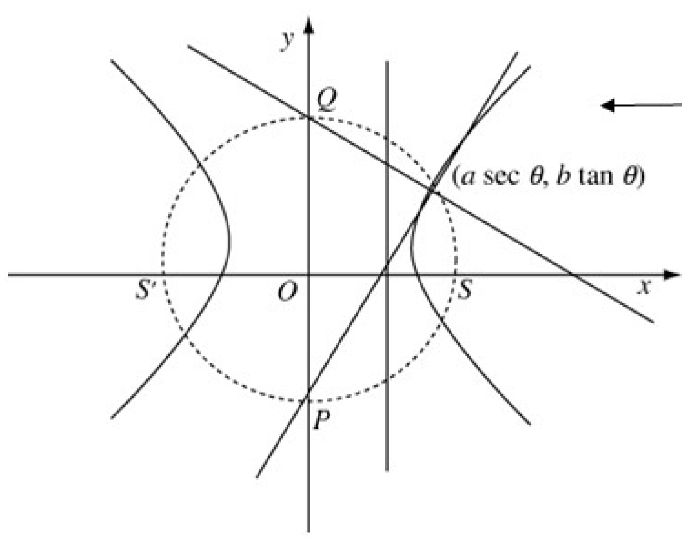
$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

When you multiply the bracket out, $\sin\theta\sec\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\tan\theta$$

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta \quad ②$$

b



This problem will be solved using the property that the angle in a semi-circle is a right angle and you need to show that PS and QS are perpendicular. All five of the points, P, Q, $(a\sec\theta, b\tan\theta)$ and the two foci lie on the same circle.

Substitute $x = 0$ into ①

$-ay\sin\theta = ab\cos\theta \Rightarrow y = -b\cot\theta$

To find the coordinates of P, you substitute $x = 0$ into the equation of the tangent found in part **a**.

The coordinates of P are $(0, -b\cot\theta)$.

Substitute $x = 0$ into ②

$by = (a^2 + b^2)\tan\theta \Rightarrow y = \dfrac{a^2+b^2}{b}\tan\theta$

To find the coordinates of Q, you substitute $x = 0$ into the equation of the normal found in part **a**.

The coordinates of Q are $\left(0, \dfrac{a^2+b^2}{b}\tan\theta\right)$.

The focus S has coordinates $(ae, 0)$

The gradient of PS is given by $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-b\cot\theta - 0}{0 - ae} = \dfrac{b}{ae}\cot\theta$

The gradient of QS is given by

$$m' = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{a^2+b^2}{b}\tan\theta - 0}{0 - ae} = \frac{-(a^2+b^2)}{abe}\tan\theta$$

$$mm' = \frac{b}{ae}\cot\theta \times -\frac{a^2+b^2}{abe}\tan\theta = -\frac{a^2+b^2}{a^2e^2}$$

The formula for the eccentricity is

$b^2 = a^2(e^2 - 1)$

$b^2 = a^2e^2 - a^2 \Rightarrow a^2e^2 = a^2 + b^2$

Hence $mm' = -\dfrac{a^2+b^2}{a^2e^2} = -\dfrac{a^2+b^2}{a^2+b^2} = -1$

So PS is perpendicular to QS and $\angle PSQ = 90°$.

By the converse of the theorem that the angle in a semi-circle is a right angle, the circle described on PQ as diameter passes through the focus S.

By symmetry, the circle also passes through the focus S'.

There is no need to repeat the calculations for PS' and QS'. It is evident from the diagram that the whole diagram is symmetrical about the y-axis, so, if the circle passes through S, it passes through S'. It is quite acceptable to appeal to symmetry to complete your proof.

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Review Exercise 1
Exercise A, Question 34

Question:

Given that $r > a > 0$ and $0 < \arcsin\left(\frac{a}{r}\right) < \frac{\pi}{2}$, show that

$$\frac{d}{dr}\left[\arcsin\left(\frac{a}{r}\right)\right] = -\frac{a}{r\sqrt{(r^2 - a^2)}}$$ **[E]**

Solution:

Let $y = \arcsin\left(\frac{a}{r}\right)$

Let $u = \frac{a}{r} = ar^{-1}$

$y = \arcsin u$

$\frac{dy}{dr} = \frac{dy}{du} \times \frac{du}{dr}$

$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}}$

$\frac{du}{dr} = -ar^{-2} = -\frac{a}{r^2}$

Hence $\frac{dy}{dr} = \frac{1}{\sqrt{(1-u^2)}} \times -\frac{a}{r^2} = -\frac{a}{r^2\sqrt{\left(1-\frac{a^2}{r^2}\right)}}$

$= -\frac{a}{r\sqrt{(r^2-a^2)}}$, as required

You can use the chain rule to differentiate $\arcsin\left(\frac{a}{r}\right)$.

You take one of the rs inside the square root sign in the denominator. In detail

$r^2\sqrt{\left(1-\frac{a^2}{r^2}\right)} = r\sqrt{r^2}\sqrt{\left(1-\frac{a^2}{r^2}\right)}$

$= r\sqrt{\left(r^2\left(1-\frac{a^2}{r^2}\right)\right)} = r\sqrt{(r^2-a^2)}$.

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Review Exercise 1
Exercise A, Question 35

Question:

Given that $y = (\arcsin x)^2$,

a prove that $(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y$,

b deduce that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$. **[E]**

Solution:

a $y = (\arcsin x)^2$

Let $u = \arcsin x$

$y = u^2$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = 2u$

$\frac{du}{dx} = \frac{1}{\sqrt{(1-x^2)}}$

This result is in the Edexcel formula booklet, which is provided for use with the paper. It is a good idea to quote any formulae you use in your solution.

Hence

$$\frac{dy}{dx} = 2u \times \frac{1}{\sqrt{(1-x^2)}} = \frac{2\arcsin x}{\sqrt{(1-x^2)}}$$

$$\sqrt{(1-x^2)}\frac{dy}{dx} = 2\arcsin x$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\arcsin x)^2$$

$$= 4y, \text{ as required}$$

Square both sides of this solution and use the given $y = (\arcsin x)^2$ to complete the solution.

b Differentiating the result of part a implicitly with respect to x

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4\frac{dy}{dx}$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 2$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2, \text{ as required}$$

Using the chain rule $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) = 2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}$.

Divide the equation throughout by $2\frac{dy}{dx}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 36

Question:

a Show that, for $x = \ln k$, where k is a positive constant, $\cosh 2x = \frac{k^4+1}{2k^2}$.

b Given that $f(x) = px - \tanh 2x$, where p is a constant, find the value of p for which $f(x)$ has a stationary value at $x = \ln 2$, giving your answer as an exact fraction. **[E]**

Solution:

a $\cosh 2x = \frac{e^{2x}+e^{-2x}}{2} = \frac{e^{2\ln k}+e^{-2\ln k}}{2}$

$= \frac{e^{\ln k^2}+e^{\ln \frac{1}{k^2}}}{2} = \frac{1}{2}\left(k^2+\frac{1}{k^2}\right)$

$= \frac{1}{2}\left(\frac{k^4+1}{k^2}\right) = \frac{k^4+1}{2k^2}$, as required

Using the law of logarithms $n\ln x = \ln x^n$, with $n = -2$, $-2\ln k = \ln k^{-2} = \ln\frac{1}{k^2}$.

b $f(x) = px - \tanh 2x$

For a stationary value

$f'(x) = p - 2\operatorname{sech}^2 2x = 0$

$p = 2\operatorname{sech}^2 2x = \frac{2}{\cosh^2 2x}$

Using the result of part a with $k = 2$

If $x = \ln 2$

$\cosh 2x = \frac{2^4+1}{2\times 2^2} = \frac{17}{8}$

Hence

$p = \frac{2}{\left(\frac{17}{8}\right)^2} = \frac{128}{289}$

There is no 'hence' in this question but using the result in part **a** shortens the working. The question requires an exact fraction for the answer and you should not use a calculator other than, possibly, for multiplying and dividing fractions.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 37

Question:

The curve with equation $y = -x + \tanh 4x, x \geq 0$, has a maximum turning point A.

a Find, in exact logarithmic form, the x-coordinate of A.

b Show that the y-coordinate of A is $\frac{1}{4}\{2\sqrt{3} - \ln(2+\sqrt{3})\}$. **[E]**

Solution:

a $y = -x + \tanh 4x$

$$\frac{dy}{dx} = -1 + 4\text{sech}^2 4x = 0$$

$$\text{sech}^2 4x = \frac{1}{4} \Rightarrow \cosh^2 4x = 4$$

$$\cosh 4x = 2$$

$$4x = \text{arcosh}2 = \ln(2+\sqrt{3})$$

$$x = \frac{1}{4}\ln(2+\sqrt{3})$$

As $\cosh x \geq 1$ for all real x, $\cosh 4x = -2$ is impossible.

For $x \geq 0$, there is only one value of x which gives a stationary value. The question tells you that the curve has a maximum point so, in this question, you need not show that this point is a maximum by, for example, examining the second derivative.

b $\tanh^2 4x = 1 - \text{sech}^2 4x = 1 - \frac{1}{4} = \frac{3}{4}$

You need a value for $\tanh 4x$ and this is easiest found using the hyperbolic identity $\text{sech}^2 x = 1 - \tanh^2 x$.

As $x \geq 0, \tanh 4x = \frac{\sqrt{3}}{2}$

At $x = \frac{1}{4}\ln(2+\sqrt{3})$

$$y = -x + \tanh 4x = -\frac{1}{4}\ln(2+\sqrt{3}) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left\{2\sqrt{3} - \ln(2+\sqrt{3})\right\}, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 38

Question:

The curve C has equation $y = \text{arcsec } e^x, x > 0, 0 \le y < \frac{1}{2}\pi$.

a Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{(e^{2x} - 1)}}$.

b Sketch the graph of C

The point A on C has x-coordinate $\ln 2$. The tangent to C at A intersects the y-axis at the point B.

c Find the exact value of the y-coordinate of B. **[E]**

Solution:

a $y = \text{arcsec } e^x$

$\sec y = e^x$

Differentiating implicitly with respect to x

$\sec y \tan y \frac{dy}{dx} = e^x$

$\frac{dy}{dx} = \frac{e^x}{\sec y \tan y}$

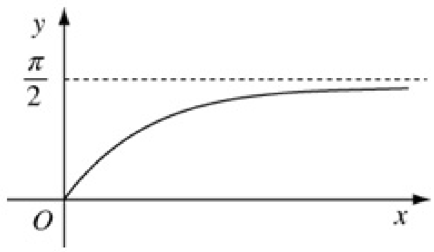
As $\sec y = e^x, \tan^2 y = \sec^2 y - 1 = e^{2x} - 1$

$\tan y = \sqrt{(e^{2x} - 1)}$

$\frac{dy}{dx} = \frac{e^x}{e^x \sqrt{(e^{2x} - 1)}} = \frac{1}{\sqrt{(e^{2x} - 1)}}$, as required.

