Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise A, Question 1

Question:

Write down the transposes of the following matrices. In each case give the dimensions of the transposed matrix.

a $\begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 4 \end{pmatrix}$

b $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

c $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Solution:

a $\begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 2 & 4 \end{pmatrix}$ dimension 3×2

b $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ dimension 2×2

c $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$ dimension 3×3

d $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$ dimension 1×3

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Further matrix algebra
Exercise A, Question 2

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix}$.

a Write down $\mathbf{A}^T$.

b Find $\mathbf{AA}^T$.

c Find $\mathbf{A}^T\mathbf{A}$.

Solution:

a $\mathbf{A}^T = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$

b
$$\mathbf{AA}^T = \begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 4+16 & -6+24 \\ -6+24 & 9+36 \end{pmatrix} = \begin{pmatrix} 20 & 18 \\ 18 & 45 \end{pmatrix}$$

c
$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 4+9 & 8-18 \\ 8-18 & 16+36 \end{pmatrix} = \begin{pmatrix} 13 & -10 \\ -10 & 52 \end{pmatrix}$$

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Further matrix algebra
Exercise A, Question 3

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix}$.

a Find $\mathbf{BA}$.

b Verify that $\mathbf{A}^T\mathbf{B}^T = (\mathbf{BA})^T$.

Solution:

a $\mathbf{BA} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3-12 & 2+6 \\ 0+8 & 0-4 \end{pmatrix}$

$= \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$

b From **a**

$(\mathbf{BA})^T = \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$ ($\mathbf{BA}$ is symmetric)

$\mathbf{A}^T\mathbf{B}^T = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -4 \end{pmatrix}$

$= \begin{pmatrix} 3-12 & 0+8 \\ 2+6 & 0-4 \end{pmatrix}$

$= \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix} = (\mathbf{BA})^T$, as required.

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Further matrix algebra
Exercise A, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -4 & 8 \\ 4 & -7 & -4 \\ 8 & 4 & 1 \end{pmatrix}$.

a Write down $\mathbf{A}^T$.

b Show that $\mathbf{AA}^T = 81\mathbf{I}$.

Solution:

a $\mathbf{A}^T = \begin{pmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{pmatrix}$

b $$\mathbf{AA}^T = \begin{pmatrix} 1 & -4 & 8 \\ 4 & -7 & -4 \\ 8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+16+64 & 4+28-32 & 8-16+8 \\ 4+28-32 & 16+49+16 & 32-28-4 \\ 8-16+8 & 32-28-4 & 64+16+1 \end{pmatrix}$$

$$= \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = 81\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 81\mathbf{I}\text{, as required.}$$

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Further matrix algebra
Exercise A, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}0 & 3 & 5\\ -3 & 0 & -1\\ -5 & 1 & 0\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}-4 & 1 & -1\\ 1 & 5 & 2\\ -3 & 0 & 3\end{pmatrix}$.

Given that $\mathbf{C}=\mathbf{AB}$,

a find $\mathbf{C}$,

b verify that the matrix $\mathbf{C}$ is symmetric.

Solution:

a $\mathbf{C}=\begin{pmatrix}0 & 3 & 5\\ -3 & 0 & -1\\ -5 & 1 & 0\end{pmatrix}\begin{pmatrix}-4 & 1 & -1\\ 1 & 5 & 2\\ -3 & 0 & 3\end{pmatrix}$

$=\begin{pmatrix}0+3-15 & 0+15+0 & 0+6+15\\ 12+0+3 & -3+0+0 & 3+0+-3\\ 20+1+0 & -5+5+0 & 5+2+0\end{pmatrix}$

$=\begin{pmatrix}-12 & 15 & 21\\ 15 & -3 & 0\\ 21 & 0 & 7\end{pmatrix}$

b $\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-12 & 15 & 21\\ 15 & -3 & 0\\ 21 & 0 & 7\end{pmatrix}=\mathbf{C}$

Hence the matrix $\mathbf{C}$ is symmetric.

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Further matrix algebra
Exercise A, Question 6

Question:

The matrix $\mathbf{A}=\begin{pmatrix}0 & 3 & 5\\ 2 & 0 & -1\\ 1 & 1 & 0\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}$.

a Find $\mathbf{AB}$.

b Verify that $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

Solution:

a $\mathbf{AB}=\begin{pmatrix}0 & 3 & 5\\ 2 & 0 & -1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}$

$$=\begin{pmatrix}0+0-5 & 0+3+0 & 0+0+15\\ 2+0+1 & 2+0+0 & -2+0-3\\ 1+0+0 & 1+1+0 & -1+0+0\end{pmatrix}$$

$$=\begin{pmatrix}-5 & 3 & 15\\ 3 & 2 & -5\\ 1 & 2 & -1\end{pmatrix}$$

b From **a**

$$(\mathbf{AB})^{\mathrm{T}}=\begin{pmatrix}-5 & 3 & 1\\ 3 & 2 & 2\\ 15 & -5 & -1\end{pmatrix}$$

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1 & 0 & -1\\ 1 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}\begin{pmatrix}0 & 2 & 1\\ 3 & 0 & 1\\ 5 & -1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0+0-5 & 2+0+1 & 1+0+0\\ 0+3+0 & 2+0+0 & 1+1+0\\ 0+0+15 & -2+0-3 & -1+0+0\end{pmatrix}$$

$$=\begin{pmatrix}-5 & 3 & 1\\ 3 & 2 & 2\\ 15 & -5 & -1\end{pmatrix}=(\mathbf{AB})^{\mathrm{T}}\text{, as required.}$$

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Further matrix algebra
Exercise B, Question 1

Question:

Find the values of the determinants.

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

b $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$

c $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix}$

d $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix}$

Solution:

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$

$= 1(6-0) - 0 + 0 = 6$

b $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 0\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} + 0\begin{vmatrix} 5 & -2 \\ 2 & 1 \end{vmatrix}$

$= 0 - 4(20-6) + 0 = -56$

c $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$

$= 1(8-5) - 0 + 1(10-12)$

$= 3 - 2 = 1$

d $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} + 4\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix}$

$= 2(10-10) + 3(10-10) + 4(10-10) = 0$

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Further matrix algebra
Exercise B, Question 2

Question:

Find the values of the determinants.