$\tan y = -\sqrt{(e^{2x} - 1)}$ is, in general, possible. In this case, the question specifies that $x > 0$ and $0 < y < \frac{1}{2}\pi$ and, with these ranges, $\text{arcsec } e^x$ is an increasing function of x and so $\frac{dy}{dx}$ is positive ($\tan y$ is positive).

b



In your sketch, you must show any important features of the curve. In this case, you need to show that the curve starts at the origin and that the line $y = \frac{\pi}{2}$ is an asymptote to the curve.

c At $x = \ln 2$, the gradient of the curve is given by

$m = \frac{dy}{dx} = \frac{1}{\sqrt{(e^{2x} - 1)}} = \frac{1}{\sqrt{(e^{2\ln 2} - 1)}}$

$= \frac{1}{\sqrt{(e^{\ln 4} - 1)}} = \frac{1}{\sqrt{(4 - 1)}} = \frac{1}{\sqrt{3}}$

At $x = \ln 2$,

$y = \text{arcsec } e^x = \text{arcsec } e^{\ln 2} = \text{arcsec} 2 = \frac{\pi}{3}$

$\text{arcsec} 2 = \arccos \frac{1}{2} = \frac{\pi}{3}$. In questions involving calculus you must use radians.

An equation of the tangent is

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{3} = \frac{1}{\sqrt{3}}(x - \ln 2)$

There is no need to simplify this equation. You only need to find the value of y at $x = 0$.

At B, $x = 0$

$y = \frac{\pi}{3} - \frac{\ln 2}{\sqrt{3}} = \frac{1}{3}(\pi - \sqrt{3} \ln 2)$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 39

Question:

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{(x^2-2x+17)}} \right) dx$, giving your answer as an exact logarithm. **[E]**

Solution:

$x^2-2x+17 = x^2-2x+1+16 = (x-1)^2+4^2$

It is usually a good idea to begin any integration involving the square root of a quadratic by completing the square.

Hence

$$\int_1^4 \frac{1}{\sqrt{(x^2-2x+17)}} dx = \int_1^4 \frac{1}{\sqrt{((x-1)^2+4^2)}} dx$$

$$= \left[\text{arsinh}\frac{x-1}{4} \right]_1^4 = \text{arsinh}\frac{3}{4} - \text{arsinh}\,0$$

$$= \ln\left(\frac{3}{4} + \sqrt{\left(\frac{9}{16}+1\right)} \right) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{25}{16}\right)} \right)$$

$$= \ln\left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$$

This is a direct application of the formula

$$\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \text{arsinh}\left(\frac{x}{a}\right)$$

which is given in the Edexcel formulae booklet. You would need to be careful to adapt this formula correctly if the coefficient of x^2 in the quadratic was not 1.

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 40

Question:

Evaluate $\int_1^3 \frac{1}{\sqrt{(x^2+4x-5)}}\,dx$, giving your answer as an exact logarithm. **[E]**

Solution:

$x^2+4x-5 = x^2+4x+4-9 = (x+2)^2-3^2$

Hence

$$\int_1^3 \frac{1}{\sqrt{(x^2+4x-5)}}dx = \int_1^3 \frac{1}{\sqrt{((x+2)^2-3^2)}}dx$$

$$= \left[\text{arcosh}\,\frac{x+2}{3}\right]_1^3 = \text{arcosh}\,\frac{5}{3} - \text{arcosh}\,1$$

$$= \ln\left(\frac{5}{3}+\sqrt{\left(\frac{25}{9}-1\right)}\right) = \ln\left(\frac{5}{3}+\sqrt{\left(\frac{16}{9}\right)}\right)$$

$$= \ln\left(\frac{5}{3}+\frac{4}{3}\right) = \ln 3$$

$\text{arcosh}\,1 = 0$

To obtain the answer as an exact logarithm, you can use the formula $\text{arcosh}\,x = \ln\left(x+\sqrt{(x^2-1)}\right)$. If you forget this, or can't remember the sign, you can find it in the Edexcel formulae booklet which is provided for use with the paper. This booklet contains many of the formulae needed for the calculus topics in the FP3 module.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 41

Question:

Use the substitution $x = \frac{a}{\sinh\theta}$, where a is a constant, to show that, for $x > 0, a > 0$,

$$\int \frac{1}{x\sqrt{(x^2 + a^2)}}\,dx = -\frac{1}{a}\text{arsinh}\left(\frac{a}{x}\right) + \text{constant}.$$ **[E]**

Solution:

$$x = \frac{a}{\sinh\theta} = a(\sinh\theta)^{-1}$$

$$\frac{dx}{d\theta} = -a(\sinh\theta)^{-2}\cosh\theta = -\frac{a\cosh\theta}{\sinh^2\theta}$$

When substituting remember to substitute for the dx as well as the rest of the integral.

$$\int \frac{1}{x\sqrt{(x^2+a^2)}}\,dx = \int \frac{1}{\frac{a}{\sinh\theta}\sqrt{\left(\frac{a^2}{\sinh^2\theta}+a^2\right)}} \times \frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{\frac{-a\cosh\theta}{\sinh^2\theta}}{\frac{a^2\sqrt{1+\sinh^2\theta}}{\sinh^2\theta}}\,d\theta = \frac{-1}{a}\int \frac{\cosh\theta}{\cos\theta}\,d\theta$$

Use $1+\sinh^2\theta = \cosh^2\theta$ to simplify this expression.

$$= -\frac{1}{a}\int \frac{\cosh\theta}{\cosh\theta}\,d\theta = -\frac{1}{a}\int 1\,d\theta$$

$$= -\frac{1}{a}\theta + \text{constant}$$

$$= -\frac{1}{a}\text{arsinh}\left(\frac{a}{x}\right) + \text{constant, as required.}$$

As $x = \frac{a}{\sinh\theta}$, then $\sinh\theta = \frac{a}{x}$ and $\theta = \text{arsinh}\left(\frac{a}{x}\right)$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 42

Question:

a Prove that the derivative of artanh x, $-1 < x < 1$, is $\frac{1}{1-x^2}$.

b Find $\int \text{artanh}\,x \, dx$. **[E]**

Solution:

a Let $y = \text{artanh}\,x$

$\tanh y = x$

Differentiate implicitly with respect to x

$$\text{sech}^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y}$$

$$= \frac{1}{1-x^2}, \text{ as required}$$

To differentiate a function f(y) with respect to x you use a version of the chain rule $\frac{d}{dx}(f(y)) = f'(y) \times \frac{dy}{dx}$.

b Using integration by parts and the result in part **a**

$$\int \text{artanh}\,x \, dx = \int 1 \times \text{artanh}\,x \, dx$$

$$= x\,\text{artanh}\,x - \int \frac{x}{1-x^2} dx$$

$$= x\,\text{artanh}\,x + \frac{1}{2}\ln\left(1-x^2\right) + A$$

You use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ with $u = \text{artanh}\,x$ and $\frac{dv}{dx} = 1$. You know $\frac{du}{dx}$ from part **a**.

This solution uses the result $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. So $\int \frac{-2x}{1-x^2} dx = \ln\left(1-x^2\right)$ and you multiply this by $-\frac{1}{2}$ to complete the solution. This is a question where there are a number of possible alternative forms of the answer.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 43

Question:

a Find $\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx$.

b Find, to 3 decimal places, the value of $\int_0^{0.3} \frac{1+x}{\sqrt{(1-4x^2)}}\,dx$. **[E]**

Solution:

a $\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx = \int \frac{1}{\sqrt{(1-4x^2)}}\,dx + \int \frac{x}{\sqrt{(1-4x^2)}}\,dx$

You must treat this integral as two separate integrals added together. Both integrals have been solved here using substitution. This is a safe method of solution but you may be able to shorten the working by adapting standard formulae or inspection.

Let $2x = \sin\theta$, then $2\frac{dx}{d\theta} = \cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\cos\theta$

$$\int \frac{1}{\sqrt{(1-4x^2)}}\,dx = \int \frac{1}{\sqrt{(1-\sin^2\theta)}}\frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{1}{\cos\theta}\times\frac{1}{2}\cos\theta\,d\theta = \int \frac{1}{2}\,d\theta$$

$$= \frac{1}{2}\theta + A = \frac{1}{2}\arcsin 2x + A$$

Let $u^2 = 1-4x^2$, then differentiating implicitly with respect to x

$$2u\frac{du}{dx} = -8x \Rightarrow x\frac{dx}{du} = -\frac{1}{4}u$$

$$\int \frac{x}{\sqrt{(1-4x^2)}}\,dx = \int \frac{1}{u}\times x\frac{dx}{du}\,du = \int \frac{1}{u}\times -\frac{1}{4}u\,du$$

$$= \int -\frac{1}{4}\,du = -\frac{1}{4}u + B = -\frac{1}{4}\sqrt{(1-4x^2)} + B$$

Combining the integrals

$$\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx = \frac{1}{2}\arcsin 2x - \frac{1}{4}\sqrt{(1-4x^2)} + C$$

b $\int_0^{0.3} \frac{1+x}{\sqrt{(1-4x^2)}}\,dx = \left[\frac{1}{2}\arcsin 2x - \frac{1}{4}\sqrt{(1-4x^2)}\right]_0^{0.3}$

$$= \frac{1}{2}\arcsin 0.6 - \frac{1}{4}\sqrt{(1-4\times 0.09)} - \left(0-\frac{1}{4}\right)$$

$$= \frac{1}{2}\arcsin 0.6 - 0.2 + \frac{1}{4}$$

$$= 0.372 \quad (3\text{ d.p.})$$

You can use your calculator at any stage to evaluate this definite integral. The calculator must be in radian mode.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 44

Question:

a Given that $y = \arctan 3x$, and assuming the derivative of tan x, prove that

$$\frac{dy}{dx} = \frac{3}{1+9x^2} .$$

b Show that $\int_0^{\frac{\sqrt{3}}{3}} 6x \arctan 3x = \frac{1}{9}(4\pi - 3\sqrt{3})$. **[E]**

Solution:

a $y = \arctan 3x$

$\tan y = 3x$

Differentiating implicitly with respect to x

$$\sec^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\sec^2 y} = \frac{3}{1+\tan^2 y}$$

$$= \frac{3}{1+9x^2}\text{, as required}$$

b Using integration by parts and the result in part **a**

$$\int 6x\arctan 3x\,dx = 3x^2\arctan 3x - \int 3x^2 \times \frac{3}{1+9x^2}\,dx$$

$$= 3x^2\arctan 3x - \int \frac{9x^2+1-1}{1+9x^2}\,dx$$

$$= 3x^2\arctan 3x - \int 1\,dx + \int \frac{1}{1+9x^2}\,dx$$

$$= 3x^2\arctan 3x - x + \frac{1}{3}\arctan 3x$$

You use $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ with $u = \text{artanh}\,3x$ and $\frac{dv}{dx} = 6x$.

You know $\frac{du}{dx}$ from part **a**.

You have to integrate $\frac{9x^2}{1+9x^2}$. As the degree of the numerator is equal to the degree of the denominator, you must divide the denominator into the numerator before integrating.

The adaptation of the formula given in the Edexcel formulae booklet, $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan\left(\frac{x}{a}\right)$ to this integral is not straightforward.

$$\int \frac{1}{1+9x^2}dx = \frac{1}{9}\int \frac{1}{\frac{1}{9}+x^2}dx$$

$$= \frac{1}{9}\times\frac{1}{\frac{1}{3}}\arctan\left(\frac{x}{\frac{1}{3}}\right) = \frac{1}{3}\arctan 3x.$$

You may prefer to find such an integral using the substitution $3x = \tan\theta$.

$$\left[3x^2\arctan 3x - x + \frac{1}{3}\arctan 3x\right]_0^{\frac{\sqrt{3}}{3}}$$

$$= 3\times\left(\frac{\sqrt{3}}{3}\right)^2\arctan\sqrt{3} - \frac{\sqrt{3}}{3} + \frac{1}{3}\arctan\sqrt{3}$$

$$= \frac{4}{3}\arctan\sqrt{3} - \frac{\sqrt{3}}{3}$$

$$= \frac{4}{3}\times\frac{\pi}{3} - \frac{\sqrt{3}}{3} = \frac{1}{9}\left(4\pi - 3\sqrt{3}\right)\text{, as required}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 45

Question:

a Starting from the definition of $\sinh x$ in terms of e^x, prove that
$\text{arsinh}x = \ln[x + \sqrt{(x^2+1)}]$.

b Prove that the derivative of $\text{arsinh}\,x$ is $(1+x^2)^{-\frac{1}{2}}$.

c Show that the equation $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$ is satisfied when
$y = (\text{arsinh}x)^2$.

d Use integration by parts to find $\int_0^1 \text{arsinh}x \, dx$, giving your answer in terms of a natural logarithm. **[E]**

Solution:

a Let $y = \text{arsinh}x$ then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$$2x = e^y - e^{-y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x + \sqrt{(4x^2+4)}}{2}$$

$$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$$

> You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

> The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

Taking the natural logarithms of both sides, $y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

b $y = \text{arsinh}x$

$\sinh y = x$

Differentiating implicitly with respect to x

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\cosh^2 y = 1 + \sinh^2 y = 1 + x^2 \Rightarrow \cosh y = \sqrt{(1+x^2)}$$

Hence $\dfrac{d}{dx}(\text{arsinh}x) = \dfrac{1}{\sqrt{(1+x^2)}} = (1+x^2)^{-\frac{1}{2}}$, as required.

> arsinh x is an increasing function of x for all x. So its gradient is always positive and you need not consider the negative square root.

c $y = (\text{arsinh}x)^2$

$$\frac{dy}{dx} = 2\text{arsinh}x(1+x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2(1+x^2)^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + 2\text{arsinh}x \times \left(-\frac{1}{2}\right)(2x)(1+x^2)^{-\frac{3}{2}}$$

$$= 2(1+x^2)^{-1} - 2x\text{arsinh}x(1+x^2)^{-\frac{3}{2}}$$

> You use the product rule for differentiation $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ with $u = 1$ 2arsinh x and $v = (1+x^2)^{-\frac{1}{2}}$.

Substituting for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ into

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2$$

$$= (1+x^2)\left(2(1+x^2)^{-1} - 2x\text{arsinh}x(1+x^2)^{-\frac{3}{2}}\right) + x \times 2\text{arsinh}x(1+x^2)^{-\frac{1}{2}} - 2$$

$$= 2 - 2x\text{arsinh}x(1+x^2)^{-\frac{1}{2}} + 2x\text{arsinh}x(1+x^2)^{-\frac{1}{2}} - 2$$

$= 0$, as required.

d $\displaystyle\int_0^1 \text{arsinh}x\,dx = \int_0^1 1 \times \text{arsinh}x\,dx$

$$= [x\text{arsinh}x]_0^1 - \int_0^1 \frac{x}{\sqrt{(1+x^2)}}dx$$

$$= \text{arsinh}1 - \left[\sqrt{(1+x^2)}\right]_0^1$$

$$= \ln(1+\sqrt{2}) - \sqrt{2} + 1$$

> $\dfrac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right) = \dfrac{1}{2} \times 2x \times (1+x^2)^{-\frac{1}{2}} = \dfrac{x}{\sqrt{(1+x^2)}}$
> so $\displaystyle\int \frac{x}{\sqrt{(1+x^2)}}dx = \sqrt{(1+x^2)}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 46

Question:

a Using the substitution $u = e^x$, find $\int \text{sech } x \, dx$.

b Sketch the curve with equation $y = \text{sech } x$.

The finite region R is bounded by the curve with equation $y = \text{sech } x$, the lines $x = 2$, $x = -2$ and the x-axis.

c Using your result from **a**, find the area of R, giving your answer to 3 decimal places. **[E]**

Solution:

a $u = e^x \Rightarrow \frac{du}{dx} = e^x = u$

Hence

$$\frac{dx}{du} = \frac{1}{u}$$

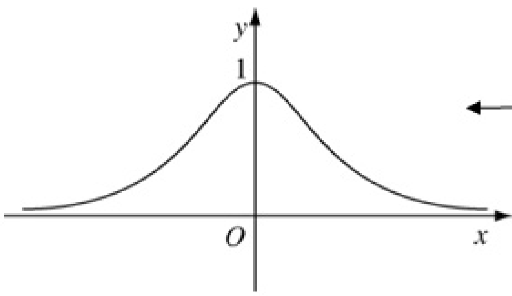
$$\int \operatorname{sech} x \, dx = \int \frac{2}{e^x + e^{-x}} \times \frac{dx}{du} \, du$$

$$= \int \frac{2}{u + \frac{1}{u}} \times \frac{1}{u} \, du = \int \frac{2}{u^2 + 1} \, du$$

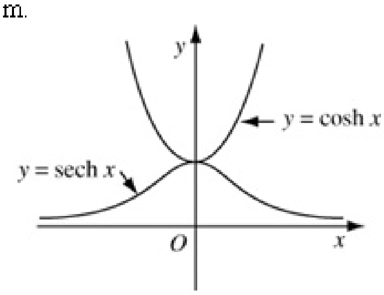
$$= 2 \arctan u + A$$

$$= 2 \arctan \left(e^x \right) + A$$

b



The specification requires you to know the graphs of cosh and sech. The sketch below illustrates the relation between them.



c Using the symmetry of the curve in **b**, the area, A, of R is given by

$$A = 2\int_0^2 \operatorname{sech} x \, dx = \left[4\arctan \left(e^x \right) \right]_0^2$$

$$= 4\arctan \left(e^2 \right) - 4\arctan 1$$

$$= 4\arctan \left(e^2 \right) - \pi = 2.604 \quad \text{(3 d.p.)}$$

The curve is symmetric, so that the area bounded by the lines $x = -2$ and $x = 2$ is twice the area between the y-axis and the line $x = 2$.

Using the result from part **a**.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 47

Question:

a Prove that $\text{arsinh}x = \ln[x + \sqrt{(x^2+1)}]$.

b **i** Find, to 3 decimal places, the coordinates of the stationary points on the curve with equation $y = x - 2\text{arsinh}x$.

ii Determine the nature of each stationary point.

iii Hence, sketch the curve with equation $y = x - 2\text{arsinh}x$.

c Evaluate $\int_{-2}^{0} (x - 2\text{arsinh}x)\,dx$. **[E]**

Solution:

a Let $y = \text{arsinh}x$ then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \dfrac{2x + \sqrt{(4x^2 + 4)}}{2}$

$= \dfrac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The negative sign can be ignored in the quadratic formula as it gives e^y negative less possible.

The specification requires you to prove this and similar results. Your preparation for the examination should include learning how to prove the formulae which express arsinh x, arcosh x and artanh x as natural logarithms.

b i $y = x - 2\text{arsinh}x$

$\dfrac{dy}{dx} = 1 - \dfrac{2}{\sqrt{(1+x^2)}} = 0$

$\sqrt{(1+x^2)} = 2 \Rightarrow 1 + x^2 = 4 \Rightarrow x = \pm\sqrt{3}$

At $x = \sqrt{3}$,

$y = \sqrt{3} - 2\text{arsinh}\sqrt{3} = \sqrt{3} - 2\ln\left(\sqrt{3} + \sqrt{(3+1)}\right)$

$= \sqrt{3} - 2\ln\left(2 + \sqrt{3}\right) = -0.902$ (3 d.p.)

At $x = -\sqrt{3}$,

$y = -\sqrt{3} - 2\text{arsinh}\left(-\sqrt{3}\right) = -\sqrt{3} - 2\ln\left(-\sqrt{3} + \sqrt{(3+1)}\right)$

$= -\sqrt{3} - 2\ln\left(2 - \sqrt{3}\right) = 0.902$ (3 d.p.)

To 3 decimal places the coordinates of the stationary points are $(1.732, -0.902), (-1.732, 0.902)$.

ii $\dfrac{dy}{dx} = 1 - 2\left(1+x^2\right)^{-\frac{1}{2}}$

$\dfrac{d^2y}{dx^2} = -2\left(-\dfrac{1}{2}\right)(2x)\left(1+x^2\right)^{-\frac{3}{2}}$

$= \dfrac{2x}{\left(1+x^2\right)^{\frac{3}{2}}}$

At $x = \sqrt{3}$,

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{3}}{(1+3)^{\frac{3}{2}}} = \frac{\sqrt{3}}{4} > 0 \Rightarrow \text{minimum}$$

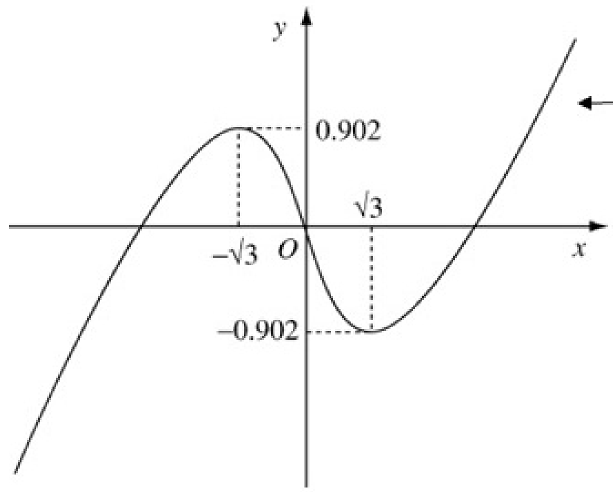
At $x = -\sqrt{3}$,

$$\frac{d^2y}{dx^2} = \frac{-2\sqrt{3}}{(1+3)^{\frac{3}{2}}} = -\frac{\sqrt{3}}{4} < 0 \Rightarrow \text{maximum}$$

These calculations show you that the curve has a maximum point in the second quadrant and a minimum point in the fourth quadrant. This helps you to sketch the graph correctly.