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$

b $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$

c $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

Solution:

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix} = 4\begin{vmatrix} -2 & 0 \\ 4 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix}$

$= 4(4-0) - 3(-4-0) - 1(8-0)$

$= 16 + 12 - 8 = 20$

b $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix} = 3\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -3 \\ 7 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix}$

$= 3(-4+6) + 2(-16+21) + 1(8-7)$

$= 6 + 10 + 1 = 17$

c $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix} = 5\begin{vmatrix} 4 & 2 \\ -4 & -3 \end{vmatrix} - (-2)\begin{vmatrix} 6 & 2 \\ -2 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 6 & 4 \\ -2 & -4 \end{vmatrix}$

$= 5(-12+8) + 2(-18+4) - 3(-24+8)$

$= 5 \times (-4) + 2 \times (-14) - 3 \times (-16)$

$= -20 - 28 + 48 = 0$

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Further matrix algebra
Exercise B, Question 3

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{pmatrix}$.

Given that $\mathbf{A}$ is singular, find the value of the constant k.

Solution:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & k \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2k+1 & k \\ 1 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2k+1 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 2(3-0) - 1(2k+1-k) - 4(0-3)$$

$$= 6 - k - 1 + 12 = 17 - k$$

As $\mathbf{A}$ is singular,

$$\det(\mathbf{A}) = 0$$

$$17 - k = 0$$

$$k = 17$$

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Further matrix algebra
Exercise B, Question 4

Question:

The matrix $\mathbf{A}=\begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant.

Given that the determinant of $\mathbf{A}$ is 8, find the possible values of k.

Solution:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 1 & k+3 \end{vmatrix} - (-1)\begin{vmatrix} k & 4 \\ -2 & k+3 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 2(2k+6-4)+1(k^2+3k+8)+3(k+4)$$

$$= 4k+4+k^2+3k+8+3k+12$$

$$= k^2+10k+24$$

As $\det(\mathbf{A})=8$

$$k^2+10k+24 = 8$$

$$k^2+10k+16 = 0$$

$$(k+8)(k+2) = 0$$

$$k = -8, -2$$

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Further matrix algebra
Exercise B, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 5 & 3\\ -2 & 0 & 4\\ 3 & 10 & 8\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 2\\ 0 & -2 & -1\end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.
b Find $\mathbf{AB}$.
c Show that $\mathbf{AB}$ is also singular.

Solution:

a $\det(\mathbf{A}) = \begin{vmatrix}2 & 5 & 3\\ -2 & 0 & 4\\ 3 & 10 & 8\end{vmatrix}$

$= 2\begin{vmatrix}0 & 4\\ 10 & 8\end{vmatrix} - 5\begin{vmatrix}-2 & 4\\ 3 & 8\end{vmatrix} + 3\begin{vmatrix}-2 & 0\\ 3 & 10\end{vmatrix}$

$= 2(0-40)-5(-16-12)+3(-20-0)$

$= -80+140-60=0$

Hence $\mathbf{A}$ is singular.

b $\mathbf{AB} = \begin{pmatrix}2 & 5 & 3\\ -2 & 0 & 4\\ 3 & 10 & 8\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 2\\ 0 & -2 & -1\end{pmatrix}$

$= \begin{pmatrix}2+5+0 & 2+10-6 & 0+10-3\\ -2+0+0 & -2+0-8 & 0+0-4\\ 3+10+0 & 3+20-16 & 0+20-8\end{pmatrix}$

$= \begin{pmatrix}7 & 6 & 7\\ -2 & -10 & -4\\ 13 & 7 & 12\end{pmatrix}$

c $\det(\mathbf{AB}) = \begin{vmatrix}7 & 6 & 7\\ -2 & -10 & -4\\ 13 & 7 & 12\end{vmatrix}$

$= 7\begin{vmatrix}-10 & -4\\ 7 & 12\end{vmatrix} - 6\begin{vmatrix}-2 & -4\\ 13 & 12\end{vmatrix} + 7\begin{vmatrix}-2 & -10\\ 13 & 7\end{vmatrix}$

$= 7(-120+28)-6(-24+52)+7(-14+130)$

$= 7\times(-92)-6\times 28+7\times 116$

$= -644-168+812=0$

Hence $\mathbf{AB}$ is also singular.

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Further matrix algebra
Exercise B, Question 6

Question:

The matrix $A = \begin{pmatrix} 4 & 5 & -2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{pmatrix}$.

a Find det (A).

b Write down A^T.

c Verify that $\det(A^T) = \det(A)$.

Solution:

a $\det(A) = \begin{vmatrix} 4 & 5 & -2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{vmatrix}$

$= 4\begin{vmatrix} -3 & 2 \\ -4 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & -3 \\ 2 & -4 \end{vmatrix}$

$= 4(-9+8) - 5(6-4) - 2(-8+6)$

$= -4 - 10 + 4 = -10$

b $A^T = \begin{pmatrix} 4 & 2 & 2 \\ 5 & -3 & -4 \\ -2 & 2 & 3 \end{pmatrix}$

c $\det(A^T) = \begin{vmatrix} 4 & 2 & 2 \\ 5 & -3 & -4 \\ -2 & 2 & 3 \end{vmatrix}$

$= 4\begin{vmatrix} -3 & -4 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & -4 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & -3 \\ -2 & 2 \end{vmatrix}$

$= 4(-9+8) - 2(15-8) + 2(10-6)$

$= -4 - 14 + 8 = -10$

$= \det(A)$, as required.

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Further matrix algebra
Exercise B, Question 7

Question:

a Show that, for all values of a, b and c, the matrix $\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$ is singular.

b Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non-singular.

Solution:

a $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} - a\begin{vmatrix} -a & c \\ b & 0 \end{vmatrix} + (-b)\begin{vmatrix} -a & 0 \\ b & -c \end{vmatrix}$

$= 0 - a(0 - cb) - b(ac - 0)$

$= abc - abc = 0$

Hence the matrix is singular for all a, b and c.

b $\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix} - (-2)\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$

$= 2(x^2 + 6) + 2(3x - 2) + 4(9 + x)$

$= 2x^2 + 12 + 6x - 4 + 36 + 4x$

$= 2x^2 + 10x + 44$

$= 2(x^2 + 5x) + 44$

$= 2\left(x^2 + 5x + \left(\frac{5}{2}\right)^2\right) + 44 - 2\times\left(\frac{5}{2}\right)^2$

$= 2\left(x + \frac{5}{2}\right)^2 + 31\frac{1}{2} \geq 31\frac{1}{2}$, for all real x.

Hence the determinant cannot be zero and the matrix is non-singular for all real x.

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Further matrix algebra
Exercise B, Question 8

Question:

Find all the values of x for which the matrix $\begin{pmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{pmatrix}$ is singular.