Hence $(1.732, -0.902)$ is a minimum point and $(-1.732, 0.902)$ is a maximum point.

iii



The curve has rotational symmetry about the origin.

c $\int \text{arsinh}\,x\,dx = \int 1 \times \text{arsinh}\,x\,dx$

$$= x\,\text{arsinh}\,x - \int \frac{x}{\sqrt{(1+x^2)}}\,dx$$

$$= x\,\text{arsinh}\,x - \sqrt{(1+x^2)}$$

Integrating arsinh x is not easy in itself and it is a good idea to work this out separately before attempting the whole integral. You integrate arsinh x using parts.

Hence $\int_{-2}^{0} (x - 2\,\text{arsinh}\,x)\,dx$

$$= \left[\frac{x^2}{2} - 2x\,\text{arsinh}\,x + 2\sqrt{(1+x^2)}\right]_{-2}^{0}$$

$$= (2) - (2 + 4\,\text{arsinh}\,(-2) + 2\sqrt{5})$$

$$= -4\ln(-2+\sqrt{5}) - 2\sqrt{5}$$

$$= 1.302 \text{ (3 d.p.)}$$

This exact answer is an acceptable answer to the question but reference to the graph shows the answer should be positive. This is not obvious from the expression and it is worthwhile evaluating it to check your work.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 48

Question:

Use the substitution $e^x = t - \frac{3}{5}$, or otherwise, to find $\int \frac{1}{3+5\cosh x} dx$. **[E]**

Solution:

If $e^x = t - \frac{3}{5}$,

then $e^x \frac{dx}{dt} = 1$

and $e^{2x} = \left(t - \frac{3}{5}\right)^2 = t^2 - \frac{6}{5}t + \frac{9}{25}$

Differentiate the equation implicitly with respect to t.

$$\int \frac{1}{3+5\cosh x} dx = \int \frac{1}{3+5\left(\frac{e^x + e^{-x}}{2}\right)} dx$$

Multiply the numerator and denominator of the right hand side of this equation by $2e^x$.

$$= \int \frac{2e^x}{6e^x + 5e^{2x} + 5} dx$$

$$= \int \frac{2}{5e^{2x} + 6e^x + 5}\left(e^x \frac{dx}{dt}\right) dt$$

$$= \int \frac{2}{5\left(t^2 - \frac{6}{5}t + \frac{9}{25}\right) + 6\left(t - \frac{3}{5}\right) + 5}(1)\, dt$$

$$= \int \frac{2}{5t^2 - 6t + \frac{9}{5} + 6t - \frac{18}{5} + 5} dt$$

$$= \int \frac{2}{5t^2 + \frac{16}{5}} dt = \frac{2}{5}\int \frac{1}{t^2 + \frac{16}{25}} dt$$

Using the standard result $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$ with $a = \frac{4}{5}$.

$$= \frac{2}{5} \times \frac{1}{\frac{4}{5}} \arctan\left(\frac{t}{\frac{4}{5}}\right) + A$$

Remember to return to the original variable, which is x not t.

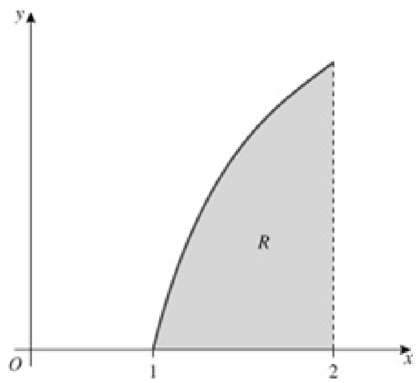
$$= \frac{1}{2}\arctan\left(\frac{5}{4}\left(e^x + \frac{3}{5}\right)\right) + A = \frac{1}{2}\arctan\left(\frac{5e^x + 3}{4}\right) + A$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 49

Question:



The figure above shows a sketch of the curve with equation $y = x \text{arcosh} x, 1 \le x \le 2$.
The region R, shaded in the figure, is bounded by the curve, the x-axis and the line $x = 2$.

Show that the area of R is $\frac{7}{4}\ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$. **[E]**

Solution:

$$\int x\,\text{arcosh}\,x\,dx = \frac{x^2}{2}\text{arcosh}\,x - \int \frac{x^2}{2\sqrt{(x^2-1)}}dx$$

This solution uses integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$, with $u = \text{arcosh}\,x$ and $\frac{dv}{dx} = x$.
There are other possible approaches to this question, for example, substituting $u = \text{arcosh}\,x$.

To find the remaining integral, let $x = \cosh\theta$.

$$\frac{dx}{d\theta} = \sinh\theta$$

$$\int \frac{x^2}{2\sqrt{(x^2-1)}}dx = \int \frac{\cosh^2\theta}{2\sqrt{(\cosh^2\theta-1)}}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{\cosh^2\theta}{2\sinh\theta}\sinh\theta\,d\theta = \frac{1}{2}\int \cosh^2\theta\,d\theta$$

$$= \frac{1}{4}\int(\cosh 2\theta + 1)d\theta$$

Using the identity $\cosh 2\theta = 2\cosh^2\theta - 1$.

$$= \frac{\sinh 2\theta}{8} + \frac{\theta}{4} = \frac{\sinh\theta\cosh\theta}{4} + \frac{\theta}{4}$$

$$= \frac{\left[\sqrt{(x^2-1)}\right]x}{4} + \frac{1}{4}\text{arcosh}\,x$$

$\sinh\theta = \sqrt{(\cosh^2\theta - 1)} = \sqrt{(x^2-1)}$

Hence the area, A, of R is given by

$$A = \left[\frac{x^2}{2}\text{arcosh}\,x - \frac{1}{4}x\sqrt{(x^2-1)} - \frac{1}{4}\text{arcosh}\,x\right]_1^2$$

$$= \left[\left(\frac{x^2}{2} - \frac{1}{4}\right)\text{arcosh}\,x - \frac{1}{4}x\sqrt{(x^2-1)}\right]_1^2$$

$$= \left[\frac{7}{4}\text{arcosh}\,2 - \frac{\sqrt{3}}{2}\right] - [0]$$

As $\text{arcosh}\,1 = 0$ and $\sqrt{(1^2-1)} = 0$, both terms are zero at the lower limit.

$$= \frac{7}{4}\ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 50

Question:

$4x^2+4x+5 \equiv (px+q)^2+r$

a Find the values of the constants p, q and r.

b Hence, or otherwise, find $\int \frac{1}{4x^2+4x+5}\,dx$.

c Show that $\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \ln[(2x+1)+\sqrt{(4x^2+4x+5)}]+k$, where k is an arbitrary constant. **[E]**

Solution:

a $4x^2+4x+5 = (px+q)^2+r$

$= p^2x^2+2pqx+q^2+r$

Equating coefficients of x^2

$4 = p^2 \Rightarrow p = 2$

Equating coefficients of x

$4 = 2pq = 4q \Rightarrow q = 1$

Equating constant coefficients

$5 = q^2+r = 1+r \Rightarrow r = 4$

$p = 2, q = 1, r = 4$

> The conditions of the question allow $p = -2$ as an answer, but the negative sign would make the integrals following awkward, so choose the positive root.

b $\int \frac{1}{4x^2+4x+5}\,dx = \int \frac{1}{(2x+1)^2+4}\,dx$

If you know a formula of the type $\int \frac{1}{a^2x^2+b^2}\,dx = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right)$, or you are confident at writing down integrals by inspection, you may be able to find this integral without working. It is, however, very easy to make errors with the constant and get, for example, the common error $\frac{1}{2}\arctan\left(\frac{2x+1}{2}\right)+C$.

Let $2x+1 = 2\tan\theta$

$$2\frac{dx}{d\theta} = 2\sec^2\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$$

$$\int \frac{1}{(2x+1)^2+4}\,dx = \int \frac{1}{4\tan^2\theta+4}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{1}{4\cancel{\sec^2\theta}}\left(\cancel{\sec^2\theta}\right)d\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\arctan\left(\frac{2x+1}{2}\right)+C$$

c $\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \int \frac{2}{\sqrt{((2x+1)^2+4)}}\,dx$

As in part **b**, you may be able to write down this integral without working.

Let $2x+1 = 2\sinh\theta$

$$2\frac{dx}{d\theta} = 2\cosh\theta \Rightarrow \frac{dx}{d\theta} = \cosh\theta$$

$$\int \frac{2}{\sqrt{((2x+1)^2+4)}}\,dx = \int \frac{2}{\sqrt{(4\sinh^2\theta+4)}}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{2}{2\cosh\theta}(\cosh\theta)\,d\theta = \int 1\,d\theta$$

$$= \theta + C = \operatorname{arsinh}\left(\frac{2x+1}{2}\right)+C$$

Using $\operatorname{arsinh}x = \ln\left(x+\sqrt{(x^2+1)}\right)$

$$\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{(2x+1)^2}{4}+1\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{4x^2+4x+1+4}{4}\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\frac{1}{2}\sqrt{(4x^2+4x+5)}\right]+C$$

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]-\ln 2+C$$

$-\ln 2 +$ an arbitrary constant is another arbitrary constant.

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]+k, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 51

Question:

Using the substitution $x = 2\cosh^2 t - \sinh^2 t$, evaluate $\int_2^3 (x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}\,dx$. **[E]**

Solution:

If $x = 2\cosh^2 t - \sinh^2 t$ then

$$x - 1 = 2\cosh^2 t - \sinh^2 t - 1$$
$$= 2\cosh^2 t - \left(1 + \sinh^2 t\right)$$
$$= 2\cosh^2 t - \cosh^2 t = \cosh^2 t$$

Simplify using $1 + \sinh^2 t = \cosh^2 t$.

$$x - 2 = 2\cosh^2 t - \sinh^2 t - 2$$
$$= 2\left(\cosh^2 t - 1\right) - \sinh^2 t$$
$$= 2\sinh^2 t - \sinh^2 t = \sinh^2 t$$

Simplify using $\cosh^2 t - 1 = \sinh^2 t$.

$$\frac{dx}{dt} = 4\cosh t \sinh t - 2\cosh t \sinh t = 2\cosh t \sinh t$$

Substituting into the integral

$$\int (x-1)^{\frac{1}{2}} (x-2)^{\frac{1}{2}}\, dx = \int \left(\cosh^2 t\right)^{\frac{1}{2}} \left(\sinh^2 t\right)^{\frac{1}{2}} \frac{dx}{dt}\, dt$$
$$= \int \cosh t \sinh t \left(2\cosh t \sin t\right) dt$$
$$= \int 2\left(\cosh t \sinh t\right)^2 dt$$
$$= \frac{1}{2}\int \sinh^2 2t\, dt = \frac{1}{4}\int \left(\cosh 4t - 1\right) dt$$
$$= \frac{1}{16}\sinh 4t - \frac{t}{4}$$

To find the integral you need the hyperbolic identities $\sinh 2t = 2\sinh t \cosh t$ and $\cosh 4t = 1 + 2\sinh^2 2t$.