Solution:

$$\begin{vmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{vmatrix} = (x-3)\begin{vmatrix} x & -2 \\ -1 & x+1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -2 \\ -2 & x+1 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ -2 & -1 \end{vmatrix}$$

$$= (x-3)(x^2+x-2)+2(x+1-4)+0$$

$$= x^3+x^2-2x-3x^2-3x+6+2x-6$$

$$= x^3-2x^2-3x$$

For the matrix to be singular, the determinant must be zero.

$$x^3-2x^2-3x = x(x^2-2x-3) = x(x-3)(x+1) = 0$$

$$x = -1, 0, 3$$

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Further matrix algebra
Exercise C, Question 1

Question:

Find the inverses of these matrices.

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

Solution:

a Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1(4-1) - 0 + 0 = 3$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

b By inspection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

c Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} - 0\begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} + 0\begin{vmatrix} 0 & \frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix}$$

$$= 1\left(\frac{9}{25} + \frac{6}{25}\right) - 0 + 0 = 1$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & \frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{3}{5} & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 2

Question:

Find the inverses of these matrices.

a $\begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$

b $\begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

c $\begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

Solution:

a Let $\mathbf{A} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix}$$
$$= 1(-4-0)+3(0-3)+2(0+6)$$
$$= -4-9+12 = -1$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & -2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} -4 & -3 & 6 \\ -6 & -4 & 9 \\ 1 & 1 & -2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -4 & 3 & 6 \\ 6 & -4 & -9 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -6 & -1 \\ -3 & 4 & 1 \\ -6 & 9 & 2 \end{pmatrix}$$

b Let $\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

$$\det(\mathbf{A}) = 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= 2(-2-1) - 3(3-2) + 2(3+4)$$

$$= -6 - 3 + 14 = 5$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & 7 \\ 1 & -2 & -4 \\ 7 & -4 & -13 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{5}\begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{7}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{4}{5} & -\frac{13}{5} \end{pmatrix}$$

c Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

$$\det(\mathbf{A}) = 3\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + (-7)\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}$$

$$= 3(6-2) - 2(-2-0) - 7(2-0)$$

$$= 12 + 4 - 14 = 2$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 2 \\ -10 & -6 & -6 \\ -19 & -10 & -11 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix} = \begin{pmatrix} 2 & -5 & -\frac{19}{2} \\ 1 & -3 & -5 \\ 1 & -3 & -\frac{11}{2} \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 3

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^{-1}$.

b Find $\mathbf{B}^{-1}$.

Given that $(\mathbf{AB})^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$,

c verify that $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

Solution:

a $\det(\mathbf{A}) = 1\begin{vmatrix}1&0\\0&1\end{vmatrix} - 0\begin{vmatrix}0&0\\2&1\end{vmatrix} + 1\begin{vmatrix}0&1\\2&0\end{vmatrix}$

$= 1 - 0 - 2 = -1$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}1&0\\0&1\end{vmatrix} & \begin{vmatrix}0&0\\2&1\end{vmatrix} & \begin{vmatrix}0&1\\2&0\end{vmatrix} \\ \begin{vmatrix}0&1\\0&1\end{vmatrix} & \begin{vmatrix}1&1\\2&1\end{vmatrix} & \begin{vmatrix}1&0\\2&0\end{vmatrix} \\ \begin{vmatrix}0&1\\1&0\end{vmatrix} & \begin{vmatrix}1&1\\0&0\end{vmatrix} & \begin{vmatrix}1&0\\0&1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

b $\det(\mathbf{B}) = 2\begin{vmatrix}0&1\\2&1\end{vmatrix} - 1\begin{vmatrix}1&1\\1&1\end{vmatrix} + (-1)\begin{vmatrix}1&0\\1&2\end{vmatrix}$

$= -4 - 0 - 2 = -6$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}0&1\\2&1\end{vmatrix} & \begin{vmatrix}1&1\\1&1\end{vmatrix} & \begin{vmatrix}1&0\\1&2\end{vmatrix} \\ \begin{vmatrix}1&-1\\2&1\end{vmatrix} & \begin{vmatrix}2&-1\\1&1\end{vmatrix} & \begin{vmatrix}2&1\\1&2\end{vmatrix} \\ \begin{vmatrix}1&-1\\0&1\end{vmatrix} & \begin{vmatrix}2&-1\\1&1\end{vmatrix} & \begin{vmatrix}2&1\\1&0\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 2 \\ 3 & 3 & 3 \\ 1 & 3 & -1 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -2 & 0 & 2 \\ -3 & 3 & -3 \\ 1 & -3 & -1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-6}\begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

c $$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}+0-\frac{1}{3} & 0+\frac{1}{2}+0 & \frac{1}{3}+0+\frac{1}{6} \\ 0+0+1 & 0-\frac{1}{2}+0 & 0+0-\frac{1}{2} \\ \frac{1}{3}+0+\frac{1}{3} & 0+\frac{1}{2}+0 & -\frac{1}{3}+0-\frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = (\mathbf{AB})^{-1}, \text{ as required.}$$

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Further matrix algebra
Exercise C, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

a Show that $\det(\mathbf{A}) = 3(k+1)$

b Given that $k \neq -1$, find $\mathbf{A}^{-1}$.

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Further matrix algebra
Exercise C, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}$

Given that $\mathbf{A}=\mathbf{A}^{-1}$, find the values of the constants a, b and c.

Solution:

$\mathbf{A}=\mathbf{A}^{-1}$

Multiplying throughout by $\mathbf{A}$

$\mathbf{AA}=\mathbf{AA}^{-1}$

$\mathbf{A}^2=\mathbf{I}$

$$\mathbf{A}^2=\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}$$

$$=\begin{pmatrix}ab+33 & -2a-8 & 8a+4c+20\\ 16-2b & ab+33 & 4b+8c-56\\ -2b+2c+10 & 2a-2c+14 & c^2-8\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Equating the second elements in the first row

$-2a-8=0\Rightarrow a=-4$

Equating the first elements in the second row

$16-2b=0\Rightarrow b=8$

Equating the first elements in the third row and using $b=8$

$-2b+2c+10=0\Rightarrow -16+2c+10=0$

$2c=6\Rightarrow c=3$

$a=-4, b=8, c=3$

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Further matrix algebra
Exercise C, Question 6

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = \mathbf{I}$.

b Hence find $\mathbf{A}^{-1}$.