For the limits

At $x = 2$

$$2 = 2\cosh^2 t - \sinh^2 t = \cosh^2 t + \left(\cosh^2 t - \sinh^2 t\right)$$
$$2 = \cosh^2 t + 1 \Rightarrow \cosh t = 1 \Rightarrow t = 0$$

At $x = 3$

$$3 = 2\cosh^2 t - \sinh^2 t = \cosh^2 t + \left(\cosh^2 t - \sinh^2 t\right)$$
$$3 = \cosh^2 t + 1 \Rightarrow \cosh^2 t = 2$$
$$\cosh t = \sqrt{2} \Rightarrow t = \ln\left(\sqrt{2} + 1\right)$$
$$\sinh t = \sqrt{\left(\cosh^2 t - 1\right)} = \sqrt{(2-1)} = 1$$

Hence at $x = 3$

Using the formula

$$\text{arcosh}\, x = \ln\left(x + \sqrt{\left(x^2 - 1\right)}\right),$$
$$\text{arcosh}\sqrt{2} = \ln\left(\sqrt{2} + \sqrt{(2-1)}\right)$$
$$= \ln\left(\sqrt{2} + 1\right)$$

$$\frac{1}{16}\sinh 4t = \frac{1}{8}\sinh 2t \cosh 2t$$
$$= \frac{1}{8}(2\sinh t \cosh t)(1+2\sinh^2 t)$$
$$= \frac{1}{8}(2\sqrt{2})(1+2) = \frac{3\sqrt{2}}{4}$$

The evaluation of $\frac{1}{16}\sinh 4t$ at the upper limit requires the use of two hyperbolic double angle formulae and it is a good idea to work this out as a separate calculation before attempting the complete integral.

Hence

$$\int_2^3 (x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}\,dx = \left[\frac{1}{16}\sinh 4t - \frac{t}{4}\right]_0^{\ln(\sqrt{2}+1)}$$
$$= \frac{3\sqrt{2}}{4} - \frac{1}{4}\ln(\sqrt{2}+1)$$

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Review Exercise 1
Exercise A, Question 52

Question:

$f(x) = \arcsin x$

a Show that $f'(x) = \dfrac{1}{\sqrt{(1-x^2)}}$.

b Given that $y = \arcsin 2x$, obtain $\dfrac{dy}{dx}$ as an algebraic fraction.

c Using the substitution $x = \dfrac{1}{2}\sin\theta$, show that $\displaystyle\int_0^{\frac{1}{4}} \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}}\,dx = \frac{1}{48}(6-\pi\sqrt{3})$. **[E]**

Solution:

a Let $y = f(x) = \arcsin x$

$\sin y = x$

Differentiating implicitly with respect to x

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$$

$$f'(x) = \frac{1}{\sqrt{(1-x^2)}}, \text{ as required}$$

Unless otherwise stated, $\arcsin x$ is taken to have the range $-\frac{\pi}{2} < \arcsin x < \frac{\pi}{2}$. These are the principal values of $\arcsin x$. In this range, $\arcsin x$ is an increasing function of x, $\frac{dy}{dx}$ is positive and you can take the positive value of the square root.

b $y = \arcsin 2x$

Let $u = 2x, \frac{du}{dx} = 2$

$y = \arcsin u$

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{(1-u^2)}} \times 2 = \frac{2}{\sqrt{(1-4x^2)}}$$

c $x = \frac{1}{2}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\cos\theta$

At $x = \frac{1}{4}, \frac{1}{4} = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

At $x = 0, 0 = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

In this question it is convenient to carry out the substitution without returning to the original variable x. So at some stage you must change the x limits to θ limits.

$$\int \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}} dx = \int \frac{\frac{1}{2}\sin\theta \arcsin(\sin\theta)}{\sqrt{(1-\sin^2\theta)}} \left(\frac{dx}{d\theta}\right) d\theta$$

$$= \int \frac{\frac{1}{2}\sin\theta \times \theta}{\cos\theta}\left(\frac{1}{2}\cos\theta\right) d\theta$$

By definition, $\arcsin(\sin\theta) = \theta$.

$$= \frac{1}{4}\int \theta \sin\theta \, d\theta$$

You use integration by parts, $\int u\frac{dv}{d\theta} = uv - \int v\frac{du}{d\theta}\,d\theta$, with $u = \theta$ and $\frac{dv}{d\theta} = \sin\theta$.

$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\int \cos\theta \, d\theta$$

$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta$$

Hence

$$\int_0^{\frac{1}{4}} \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}} dx = \left[-\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left[-\frac{\pi}{24}\cos\frac{\pi}{6} + \frac{1}{4}\sin\frac{\pi}{6}\right] - [0]$$

$$= -\frac{\pi}{24} \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{48}\left(6 - \pi\sqrt{3}\right), \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 53

Question:

a Show that $\text{artanh}\left(\sin\frac{\pi}{4}\right)=\ln(1+\sqrt{2})$.

b Given that $y=\text{artanh}(\sin x)$, show that $\frac{dy}{dx}=\sec x$.

c Find the exact value of $\int_0^{\frac{\pi}{4}}\sin x\,\text{artanh}(\sin x)\,dx$. **[E]**

Solution:

a Using $\text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$$\text{artanh}\left(\sin\frac{\pi}{4}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)$$

Multiply the numerator and denominator of this fraction by $\sqrt{2}$.

$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Rationalise the denominator of the fraction by multiplying the numerator and denominator by $(\sqrt{2}+1)$.

$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{1}{2}\ln\left(\frac{(\sqrt{2}+1)^2}{1}\right)$$

$$= 2\times\frac{1}{2}\ln(1+\sqrt{2}) = \ln(1+\sqrt{2}), \text{ as required}$$

Use the property of logarithms $\ln a^2 = 2\ln a$.

b $y = \text{artanh}(\sin x)$

Let $u = \sin x$, then $\frac{du}{dx} = \cos x$

$y = \text{artanh}u$

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

$$= \frac{1}{1-u^2}\times\cos x$$

The formula $\frac{d}{dx}(\text{artanh}x) = \frac{1}{1-x^2}$ is given in the Edexcel formulae booklet which is provided for use with the paper.

$$= \frac{1}{1-\sin^2 x}\times\cos x = \frac{\cos x}{\cos^2 x}$$

$$= \frac{1}{\cos x} = \sec x, \text{ as required.}$$

c $\int \sin x\,\text{artanh}(\sin x)\,dx$

From part **a**, you know that you can differentiate artanh $(\sin x)$ and this suggests using integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$, with $u = \text{artanh}(\sin x)$ and $\frac{dv}{dx} = \sin x$.

$$= -\text{artanh}(\sin x)\cos x + \int \cos x\sec x\,dx$$

$$= -\text{artanh}(\sin x)\cos x + \int 1\,dx$$

$$= -\text{artanh}(\sin x)\cos x + x$$

Hence

$$\int_0^{\frac{\pi}{4}} \sin x\,\text{artanh}(\sin x)\,dx = \left[-\text{artanh}(\sin x)\cos x + x\right]_0^{\frac{\pi}{4}}$$

$$= \left[-\text{artanh}\left(\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} + \frac{\pi}{4}\right] - [0]$$

You evaluate the upper limit using the result proved in part **a** that $\text{artanh}\left(\sin\frac{\pi}{4}\right) = \ln(1+\sqrt{2})$.

$$= -\ln(1+\sqrt{2})\times\frac{1}{\sqrt{2}} + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{\sqrt{2}}{2}\ln(1+\sqrt{2})$$

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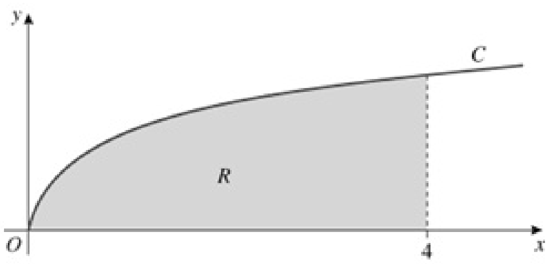
Review Exercise 1
Exercise A, Question 54

Question:



The figure shows part of the curve C with equation $y = \text{arsinh}(\sqrt{x}), x \geq 0$.

a Find the gradient of C at the point where $x = 4$.

The region R, shown shaded in the figure, is bounded by C, the x-axis and the line $x = 4$.

b Using the substitution $x = \sinh^2 \theta$, or otherwise, show that the area of R is $k \ln(2 + \sqrt{5}) - \sqrt{5}$, where k is a constant. **[E]**

Solution:

a $y = \text{arsinh}\left(\sqrt{x}\right)$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$y = \text{arsinh}u$

Using the chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{\sqrt{(u^2+1)}} \times \frac{1}{2}x^{-\frac{1}{2}}$

As $u = x^{\frac{1}{2}}$, then $u^2 = x$ and $\sqrt{(u^2+1)} = \sqrt{(x+1)}$.

$= \frac{1}{2\sqrt{x}\sqrt{(x+1)}}$

At $x = 4$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4}\sqrt{(4+1)}} = \frac{1}{4\sqrt{5}} = \frac{\sqrt{5}}{20}$

b If $x = \sinh^2\theta, \frac{dx}{d\theta} = 2\sinh\theta\cosh\theta = \sinh 2\theta$

$\int \text{arsinh}\sqrt{x}\,dx = \int \text{arsinh}\left(\sqrt{\left(\sinh^2\theta\right)}\right) \times \frac{dx}{d\theta}\,d\theta$

$= \int \text{arsinh}(\sinh\theta) \times \sinh 2\theta\, d\theta$

By definition $\text{arsinh}(\sinh\theta) = \theta$.

$= \int \theta\sinh 2\theta\, d\theta$

You use integration by parts, $\int u\frac{dv}{d\theta}\,d\theta = uv - \int v\frac{du}{d\theta}\,d\theta$, with $u = \theta$ and $\frac{dv}{d\theta} = \sinh 2\theta$.

$= \frac{\theta\cosh 2\theta}{2} - \int \frac{\cosh 2\theta}{2}\,d\theta$

$= \frac{\theta\cosh 2\theta}{2} - \frac{\sinh 2\theta}{4}$

$= \frac{\theta\left(1+2\sinh^2\theta\right)}{2} - \frac{2\sinh\theta\cosh\theta}{4}$

This solution uses double angle formulae to transform the expression back to the original variable x before substituting in the limits.

$= \frac{\text{arsinh}\left(\sqrt{x}\right)(1+2x)}{2} - \frac{\sqrt{x}\sqrt{(1+x)}}{2}$

Hence the area, A, of R is given by

$A = \left[\frac{\text{arsinh}\left(\sqrt{x}\right)(1+2x)}{2} - \frac{\sqrt{x}\sqrt{(1+x)}}{2}\right]_0^4$

$= \left[\frac{\text{arsinh}(2)(9)}{2} - \frac{2\sqrt{(5)}}{2}\right] - [0]$

$= \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$

This is the required result with $k = \frac{9}{2}$.

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Review Exercise 1
Exercise A, Question 55

Question:

$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx, n \geq 0$.

a Prove that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}, n \geq 2$.

b Find an exact expression for I_6. **[E]**

Solution:

a $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$

$= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x \, dx$

$= \left(\frac{\pi}{2}\right)^n + \left[nx^{n-1} \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \cos x \, dx$

$= \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$, as required

You use integration by parts, $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = x^n$ and $\frac{dv}{dx} = \cos x$. Then $v = \sin x$.

You repeat integration by parts this time with $u = nx^{n-1}$ and $\frac{dv}{dx} = \sin x$.

This expression is zero at both the lower and upper limit.

b $I_6 = \left(\frac{\pi}{2}\right)^6 - 6 \times 5 I_4$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\left(\frac{\pi}{2}\right)^4 - 4 \times 3 I_2\right)$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360 I_2$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\left(\frac{\pi}{2}\right)^2 - 2 \times 1 I_0\right)$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720 I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1 - 0 = 1$

Hence

$I_6 = \left(\frac{\pi}{2}\right)^6 - 360\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$

This is the result of part a with 6 substituted for n. You have now reduced the integral to $n = 4$. You then repeat the procedure until you reach an integral which you can evaluate directly.

$I_0 = \int_0^{\frac{\pi}{2}} x^0 \cos x \, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx$ and as the integral of $\cos x$ is $\sin x$ you can work this out without further use of the reduction formula.

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Review Exercise 1
Exercise A, Question 56

Question:

Given that $I_n = \int_0^4 x^n \sqrt{(4-x)}\,dx, n \geq 0$.

a show that $I_n = \frac{8n}{2n+3} I_{n-1}, n \geq 1$.

Given that $\int_0^4 \sqrt{(4-x)}\,dx = \frac{16}{3}$,

b use the result in part **a** to find the exact value of $\int_0^4 x^2 \sqrt{(4-x)}\,dx$. **[E]**

Solution:

a $I_n = \int_0^4 x^n \sqrt{(4-x)}\,dx$

$= \left[-\frac{2}{3}(4-x)^{\frac{3}{2}} x^n\right]_0^4 + \frac{2}{3}\int_0^4 nx^{n-1}(4-x)^{\frac{3}{2}}\,dx$

$= \frac{2}{3}\int_0^4 nx^{n-1}(4-x)(4-x)^{\frac{1}{2}}\,dx$

You use integration by parts, $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, with $u = x^n$ and $\frac{dv}{dx} = (4-x)^{\frac{1}{2}}$. Then

$v = \int (4-x)^{\frac{1}{2}}\,dx = \frac{(4-x)^{\frac{3}{2}}}{-1\times\frac{3}{2}} = -\frac{2}{3}(4-x)^{\frac{3}{2}}$.

$= \frac{2}{3}\int_0^4 nx^{n-1}4(4-x)^{\frac{1}{2}}\,dx - \frac{2}{3}\int_0^4 nx^{n-1}x(4-x)^{\frac{1}{2}}\,dx$

$= \frac{8n}{3}\int_0^4 x^{n-1}(4-x)^{\frac{1}{2}}\,dx - \frac{2n}{3}\int_0^4 x^n(4-x)^{\frac{1}{2}}\,dx$

$= \frac{8n}{3}I_{n-1} - \frac{2n}{3}I_n$

You split this integral into two separate integrals using

$(4-x)^{\frac{3}{2}} = (4-x)^1(4-x)^{\frac{1}{2}}$

$= (4-x)(4-x)^{\frac{1}{2}}$

$= 4(4-x)^{\frac{1}{2}} - x(4-x)^{\frac{1}{2}}$

Hence

$I_n + \frac{2n}{3}I_n =$ so $\frac{3+2n}{3}I_n = \frac{8n}{3}I_{n-1}$

$I_n = \frac{8n}{2n+3}I_{n-1}$, as required.

Collect the terms in I_n on one side of the equation and solve for I_n in terms of n and I_{n-1}.

b $I_2 = \frac{8\times 2}{2\times 2+3}I_1 = \frac{16}{7}I_1$

$= \frac{16}{7}\times\frac{8\times 1}{2\times 1+3}I_0 = \frac{16}{7}\times\frac{8}{5}I_0$

$= \frac{16}{7}\times\frac{8}{5}\times\frac{16}{3} = \frac{2048}{105}$

This is the result of part **a** with 2 substituted for n. You have now reduced the integral to $n = 1$. You then repeat the procedure reaching $n = 0$ and, in this question, you have been given I_0.

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Review Exercise 1
Exercise A, Question 57

Question:

Given that $y = \sinh^{n-1} x \cosh x$,

a show that $\frac{dy}{dx} = (n-1)\sinh^{n-2} x + n\sinh^n x$.

The integral I_n is defined by $I_n = \int_0^{\text{arsinh} 1} \sinh^n x \, dx, n \geq 0$.

b Using the result in part **a**, or otherwise, show that $nI_n = \sqrt{2} - (n-1)I_{n-2}, n \geq 2$.

c Hence find the value of I_4. **[E]**

Solution:

a $y = \sinh^{n-1} x \cosh x$

$$\frac{dy}{dx} = (n-1)\sinh^{n-2} x \cosh x \times \cosh x + \sinh^{n-1} x \times \sinh x$$

Using the product rule for differentiation.

$$= (n-1)\sinh^{n-2} x \cosh^2 x + \sinh^n x$$
$$= (n-1)\sinh^{n-2} x\left(1+\sinh^2 x\right) + \sinh^n x$$
$$= (n-1)\sinh^{n-2} x + (n-1)\sinh^n x + \sinh^n x$$
$$= (n-1)\sinh^{n-2} x + n\sinh^n x, \text{ as required.}$$

You use the identity $\cosh^2 x - \sinh^2 x = 1$ to write this expression in terms of the powers of $\sinh x$ only.

b Integrating the result of part **a** throughout with respect to x.

$$\int \frac{dy}{dx}\,dx = \int (n-1)\sinh^{n-2} x\,dx + \int n\sinh^n x\,dx$$

$$y = \int (n-1)\sinh^{n-2} x\,dx + \int n\sinh^n x\,dx$$

As integration is the reverse process of differentiation $\int \frac{dy}{dx}\,dx = y$ and, in this question, $y = \sinh^{n-1} x \cosh x$.

$$\sinh^{n-1} x \cosh x = \int (n-1)\sinh^{n-2} x\,dx + \int n\sinh^n x\,dx$$

Between the limits 0 and arsinh 1

$$\left[\sinh^{n-1} x \cosh x\right]_0^{\text{arsinh}1} = \int_0^{\text{arsinh}1} (n-1)\sinh^{n-2} x\,dx + \int_0^{\text{arsinh}1} n\sinh^n x\,dx$$

$$1\times\sqrt{2} - 0 = (n-1)I_{n-2} + nI_n$$
$$nI_n = \sqrt{2} - (n-1)I_{n-2}, \text{ as required.}$$

$x = \text{arsinh}1 \Rightarrow \sinh x = 1$ and, as $\cosh^2 x = 1 + \sinh^2 x$, then $\cosh^2 1 = 1 + \sinh^2 x = 2 \Rightarrow \cosh 1 = \sqrt{2}$

c From part **b** $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$

Hence

$$I_4 = \frac{\sqrt{2}}{4} - \frac{3}{4} I_2$$
$$= \frac{\sqrt{2}}{4} - \frac{3}{4}\left(\frac{\sqrt{2}}{2} - \frac{1}{2} I_0\right) = \frac{3}{8} I_0 - \frac{\sqrt{2}}{8}$$

$$I_0 = \int_0^{\text{arsinh}1} \sinh^0 x\,dx = \int_0^{\text{arsinh}1} 1\,dx$$
$$= [x]_0^{\text{arsinh}1} = \text{arsinh}1 - 0 = \ln\left(1+\sqrt{2}\right)$$

Using $\text{arsinh}x = \ln\left[x + \sqrt{\left(x^2+1\right)}\right]$ with $x = 1$.

Hence

$$I_4 = \frac{1}{8}\left(3\ln\left(1+\sqrt{2}\right) - \sqrt{2}\right)$$

It is usual to give values involving inverse hyperbolic functions in terms of natural logarithms but, as this question specifies no form of the answer, $\frac{1}{8}\left(3\,\text{arsinh}1 - \sqrt{2}\right)$ would be acceptable.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 58

Question:

Given that $I_n = \int_0^8 x^n (8-x)^{\frac{1}{3}}\,dx, n \geq 0$,

a show that $I_n = \frac{24n}{3n+4} I_{n-1}, n \geq 1$.

b Hence find the exact value of $\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\,dx$. **[E]**

Solution:

a $I_n = \int_0^8 x^n (8-x)^{\frac{1}{3}}\,dx$

$= \left[x^n \left(-\frac{3}{4}\right)(8-x)^{\frac{4}{3}} \right]_0^8 - \int_0^8 nx^{n-1}\left(-\frac{3}{4}\right)(8-x)^{\frac{4}{3}}\,dx$

$= \frac{3n}{4}\int_0^8 x^{n-1}(8-x)(8-x)^{\frac{1}{3}}\,dx$

$= \frac{3n}{4}\int_0^8 x^{n-1}(8)(8-x)^{\frac{1}{3}}\,dx - \frac{3n}{4}\int_0^8 x^{n-1}(x)(8-x)^{\frac{1}{3}}\,dx$

$= 6n\int_0^8 x^{n-1}(8-x)^{\frac{1}{3}}\,dx - \frac{3n}{4}\int_0^8 x^n (8-x)^{\frac{1}{3}}\,dx$

$I_n = 6nI_{n-1} - \frac{3n}{4}I_n$

$\left(1+\frac{3n}{4}\right)I_n = \therefore \frac{4+3n}{4}I_n = 6nI_{n-1}$

$I_n = \frac{24n}{3n+4}I_{n-1}$

You use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, with $u = x^n$ and $\frac{dv}{dx} = (8-x)^{\frac{1}{3}}$.

$v = \int (8-x)^{\frac{1}{3}}\,dx = \frac{(8-x)^{\frac{4}{3}}}{-\frac{4}{3}}$

$= -\frac{3}{4}(8-x)^{\frac{4}{3}}$

You split this integral into two separate integrals using

$(8-x)^{\frac{4}{3}} = (8-x)^1(8-x)^{\frac{1}{3}}$

$= (8-x)(8-x)^{\frac{1}{3}}$

$= 8(8-x)^{\frac{1}{3}} - x(8-x)^{\frac{1}{3}}$.