Solution:

a $\mathbf{A}^2 = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 4-4-3 & -2+3+3 & 2+0+1 \\ 8-12+0 & -4+9+0 & 4+0+0 \\ -6+12-3 & 3-9+3 & -3+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$$

$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -6+16-9 & 3-12+9 & -3+0+3 \\ -8+20-12 & 4-15+12 & -4+0+4 \\ 6-12+6 & -3+9-6 & 3+0-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

b $\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \mathbf{I}$

Comparing with the definition of an inverse

$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

$\mathbf{A}^{-1} = \mathbf{A}^2 = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$

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Further matrix algebra
Exercise C, Question 7

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$.

b Deduce that $15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$.

c Hence find $\mathbf{A}^{-1}$.

Solution:

a $\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1+3+0 & 1-3+0 & 0+1+0 \\ 3-9+0 & 3+9+3 & 0-3+2 \\ 0+9+0 & 0-9+6 & 0+3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4-6+0 & 4+6+3 & 0-2+2 \\ -6+45+0 & -6-45-3 & 0+15-2 \\ 9-9+0 & 9+9+21 & 0-3+14 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 13 & 0 \\ 39 & -54 & 13 \\ 0 & 39 & 11 \end{pmatrix}$$

$$13\mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 13 & 13 & 0 \\ 39 & -39 & 13 \\ 0 & 39 & 26 \end{pmatrix} - \begin{pmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 13 & 0 \\ 39 & -54 & 13 \\ 0 & 39 & 11 \end{pmatrix} = \mathbf{A}^3$$

Hence

$\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$, as required.

b Multiply the result of part **a** throughout by $\mathbf{A}^{-1}$

$$\mathbf{A}^3\mathbf{A}^{-1} = 13\mathbf{A}\mathbf{A}^{-1} - 15\mathbf{I}\mathbf{A}^{-1}$$
$$\mathbf{A}^2 = 13\mathbf{I} - 15\mathbf{A}^{-1}$$

Rearranging

$15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$, as required.

c Using the result of part **b**

$$15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2 = \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$$

Hence

$$\mathbf{A}^{-1} = \frac{1}{15}\begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 8

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 0 & 1\\ 4 & 3 & -2\\ 0 & 3 & -4\end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

The matrix $\mathbf{C}$ is the matrix of the cofactors of $\mathbf{A}$.

b Find $\mathbf{C}$.

c Show that $\mathbf{AC}^{\mathrm{T}}=\mathbf{0}$.

Solution:

a $\det(\mathbf{A}) = 2\begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} - 0\begin{vmatrix} 4 & -2 \\ 0 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix}$

$= 2(-12+6) - 0 + 1(12-0)$

$= -12 + 12 = 0$

Hence $\mathbf{A}$ is singular.

b The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -16 & 12 \\ -3 & -8 & 6 \\ -3 & -8 & 6 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -6 & 16 & 12 \\ 3 & -8 & -6 \\ -3 & 8 & 6 \end{pmatrix}$$

c $\mathbf{AC}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}\begin{pmatrix} -6 & 3 & -3 \\ 16 & -8 & 8 \\ 12 & -6 & 6 \end{pmatrix}$

$$= \begin{pmatrix} -12+0+12 & 6+0-6 & -6+0+6 \\ -24+48-24 & 12-24+12 & -12+24-12 \\ 0+48-48 & 0-24+24 & 0+24-24 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \text{ as required.}$$

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Further matrix algebra
Exercise D, Question 1

Question:

Given that $T:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x-y\\y+z\\2x-3z\end{pmatrix}$ and $U:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}2x-3y-z\\2y+3z\\5z\end{pmatrix}$, find matrices representing

a T
b U
c TU.

Solution:

a $\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1-0\\0+0\\2-0\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0-1\\1+0\\0-0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0-0\\0+1\\0-3\end{pmatrix} = \begin{pmatrix}0\\1\\-3\end{pmatrix}$

The matrix representing T is $\begin{pmatrix}1 & -1 & 0\\0 & 1 & 1\\2 & 0 & -3\end{pmatrix}$

b $\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}2-0-0\\0+0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0-3-0\\2+0\\0\end{pmatrix} = \begin{pmatrix}-3\\2\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0-0-1\\0+3\\5\end{pmatrix} = \begin{pmatrix}-1\\3\\5\end{pmatrix}$

The matrix representing U is $\begin{pmatrix}2 & -3 & -1\\0 & 2 & 3\\0 & 0 & 5\end{pmatrix}$

c The matrix representing TU is given by

$$\begin{pmatrix}1 & -1 & 0\\0 & 1 & 1\\2 & 0 & -3\end{pmatrix}\begin{pmatrix}2 & -3 & -1\\0 & 2 & 3\\0 & 0 & 5\end{pmatrix} = \begin{pmatrix}2+0+0 & -3-2+0 & -1-3+0\\0+0+0 & 0+2+0 & 0+3+5\\4+0+0 & -6+0+0 & -2+0-15\end{pmatrix} = \begin{pmatrix}2 & -5 & -4\\0 & 2 & 8\\4 & -6 & -17\end{pmatrix}$$

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Further matrix algebra
Exercise D, Question 2

Question:

The point with position vector $\begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix}$ is transformed by the linear transformation represented by the matrix $\begin{pmatrix} 4 & -1 & 0 \\ -2 & 2 & 3 \\ 5 & -2 & 1 \end{pmatrix}$ to the point with position vector $\begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$.

Find the values of the constants a, b and c.

Solution:

$$\begin{pmatrix} 4 & -1 & 0 \\ -2 & 2 & 3 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix} = \begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 4+3a \\ -1+a \end{pmatrix} = \begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$$

Equating the top elements

$b = 1$

Equating the middle elements

$4 + 3a = -5 \Rightarrow a = -3$

Equating the lowest elements and using $a = -3$

$-1 + a = -1 - 3 = c \Rightarrow c = -4$

$a = -3, b = 1, c = -4$

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Further matrix algebra
Exercise D, Question 3

Question:

The transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$.

The vector $\begin{pmatrix}2\\0\\0\end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix}6\\2\\4\end{pmatrix}$.

The vector $\begin{pmatrix}3\\0\\-1\end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix}-2\\3\\5\end{pmatrix}$.

The vector $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix}2\\-1\\-2\end{pmatrix}$.

Find $\mathbf{T}$.