Collect the terms in I_n on one side of the equation and solve for I_n in terms of n and I_{n-1}.

b $\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\,dx = \int_0^8 (x^2+5x)(8-x)^{\frac{1}{3}}\,dx$

$= \int_0^8 x^2(8-x)^{\frac{1}{3}}\,dx + 5\int_0^8 x(8-x)^{\frac{1}{3}}\,dx$

$= I_2 + 5I_1$

$I_0 = \int_0^8 (8-x)^{\frac{1}{3}}\,dx = \left[\frac{(8-x)^{\frac{4}{3}}}{-\frac{4}{3}}\right]_0^8$

$= \left[-\frac{3}{4}(8-x)^{\frac{4}{3}}\right]_0^8 = 0 - \left(-\frac{3}{4}\times 8^{\frac{4}{3}}\right)$

$= \frac{3}{4}\times 16 = 12$

Using the result of part **a**

$I_1 = \frac{24}{7}I_0 = \frac{24}{7}\times 12 = \frac{288}{7}$

$I_2 = \frac{48}{10}I_1 = \frac{48}{10}\times\frac{288}{7} = \frac{6912}{35}$

These fractions are awkward. Use your calculator to manipulate the fractions.

$\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\,dx = I_2 + 5I_1$

$= \frac{6912}{35} + 5\times\frac{288}{7} = \frac{2016}{5}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 59

Question:

$I_n = \int \frac{\sin nx}{\sin x} \mathrm{d}x \; n > 0, n \in \mathbb{Z}$.

a By considering $I_{n+2} - I_n$, or otherwise, show that $I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n$.

b Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} \mathrm{d}x$, giving your answer in the form $p\sqrt{2} + q\sqrt{3}$, where p and q are rational numbers to be found. **[E]**

Solution:

a $I_{n+2} - I_n = \int \frac{\sin(n+2)x}{\sin x}\,dx - \int \frac{\sin nx}{\sin x}\,dx$

$= \int \frac{\sin(n+2)x - \sin nx}{\sin x}\,dx$

$= \int \frac{2\cos(n+1)x \sin x}{\sin x}\,dx$

$= \int 2\cos(n+1)x\,dx$

$= \frac{2\sin(n+1)x}{n+1}$

Hence

$I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n$, as required.

You use the trigonometric identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ with $A = (n+2)x$ and $B = nx$. The identity can be found among the formulae for module C3 in the Edexcel formulae booklet which is provided for use in the examination. The specification for FP3 requires knowledge of the specifications for C1, C2, C3, C4 and FP1 and their associated formulae.

b Using the result in part **a**

$I_6 = \frac{2\sin 5x}{5} + I_4$

$= \frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + I_2$

$I_2 = \int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx$

$= \int 2\cos x\,dx = 2\sin x + C$

I_2 can be found directly. You should not reduce the integral to I_0 as the first line of the question specifies $n > 0$.

The constant of integration will disappear when limits are applied.

Hence

$$I_6 = \frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + 2\sin x + C$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\,dx = \left[\frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + 2\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\frac{2}{5}\times -\frac{\sqrt{3}}{2} + \frac{2}{3}\times 0 + 2\times\frac{\sqrt{3}}{2}\right) - \left(\frac{2}{5}\times -\frac{\sqrt{2}}{2} + \frac{2}{3}\times\frac{\sqrt{2}}{2} + 2\times\frac{\sqrt{2}}{2}\right)$$

$$= \frac{4}{5}\sqrt{3} - \frac{17}{15}\sqrt{2}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 60

Question:

$I_n = \int_0^1 x^n \text{e}^x \, \text{d}x$ and $J_n = \int_0^1 x^n \text{e}^{-x} \text{d}x, n \geq 0$.

a Show that, for $n \geq 1$, $I_n = \text{e} - nI_{n-1}$.

b Find a similar formula for J_n.

c Show that $J_2 = 2 - \frac{5}{\text{e}}$.

d Show that $\int_0^1 x^n \cosh x \text{d}x = \frac{1}{2}(I_n + J_n)$.

e Hence, or otherwise, evaluate $\int_0^1 x^2 \cosh x \text{d}x$, giving your answer in terms of e. **[E]**

Solution:

a $I_n = \int_0^1 x^n \, e^x \, dx$

$= \left[x^n \, e^x\right]_0^1 - \int_0^1 n x^{n-1} \, e^x \, dx$

$= e^1 - 0 - n\int_0^1 x^{n-1} e^x \, dx$

$= e - nI_{n-1}$, as required.

You use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = x^n$ and $\frac{dv}{dx} = e^x$.

b $J_n = \int_0^1 x^n \, e^{-x} \, dx$

$= \left[-x^n \, e^{-x}\right]_0^1 + \int_0^1 n x^{n-1} \, e^{-x} \, dx$

$= -e^{-1} - 0 + n\int_0^1 x^{n-1} \, e^{-x} dx$

$= -e^{-1} + nJ_{n-1}$

As you are asked to find a similar formula, it is sensible to pattern your solution to part **b** on that of part **a**. Here you use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = x^n$ and $\frac{dv}{dx} = e^{-x}$.

c $J_2 = -e^{-1} + 2J_1$

$= -e^{-1} + 2\left(-e^{-1} + J_0\right) = -3e^{-1} + 2J_0$

$J_0 = \int_0^1 x^0 e^{-x} dx = \int_0^1 e^{-x} dx$

$= \left[-e^{-x}\right]_0^1 = -e^{-1} - (-1) = 1 - e^{-1}$

You use the result of part **b** twice and evaluate J_0 directly.

Hence

$J_2 = -3e^{-1} + 2\left(1 - e^{-1}\right) = 2 - 5e^{-1}$

$= 2 - \frac{5}{e}$, as required.

d $\int_0^1 x^n \cosh x \, dx = \int_0^1 x^n \left(\frac{e^x + e^{-x}}{2}\right) dx$

$= \frac{1}{2}\int_0^1 x^n e^x dx + \frac{1}{2}\int_0^1 x^n e^{-x} dx$

$= \frac{1}{2} I_n + \frac{1}{2} J_n = \frac{1}{2}\left(I_n + J_n\right)$, as required.

e $I_2 = \mathrm{e} - 2I_1$

$= \mathrm{e} - 2(\mathrm{e} - I_0) = -\mathrm{e} + 2I_0$

$I_0 = \int_0^1 x^0 \mathrm{e}^x \, \mathrm{d}x = \int_0^1 \mathrm{e}^x \, \mathrm{d}x$

$= \left[\mathrm{e}^x\right]_0^1 = \mathrm{e}^1 - 1 = \mathrm{e} - 1$

You use the result of part **a** twice and evaluate I_0 directly.

Hence

$I_2 = -\mathrm{e} + 2(\mathrm{e} - 1) = \mathrm{e} - 2$

$\int_0^1 x^2 \cosh x \, \mathrm{d}x = \frac{1}{2}(I_2 + J_2)$

$= \frac{1}{2}\left(\mathrm{e} - 2 + 2 - \frac{5}{\mathrm{e}}\right) = \frac{1}{2}\left(\mathrm{e} - \frac{5}{\mathrm{e}}\right)$

This is the result of part **d** with $n = 2$.

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Review Exercise 1
Exercise A, Question 61

Question:

Given that $I_n = \int \sec^n x \, dx$,

a show that $(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}, n \geq 2$.

b Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^3 x \, dx$, giving your answer in terms of natural logarithms and surds. **[E]**

Solution:

a $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \times \tan x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$

$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$

$I_n + (n-2)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$

$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$, as required.

> You use $\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$, with $u = \sec^{n-2} x$ and $\frac{dv}{dx} = \sec^2 x$.
>
> Using the chain rule $\frac{d}{dx}\left(\sec^{n-2} x\right)$
>
> $= (n-2)\sec^{n-3} x \frac{d}{dx}(\sec x)$
>
> $= (n-2)\sec^{n-3} x \times \sec x \tan x$.

b From part a $I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

Substituting $n = 3$

$I_3 = \frac{\tan x \sec x}{2} + \frac{1}{2} I_1$

$I_1 = \int \sec x\,dx = \ln(\sec x + \tan x) + C$

> The formula for integrating sec x can be found among the formulae for module C4 in the Edexcel formula booklet, which is provided for use in the examination.

Hence $I_3 = \frac{\tan x \sec x}{2} + \frac{1}{2}\ln(\sec x + \tan x) + C$

With the limits $x = 0$ and $x = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} \sec^3 x\,dx = \left[\frac{\tan x \sec x}{2} + \frac{1}{2}\ln(\sec x + \tan x)\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{2} \times \sqrt{3} \times 2 + \frac{1}{2}\ln\left(2 + \sqrt{3}\right)\right) - 0$$

$$= \sqrt{3} + \frac{1}{2}\ln\left(2 + \sqrt{3}\right)$$

> $\tan\frac{\pi}{3} = \sqrt{3}$ and
>
> $\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 62

Question:

$I_n = \int_0^1 (1-x^2)^n \, dx, n \geq 0$.

a Prove that $(2n+1)I_n = 2nI_{n-1}, n \geq 1$.

b Prove by induction that $I_n \geq \left(\frac{2n}{2n+1}\right)^n$ for $n \in \mathbb{Z}^+$. **[E]**

Solution:

a $I_n = \int_0^1 (1-x^2)^n \, dx = \int_0^1 1\times(1-x^2)^n \, dx$

$= \left[x(1-x^2)^n\right]_0^1 - \int_0^1 x \times n(1-x^2)^{n-1}(-2x)\,dx$

$= 2n\int_0^1 x^2(1-x^2)^{n-1}\,dx$

$= 2n\int_0^1 (x^2-1+1)(1-x^2)^{n-1}\,dx$

$= -2n\int_0^1 (1-x^2)^n\,dx + 2n\int_0^1 (1-x^2)^{n-1}\,dx$

$= -2nI_n + 2nI_{n-1}$

$(2n+1)I_n = 2nI_{n-1}$, as required

You use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, with $u=(1-x^2)^n$, $\frac{dv}{dx}=1$ and, so, $v = x$.

You split this integral into two separate integrals using algebra.

$x^2(1-x^2)^{n-1} = (x^2-1+1)(1-x^2)^{n-1}$

$= (x^2-1)(1-x^2)^{n-1} + 1(1-x^2)^{n-1}$

$= -(1-x^2)^n + (1-x^2)^{n-1}$

b To prove $I \le \left(\frac{2n}{2n+1}\right)^n$ by mathematical induction.

Mathematical induction is a topic in the FP1 specification. The FP3 specification assumes knowledge of the FP1 specification.

Let $n = 1$

$I_1 = \int_0^1 (1-x^2)^1\,dx = \int_0^1 (1-x^2)\,dx$

$= \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3} = \left(\frac{2\times 1}{2\times 1+1}\right)^1$

Hence for $n = 1, I_n \le \left(\frac{2n}{2n+1}\right)^n$

For $n = 1$, equality holds.

Assume the inequality is true for $n = k$, that is $I_k \le \left(\frac{2k}{2k+1}\right)^k$.