Solution:

Let $\mathbf{T} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ 2d \\ 2g \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

Equating the elements

$2a = 6 \Rightarrow a = 3$

$2d = 2 \Rightarrow d = 1$

$2g = 4 \Rightarrow g = 2$

$$\begin{pmatrix} 3 & b & c \\ 1 & e & f \\ 2 & h & i \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 9-c \\ 3-f \\ 6-i \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$$

Equating the elements

$9 - c = -2 \Rightarrow c = 11$

$3 - f = 3 \Rightarrow f = 0$

$6 - i = 5 \Rightarrow i = 1$

$$\begin{pmatrix} 3 & b & 11 \\ 1 & e & 0 \\ 2 & h & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} b-11 \\ e \\ h-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

Equating the elements

$b - 11 = 2 \Rightarrow b = 13$

$e = -1$

$h - 1 = -2 \Rightarrow h = -1$

$$\mathbf{T} = \begin{pmatrix} 3 & 13 & 11 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

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Further matrix algebra
Exercise D, Question 4

Question:

The transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 5 & -4 \\ 3 & 2 & 1 \end{pmatrix}.$$

The line l_1 is transformed by T to the line l_2. The line l_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}, \text{ where } t \text{ is a real parameter.}$$

Find a vector equation of l_2.

Solution:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-t \\ 4-2t \\ 1+3t \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 2 \\ 2 & 5 & -4 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2-t \\ 4-2t \\ 1+3t \end{pmatrix} = \begin{pmatrix} 0(2-t)-1(4-2t)+2(1+3t) \\ 2(2-t)+5(4-2t)-4(1+3t) \\ 3(2-t)+2(4-2t)+1(1+3t) \end{pmatrix}$$

$$= \begin{pmatrix} -2+8t \\ 20-24t \\ 15-4t \end{pmatrix} = \begin{pmatrix} -2 \\ 20 \\ 15 \end{pmatrix} + t\begin{pmatrix} 8 \\ -24 \\ -4 \end{pmatrix}$$

An equation of l_2 is $\mathbf{r} = \begin{pmatrix} -2 \\ 20 \\ 15 \end{pmatrix} + t\begin{pmatrix} 8 \\ -24 \\ -4 \end{pmatrix}$

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Further matrix algebra
Exercise D, Question 5

Question:

The points A and B have position vectors $\begin{pmatrix}2\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\3\\4\end{pmatrix}$ respectively. The points A and B are transformed by the linear transformation T to the points A' and B' respectively.

The transformation T is represented by the matrix $\mathbf{T}$, where $\mathbf{T}=\begin{pmatrix}1&-3&4\\2&3&-2\\0&2&5\end{pmatrix}$.

a Find the position vectors of A' and B'.
b Hence find a vector equation of the line $A'B'$.

Solution:

a $\begin{pmatrix}1&-3&4\\2&3&-2\\0&2&5\end{pmatrix}\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}2-3+0\\4+3+0\\0+2+0\end{pmatrix}=\begin{pmatrix}-1\\7\\2\end{pmatrix}$

$\begin{pmatrix}1&-3&4\\2&3&-2\\0&2&5\end{pmatrix}\begin{pmatrix}-2\\3\\4\end{pmatrix}=\begin{pmatrix}-2-9+16\\-4+9-8\\0+6+20\end{pmatrix}=\begin{pmatrix}5\\-3\\26\end{pmatrix}$

The position vector of A' is $\begin{pmatrix}-1\\7\\2\end{pmatrix}$ and the position vector of B' is $\begin{pmatrix}5\\-3\\26\end{pmatrix}$.

b $\mathbf{r}=\mathbf{a}'+t(\mathbf{b}'-\mathbf{a}')$

$=\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}5-(-1)\\-3-7\\26-2\end{pmatrix}=\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}6\\-10\\24\end{pmatrix}$

A vector equation of $A'B'$ is $\mathbf{r}=\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}6\\-10\\24\end{pmatrix}$.

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Further matrix algebra
Exercise D, Question 6

Question:

The transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}.$$

The plane Π_1 is transformed by T to the plane Π_2. The plane Π_1 has Cartesian equation $x-2y+z=0$.

Find a Cartesian equation of Π_2.

Solution:

Let $y=s$ and $z=t$, then $x=2y-z=2s-t$

A parametric form of the general point on Π_1 is $\begin{pmatrix} 2s-t \\ s \\ t \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 6s-3t-2s-2t \\ -4s+2t-8s+4t \\ -4s+2t+4s \end{pmatrix} = \begin{pmatrix} 4s-5t \\ -12s+6t \\ 2t \end{pmatrix}$$

Parametric equations of Π_2 are

$x=4s-5t$ ①

$y=-12s+6t$ ②

$z=2t$ ③

From ③ $t=\frac{z}{2}$

Substituting for t in ① and ②

$x=4s-\frac{5z}{2}$ ④

$y=-12s+3z$ ⑤

3x ④ + ⑤ $3x+y=-\frac{9z}{2} \Rightarrow 6x+2y+9z=0$

A Cartesian equation of Π_2 is $6x+2y+9z=0$.

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Further matrix algebra
Exercise D, Question 7

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 4 & 5 & -3 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The plane Π_1 is transformed by T to the plane Π_2. The plane Π_1 has vector equation

$$r = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \text{ where } s \text{ and } t \text{ are real parameters.}$$

Find an equation of Π_2 in the form $\mathbf{r} \cdot \mathbf{n} = p$.

Solution:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} s+3t \\ 1-s \\ 1+2s+4t \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 & -3 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} s+3t \\ 1-s \\ 1+2s+4t \end{pmatrix} = \begin{pmatrix} 4s+12t+5-5s-3-6s-12t \\ -s-3t+2-2s+1+2s+4t \\ s+3t+1+2s+4t \end{pmatrix}$$

$$= \begin{pmatrix} 2-7s \\ 3-s+t \\ 1+3s+7t \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}$$

A vector equation of Π_2 is $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}$ ⁎

To find a vector perpendicular to both $\begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & -1 & 3 \\ 0 & 1 & 7 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 3 \\ 1 & 7 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -7 & 3 \\ 0 & 7 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -7 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= -10\mathbf{i} + 49\mathbf{j} - 7\mathbf{k} = \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix}$$

Taking the scalar product of equation ⁎ throughout with $\begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix}$ and using the property that the scalar product of perpendicular vectors is 0

$$\mathbf{r} \cdot \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix} = -20 + 147 - 7 = 120$$

A vector equation of Π_2 is $\mathbf{r} \cdot \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix} = 120$.

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Further matrix algebra
Exercise D, Question 8

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T}=\begin{pmatrix}4 & 1 & -2\\ -2 & 3 & 4\\ -1 & 0 & 2\end{pmatrix}.$$

There is a line through the origin for which every point on the line is mapped onto itself under T.
Find a vector equation of this line.