From part a, $I_n = \frac{2n}{2n+1}I_{n-1}$

With $n = k+1$ and using the induction hypothesis

$I_{k+1} = \frac{2k+2}{2k+3}I_k \le \frac{2k+2}{2k+3}\left(\frac{2k}{2k+1}\right)^k$

To complete the proof it is necessary to show that,

for $k > 0, \frac{2k}{2k+1} \le \frac{2k+2}{2k+3}$

To complete the proof the $\frac{2k}{2k+1}$ in the bracket needs to be replaced by $\frac{2k+2}{2k+3}$, which is the expression $\frac{2n}{2n+1}$ with $n = k+1$. You are also using the property that, for positive numbers, $a \le b \Rightarrow a^k \le b^k$.

$$\frac{2k}{2k+1}-\frac{2k+2}{2k+3}=\frac{2k(2k+3)-(2k+2)(2k+1)}{(2k+1)(2k+3)}$$

$$=\frac{4k^2+6k-(4k^2+6k+2)}{(2k+1)(2k+3)}$$

$$=\frac{-2}{(2k+1)(2k+3)}<0,\text{ for } k>0$$

Hence $\frac{2k}{2k+1}\le\frac{2k+2}{2k+3}$ and $I_{k+1}\le\frac{2k+2}{2k+3}\left(\frac{2k}{2k+1}\right)^k\le\frac{2k+2}{2k+3}\left(\frac{2k+2}{2k+3}\right)^k=\left(\frac{2k+2}{2k+3}\right)^{k+1}$

This is the inequality with $n=k+1$.

The inequality is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the inequality is true for all positive integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 63

Question:

A curve is defined by $x = t + \sin t, y = 1 - \cos t$, where t is a parameter.

Find the length of the curve from $t = 0$ to $t = \frac{\pi}{2}$, giving your answer in surd form. **[E]**

Solution:

$x = t + \sin t \quad y = 1 - \cos t$

$\frac{dx}{dt} = 1 + \cos t \quad \frac{dy}{dt} = \sin t$

$s = \int \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)} dt$

It is always a good idea to quote any formula you are going to use in answering a question.

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + \cos t)^2 + \sin^2 t$

$= 1 + 2\cos t + \cos^2 t + \sin^2 t$

$= 2 + 2\cos t$

$= 4\cos^2 \frac{t}{2}$

You simplify this expression using the identity $\sin^2 t + \cos^2 t = 1$ and the double angle formula $\cos 2x = 2\cos^2 x - 1$, with $x = \frac{t}{2}$.

Hence, the length of the curve is given by

$s = \int_0^{\frac{\pi}{2}} \sqrt{\left(4\cos^2 \frac{t}{2}\right)} dt = \int_0^{\frac{\pi}{2}} 2\cos \frac{t}{2} dt$

$\int 2\cos \frac{t}{2} dt = \frac{2\sin \frac{t}{2}}{\frac{1}{2}} = 4\sin \frac{t}{2}$

$= \left[4\sin \frac{t}{2}\right]_0^{\frac{\pi}{2}} = 4\sin \frac{\pi}{4}$

$= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$

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Review Exercise 1
Exercise A, Question 64

Question:

Parametric equations for the curve C are $x=\cosh t+t, y=\cosh t-t, t\geq 0$.
Show that the length of the arc of the curve C between points at which $t=0$ and $t=a$, where a is a positive constant, is $(\sqrt{2})\sinh a$. **[E]**

Solution:

$$x=\cosh t+t \quad y=\cosh t-t$$

$$\frac{dx}{dt}=1+\sinh t \quad \frac{dy}{dt}=\sinh t-1$$

$$s=\int\sqrt{\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right)}dt$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(\sinh t+1)^2+(\sinh t-1)^2$$

$$=\sinh^2 t+2\sinh t+1+\sinh^2 t-2\sinh t+1$$

$$=2\sinh^2 t+2=2\cosh^2 t$$

To save a lot of writing, it is often sensible to work out complicated expressions like $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$, and to transform them into a form where a square root can be easily found, before substituting into the integral.

Using $\cosh^2 t-\sinh^2 t=1$

Hence, the length of the curve is given by

$$s=\int_0^a\sqrt{(2\cosh^2 t)}\,dt=\sqrt{2}\int_0^a\cosh t\,dt$$

$$=\sqrt{2}[\sinh t]_0^a=\sqrt{2}(\sinh a-\sinh 0)$$

$$=\sqrt{2}\sinh a, \text{ as required.}$$

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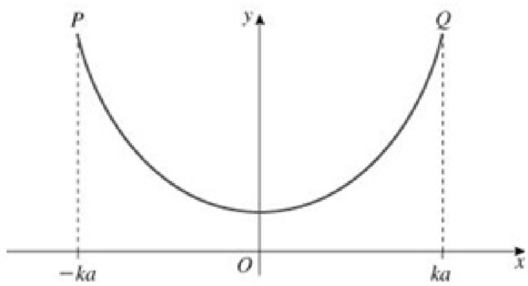
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 65

Question:



A rope is hung from points P and Q on the same horizontal line, as shown in the figure. The curve formed is modelled by the equation $y = a\cosh\left(\frac{x}{a}\right), -ka \le x \le ka$.

where a and k are constants.

a Prove that the length of the rope is $2a \sinh k$.

Given that the length of the rope is $8a$,

b find the coordinates of Q, leaving your answer in terms of natural logarithms and surds, where appropriate. **[E]**

Solution:

a $y = a\cosh\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = \frac{1}{a}\times a\sinh\left(\frac{x}{a}\right) = \sinh\left(\frac{x}{a}\right)$$

$$s = \int \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}dx$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$$

The length of the rope is given by

$$s = 2\int_0^{ka} \cosh\left(\frac{x}{a}\right)dx$$

$$= 2\left[a\sinh\left(\frac{x}{a}\right)\right]_0^{ka} = 2a(\sinh k - \sinh 0)$$

$= 2a\sinh k$, as required.

From the symmetry of the diagram, the length of the rope from P to Q is twice the length of the rope from the point where $x = 0$ to Q.

b $2a\sinh k = 8a$

$$\sinh k = 4 \Rightarrow k = \text{arsinh}4$$

$$= \ln\left(4+\sqrt{17}\right)$$

You use the formula $\text{arsinh}x = \ln\left(x+\sqrt{\left(x^2+1\right)}\right)$ to find the x-coordinate of Q in terms of a natural logarithm. The question specifies that you should give your answer in this form.

At Q, $x = ka = a\ln\left(4+\sqrt{17}\right)$ and

$$y = a\cosh\left(\frac{x}{a}\right) = a\cosh\left(\frac{ka}{a}\right) = a\cosh k$$

$$\cosh^2 k = 1+\sinh^2 k = 1+4^2 = 17 \Rightarrow \cosh k = \sqrt{17}$$

The coordinates of Q are $\left(a\ln\left(4+\sqrt{17}\right), a\sqrt{17}\right)$.

As you know that $\sinh k = 4$, you can find the value of $\cosh k$ using the identity $\cosh^2 x = 1+\sinh^2 x$.

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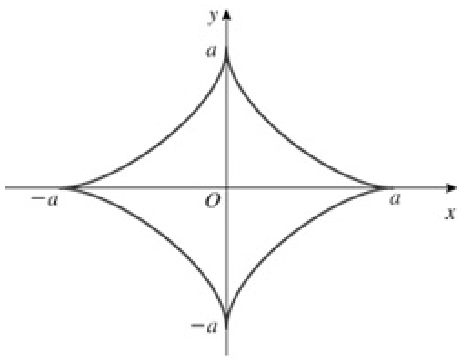
Review Exercise 1
Exercise A, Question 66

Question:



The figure shows the curve with parametric equations
$x = a\cos^3\theta, y = a\sin^3\theta, 0 \le \theta < 2\pi$.

a Find the total length of the curve.

The curve is rotated through π radians about the x-axis.

b Find the area of the surface generated. **[E]**

Solution:

a $x = a\cos^3\theta \qquad y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$s = \int \sqrt{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)} d\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2$$
$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\cos^2\theta\sin^4\theta$$
$$= 9a^2\cos^2\theta\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$
$$= 9a^2\cos^2\theta\sin^2\theta$$

Hence the length of the curve is given by

$$s = 4\times\int_0^{\frac{\pi}{2}} \sqrt{\left(9a^2\cos^2\theta\sin^2\theta\right)} d\theta$$
$$= 12a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta$$
$$= 12a\left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}}$$
$$= 12a\left(\frac{1}{2} - 0\right)$$
$$= 6a$$

> The symmetries of the diagram show that the total length of the curve is four times the length in the first quadrant. As $x\left(= a\cos^3\theta\right)$ varies from 0 to a, $\cos\theta$ varies from 0 to 1, and so θ varies from $\frac{\pi}{2}$ to 0 in that order.

> There are number of alternative ways of evaluating this integral. You could use a double angle formula.

b $A = 2\pi\int y\sqrt{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)} d\theta$

The area of the surface generated is given by

$$A = 2\times 2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta \times 3a\cos\theta\sin\theta\ d\theta$$
$$= 12\pi a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ d\theta$$
$$= 12\pi a^2\left[4\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = 12\pi a^2\left(\frac{1}{5} - 0\right)$$
$$= \frac{12}{5}\pi a^2$$

> The total area is twice the area formed by rotating the two portions of the curve on the positive side of the x-axis.
>
> You have already worked out $\sqrt{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)}$ in part **a** and there is no need to repeat the working here.

> Here the integral is found using the formula $\int \sin^n\theta\cos\theta\ d\theta = \frac{\sin^{n+1}\theta}{n+1}$ with $n = 4$. If you do not know this formula, you can find the integral using the substitution $u = \sin\theta$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 67

Question:

a By using the definition of cosh x in terms of exponentials, show that

$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$.

b The arc of the curve with equation $y = \cosh x$ from $x = 0$ to $x = \ln 2$ is rotated through 2π radians about the x-axis. Determine the area of the curved surface generated, leaving your answer in terms of π . **[E]**

Solution:

a $\cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{4} + \frac{2}{4}$

$= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2}\right) + \frac{1}{2} = \frac{1}{2}\cosh 2x + \frac{1}{2}$

$= \frac{1}{2}(\cosh 2x + 1)$, as required.

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b $y = \cosh x \Rightarrow \frac{dy}{dx} = \sinh x$

$A = 2\pi \int y \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} dx$

$= 2\pi \int_0^{\ln 2} \cosh x \sqrt{(1 + \sinh^2 x)} dx$

$= 2\pi \int_0^{\ln 2} \cosh^2 x dx$

$= 2\pi \int_0^{\ln 2} \frac{1}{2}(\cosh 2x + 1) dx = \pi \int_0^{\ln 2} (\cosh 2x + 1) dx$

You use the identity shown in part **a** to find the integral.

$= \pi \left[\frac{\sinh 2x}{2} + x\right]_0^{\ln 2}$

$= \pi [\sinh x \cosh x + x]_0^{\ln 2}$

Using the identity $\sinh 2x = 2\sinh x \cosh x$.

$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$

As, for any x, $e^{-\ln x} = e^{\ln 1 - \ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$,

then $e^{-\ln 2} = \frac{1}{2}$.

Hence the area is given by $A = \pi\left(\frac{3}{4} \times \frac{5}{4} + \ln 2\right) = \pi\left(\frac{15}{16} + \ln 2\right)$