Solution:

Let a point which is unchanged by T have coordinates $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$.

$$\begin{pmatrix}4 & 1 & -2\\ -2 & 3 & 4\\ -1 & 0 & 2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

$$\begin{pmatrix}4a+b-2c\\ -2a+3b+4c\\ -a+2c\end{pmatrix}=\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

Equating the lowest elements

$-a+2c=c \Rightarrow c=a$

Equating the top elements and substituting $c=a$

$4a+b-2a=a \Rightarrow b=-a$

(Equating the middle elements also gives $b=-a$)

The general form of a point which is unchanged is $\begin{pmatrix}a\\ -a\\ a\end{pmatrix}=a\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$

A vector equation of the line is $\mathbf{r}=t\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 1

Question:

A transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T}^{-1} = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}.$$

The point with position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the point with position vector $\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$.

a Find the values of the constants a, b and c.

A line l_1 which passes through the origin is transformed by T to the line l_2.

A vector equation of l_2 is $\mathbf{r} = t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b Find a vector equation of l_1.

Solution:

a $$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} -24-21+24 \\ 12-28+40 \\ -24-7+8 \end{pmatrix} = \begin{pmatrix} -21 \\ 24 \\ -23 \end{pmatrix}$$

$a = -21, b = 24, c = -23$

b $$\begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4-6+3 \\ -2-8+5 \\ 4-2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$$

A vector equation of l_1 is $\mathbf{r} = t\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 2

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 2 \\ -3 & 2 & 8 \end{pmatrix}.$$

a Find $\mathbf{T}^{-1}$.

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix}$.

b Find the values of the constants a, b and c.

Solution:

a $\det(\mathbf{T}) = 2\begin{vmatrix}1 & 2\\2 & 8\end{vmatrix} - 0\begin{vmatrix}0 & 2\\-3 & 8\end{vmatrix} + (-3)\begin{vmatrix}0 & 1\\-3 & 2\end{vmatrix}$

$= 2(8-4) - 0 - 3(0+3)$

$= 8 - 9 = -1$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}1 & 2\\2 & 8\end{vmatrix} & \begin{vmatrix}0 & 2\\-3 & 8\end{vmatrix} & \begin{vmatrix}0 & 1\\-3 & 2\end{vmatrix} \\ \begin{vmatrix}0 & -3\\2 & 8\end{vmatrix} & \begin{vmatrix}2 & -3\\-3 & 8\end{vmatrix} & \begin{vmatrix}2 & 0\\-3 & 2\end{vmatrix} \\ \begin{vmatrix}0 & -3\\1 & 2\end{vmatrix} & \begin{vmatrix}2 & -3\\0 & 2\end{vmatrix} & \begin{vmatrix}2 & 0\\0 & 1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 & 3 \\ 6 & 7 & 4 \\ 3 & 4 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 4 & -6 & 3 \\ -6 & 7 & -4 \\ 3 & -4 & 2 \end{pmatrix}$$

As $\mathbf{C}$ is symmetric $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 4 & -6 & 3 \\ -6 & 7 & -4 \\ 3 & -4 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 6 & -3 \\ 6 & -7 & 4 \\ -3 & 4 & -2 \end{pmatrix}$$

b $$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix} = \begin{pmatrix} -4 & 6 & -3 \\ 6 & -7 & 4 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix} = \begin{pmatrix} 20+30-48 \\ -30-35+64 \\ 15+20-32 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$a = 2, b = -1, c = 3$

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Further matrix algebra
Exercise E, Question 3

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -3 & 0 & -4 \end{pmatrix}.$$

a Find $\mathbf{T}^{-1}$.

The line l_1 is transformed by T to the line l_2. The line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$, where t is a real parameter.

b Find a vector equation of l_1.

Solution:

a $\det(\mathbf{T}) = 1\begin{vmatrix} 2 & 2 \\ 0 & -4 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ -3 & -4 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ -3 & 0 \end{vmatrix}$

$= 1(-8-0) - 1(0+6) + 2(0+6)$

$= -8-6+12 = -2$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 2 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -3 & -4 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -3 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 6 & 6 \\ -4 & 2 & 3 \\ -2 & 2 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -8 & -6 & 6 \\ 4 & 2 & -3 \\ -2 & -2 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -8 & 4 & -2 \\ -6 & 2 & -2 \\ 6 & -3 & 2 \end{pmatrix}$$

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-2}\begin{pmatrix} -8 & 4 & -2 \\ -6 & 2 & -2 \\ 6 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 3 & -1 & 1 \\ -3 & \frac{3}{2} & -1 \end{pmatrix}$$

b A general point on l_2 has coordinates $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix}$

$$\mathbf{T}^{-1}\begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 3 & -1 & 1 \\ -3 & \frac{3}{2} & -1 \end{pmatrix}\begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix}$$

$$= \begin{pmatrix} 8-4t-8+1+t \\ 6-3t-4+1+t \\ -6+3t+6-1-t \end{pmatrix} = \begin{pmatrix} 1-3t \\ 3-2t \\ -1+2t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$$

A vector equation of l_1 is $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 4

Question:

The matrix $\mathbf{T} = \begin{pmatrix} a & 1 & 2 \\ 4 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, where a is a constant.

a Find $\mathbf{T}^{-1}$, in terms of a.

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$. The point with position vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is transformed by T to the point with position vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

b Find p, q and r.

Solution:

a $\det(\mathbf{T}) = a\begin{vmatrix}0 & 0\\0 & -1\end{vmatrix} - 1\begin{vmatrix}4 & 0\\0 & -1\end{vmatrix} + 2\begin{vmatrix}4 & 0\\0 & 0\end{vmatrix}$

$= 0+4+0 = 4$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}0 & 0\\0 & -1\end{vmatrix} & \begin{vmatrix}4 & 0\\0 & -1\end{vmatrix} & \begin{vmatrix}4 & 0\\0 & 0\end{vmatrix} \\ \begin{vmatrix}1 & 2\\0 & -1\end{vmatrix} & \begin{vmatrix}a & 2\\0 & -1\end{vmatrix} & \begin{vmatrix}a & 1\\0 & 0\end{vmatrix} \\ \begin{vmatrix}1 & 2\\0 & 0\end{vmatrix} & \begin{vmatrix}a & 2\\4 & 0\end{vmatrix} & \begin{vmatrix}a & 1\\4 & 0\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 & 0 \\ -1 & -a & 0 \\ 0 & -8 & -4 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 0 & 4 & 0 \\ 1 & -a & 0 \\ 0 & 8 & -4 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 0 \\ 4 & -a & 8 \\ 0 & 0 & -4 \end{pmatrix}$$

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}} = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 4 & -a & 8 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -\frac{a}{4} & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

b $$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -\frac{a}{4} & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 + \frac{3}{4} + 0 \\ 2 - \frac{3a}{4} - 2 \\ 0 + 0 + 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3a}{4} \\ 1 \end{pmatrix}$$

$p = \frac{3}{4}, q = -\frac{3a}{4}, r = 1$

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Further matrix algebra
Exercise E, Question 5

Question:

The matrix $\mathbf{S}=\begin{pmatrix}1 & -\sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}$.

a Show that $\mathbf{SS}^{\mathrm{T}}=k\mathbf{I}$, stating the value of k.

The transformation $S:\mathbb{R}^3\to\mathbb{R}^3$ is represented by the matrix $\mathbf{S}$.

The vector $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$ is transformed by S to the vector $\begin{pmatrix}2\sqrt{2}\\ \sqrt{2}\\ -2\sqrt{2}\end{pmatrix}$.

b Find the values of the constants a, b and c.

Solution:

a $\mathbf{SS}^{\mathrm{T}}=\begin{pmatrix}1 & -\sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}=\begin{pmatrix}1+2+1 & \sqrt{2}+0-\sqrt{2} & 1-2+1\\ \sqrt{2}+0-\sqrt{2} & 2+0+2 & \sqrt{2}+0-\sqrt{2}\\ 1-2+1 & \sqrt{2}+0-\sqrt{2} & 1+2+1\end{pmatrix}$

$=\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}=4\mathbf{I}$

$k=4$

b $\mathbf{SS}^{\mathrm{T}}=4\mathbf{I}\Rightarrow\mathbf{S}^{-1}=\frac{1}{4}\mathbf{S}^{\mathrm{T}}$

$\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\frac{1}{4}\mathbf{S}^{\mathrm{T}}\begin{pmatrix}2\sqrt{2}\\ \sqrt{2}\\ -2\sqrt{2}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}2\sqrt{2}\\ \sqrt{2}\\ -2\sqrt{2}\end{pmatrix}$

$=\frac{1}{4}\begin{pmatrix}2\sqrt{2}+2-2\sqrt{2}\\ -4+0-4\\ 2\sqrt{2}-2-2\sqrt{2}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}2\\ -8\\ -2\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ -2\\ -\frac{1}{2}\end{pmatrix}$

$a=\frac{1}{2}, b=-2, c=-\frac{1}{2}$

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Further matrix algebra
Exercise E, Question 6

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 3 & a & -3 \\ b & -1 & -2 \\ -18 & 11 & c \end{pmatrix}$. Given that $\mathbf{AB} = \mathbf{I}$,

a find the values of the constants a, b and c.

The transformation $A: \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{A}$.

The plane Π_1 is transformed by A to the plane Π_2. The plane Π_2 has vector equation

$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, where s and t are real parameters.

b Find a vector equation of the plane Π_1.

Solution:

a $\mathbf{AB} = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & -3 \\ b & -1 & -2 \\ -18 & 11 & c \end{pmatrix}$

$$= \begin{pmatrix} 9+5b-18 & 3a-5+11 & -9-10+c \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equating the elements in the first row

$9+5b-18 = 1 \Rightarrow 5b = 10 \Rightarrow b = 2$

$3a-5+11 = 0 \Rightarrow 3a = -6 \Rightarrow a = -2$

$-9-10+c = 0 \Rightarrow c = 19$

$a = -2, b = 2, c = 19$

b $\mathbf{AB} = \mathbf{I} \Rightarrow \mathbf{A}^{-1} = \mathbf{B} = \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}$

The general point on Π_2 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-s \\ 1+t \\ 2s+t \end{pmatrix}$

$$\mathbf{A}^{-1}\begin{pmatrix} 1-s \\ 1+t \\ 2s+t \end{pmatrix} = \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}\begin{pmatrix} 1-s \\ 1+t \\ 2s+t \end{pmatrix} = \begin{pmatrix} 3-3s-2-2t-6s-3t \\ 2-2s-1-t-4s-2t \\ -18+18s+11+11t+38s+19t \end{pmatrix}$$

$$= \begin{pmatrix} 1-9s-5t \\ 1-6s-3t \\ -7+56s+30t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} + s\begin{pmatrix} -9 \\ -6 \\ 56 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 30 \end{pmatrix}$$

A vector equation of Π_2 is $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} + s\begin{pmatrix} -9 \\ -6 \\ 56 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 30 \end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 7

Question:

The transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 4 & 2 \\ 2 & -5 & 1 \end{pmatrix}.$$

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix}$.

Find the values of the constants a, b and c.

Solution:

$$\begin{pmatrix} -1 & 3 & 6 \\ 1 & 4 & 2 \\ 2 & -5 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -a+3b+6c \\ a+4b+2c \\ 2a-5b+c \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix}$$

Equating the elements

$-a+3b+6c=-8$ ①

$a+4b+2c=0$ ②

$2a-5b+c=3$ ③

① + ②

$7b+8c=-8$ ④

$2\times$ ① + ③

$b+13c=-13$ ⑤

$7\times$ ⑤

$7b+91c=-91$ ⑥

⑥ − ④

$83c=-83 \Rightarrow c=-1$

Substituting $c=-1$ into ⑤

$b-13=-13 \Rightarrow b=0$

Substituting $b=0$ and $c=-1$ into ②

$a+0-2=0 \Rightarrow a=2$

$a=2, b=0, c=-1$

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Further matrix algebra
Exercise E, Question 8

Question:

The matrix $\mathbf{S} = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and the matrix $\mathbf{T} = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}$.

a Find $\mathbf{S}^{-1}$.

b Show that $\mathbf{T}^2 = \mathbf{I}$.

The transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{S}$ and the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$.

The transformation U is the transformation T followed by the transformation S.

The point $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by U to the point $\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$.

c Find the values of the constants a, b and c.

Solution:

a $\det(\mathbf{S}) = 2\begin{vmatrix}2 & 1\\0 & 1\end{vmatrix} - (-1)\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} + 2\begin{vmatrix}0 & 2\\1 & 0\end{vmatrix}$

$= 2(2-0)+1(0-1)+2(0-2)$

$= 4-1-4=-1$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}2 & 1\\0 & 1\end{vmatrix} & \begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} & \begin{vmatrix}0 & 2\\1 & 0\end{vmatrix} \\ \begin{vmatrix}-1 & 2\\0 & 1\end{vmatrix} & \begin{vmatrix}2 & 2\\1 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\1 & 0\end{vmatrix} \\ \begin{vmatrix}-1 & 2\\2 & 1\end{vmatrix} & \begin{vmatrix}2 & 2\\0 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\0 & 2\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -2 \\ -1 & 0 & 1 \\ -5 & 2 & 4 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ -5 & -2 & 4 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ -2 & -1 & 4 \end{pmatrix}$$

$$\mathbf{S}^{-1} = \frac{1}{\det(\mathbf{S})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ -2 & -1 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 5 \\ -1 & 0 & 2 \\ 2 & 1 & -4 \end{pmatrix}$$

b $$\mathbf{T}^2 = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 9-24+16 & 12-28+16 & 12-24+12 \\ -18+42-24 & -24+49-24 & -24+42-18 \\ 12-24+12 & 16-28+12 & 16-24+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

c From part **b**, $\mathbf{T}^2 = \mathbf{I} \Rightarrow \mathbf{T}^{-1} = \mathbf{T}$

$$\mathbf{ST} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$(\mathbf{ST})^{-1}\mathbf{ST}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\mathbf{ST})^{1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\mathbf{S}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = \mathbf{T}\mathbf{S}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 & 5 \\ -1 & 0 & 2 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix} \begin{pmatrix} -12 +3 +10 \\ -6 +0 +4 \\ 12 -3 -8 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 -8 +4 \\ -6 +14 -6 \\ 4 -8 +3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$a = -1, b = 2, c = -1$

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Further matrix algebra
Exercise F, Question 1

Question:

Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

c $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$.

Solution:

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&4\\1&5\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}=\begin{pmatrix}2-\lambda&4\\1&5-\lambda\end{pmatrix}$$

a $\begin{vmatrix}2-\lambda&4\\1&5-\lambda\end{vmatrix}=(2-\lambda)(5-\lambda)-4$

$$=10-7\lambda+\lambda^2-4=\lambda^2-7\lambda+6=(\lambda-1)(\lambda-6)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(\lambda-1)(\lambda-6)=0\Rightarrow\lambda=1,6$

The eigenvalues are 1 and 6.

For $\lambda=1$

$\begin{pmatrix}2&4\\1&5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=1\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}2x+4y\\x+5y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$

Equating the upper elements

$2x+4y=x\Rightarrow x=-4y$

Let $y=1$, then $x=-4$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix}-4\\1\end{pmatrix}$.

For $\lambda=6$

$\begin{pmatrix}2&4\\1&5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=6\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}2x+4y\\x+5y\end{pmatrix}=\begin{pmatrix}6x\\6y\end{pmatrix}$

Equating the upper elements

$2x+4y=6x\Rightarrow y=x$

Let $x=1$, then $y=1$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix}1\\1\end{pmatrix}$.

b $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4&-1\\-1&4\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}=\begin{pmatrix}4-\lambda&-1\\-1&4-\lambda\end{pmatrix}$

$\begin{vmatrix}4-\lambda&-1\\-1&4-\lambda\end{vmatrix}=(4-\lambda)^2-1$

$$=16-8\lambda+\lambda^2-1=\lambda^2-8\lambda+15=(\lambda-3)(\lambda-5)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(\lambda-3)(\lambda-5)=0\Rightarrow\lambda=3,5$

The eigenvalues are 3 and 5.

For $\lambda=3$

$\begin{pmatrix}4&-1\\-1&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=3\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}4x-y\\-x+4y\end{pmatrix}=\begin{pmatrix}3x\\3y\end{pmatrix}$

Equating the upper elements

$4x-y=3x\Rightarrow y=x$

Let $x=1$, then $y=1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\1\end{pmatrix}$.

For $\lambda=5$

$$\begin{pmatrix}4 & -1\\-1 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=5\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}4x-y\\-x+4y\end{pmatrix}=\begin{pmatrix}5x\\5y\end{pmatrix}$$

Equating the upper elements

$4x-y=5x\Rightarrow y=-x$

Let $x=1$, then $y=-1$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix}1\\-1\end{pmatrix}$.

c $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}=\begin{pmatrix}3-\lambda & -2\\0 & 4-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda & -2\\0 & 4-\lambda\end{vmatrix}=(3-\lambda)(4-\lambda)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(4-\lambda)=0\Rightarrow\lambda=3,4$

The eigenvalues are 3 and 4.

For $\lambda=3$

$$\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=3\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}3x-2y\\4y\end{pmatrix}=\begin{pmatrix}3x\\3y\end{pmatrix}$$

Equating the lower elements

$4y=3y\Rightarrow y=0$

As x can take any non-zero value, let $x=1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\0\end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=4\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}3x-2y\\4y\end{pmatrix}=\begin{pmatrix}4x\\4y\end{pmatrix}$$

Equating the upper elements

$3x-2y=4x\Rightarrow x=-2y$

Let $y=1$, then $x=-2$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}-2\\1\end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 2

Question:

A transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A}=\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}$

a Find the eigenvalues of $\mathbf{A}$.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}-\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}=\begin{pmatrix} 3-\lambda & 4 \\ -2 & 9-\lambda \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & 4 \\ -2 & 9-\lambda \end{vmatrix}=(3-\lambda)(9-\lambda)+8$$

$$=27-12\lambda+\lambda^2+8=\lambda^2-12\lambda+35=(\lambda-5)(\lambda-7)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0 \Rightarrow (\lambda-5)(\lambda-7)=0 \Rightarrow \lambda=5,7$

The eigenvalues of $\mathbf{A}$ are 5 and 7.

b For $\lambda=5$

$$\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+4y \\ -2x+9y \end{pmatrix}=\begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$$3x+4y=5x \Rightarrow 4y=2x \Rightarrow y=\frac{1}{2}x$$

For $\lambda=7$

$$\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=7\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+4y \\ -2x+9y \end{pmatrix}=\begin{pmatrix} 7x \\ 7y \end{pmatrix}$$

Equating the upper elements

$$3x+4y=7x \Rightarrow 4y=4x \Rightarrow y=x$$

Cartesian equations of the invariant lines are $y=\frac{1}{2}x$ and $y=x$.

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Further matrix algebra
Exercise F, Question 3

Question:

Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}3-\lambda&0&0\\2&4-\lambda&2\\-2&0&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda&0&0\\2&4-\lambda&2\\-2&0&1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}4-\lambda&2\\0&1-\lambda\end{vmatrix}-0\begin{vmatrix}2&2\\-2&1-\lambda\end{vmatrix}+0\begin{vmatrix}2&4-\lambda\\-2&0\end{vmatrix}$$

$$=(3-\lambda)(4-\lambda)(1-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(4-\lambda)(1-\lambda)=0\Rightarrow\lambda=3,4,1$$

The eigenvalues are 1, 3 and 4

For $\lambda=1$

$$\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3x\\2x+4y+2z\\-2x+z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating the top elements

$3x=x\Rightarrow x=0$

Equating the middle elements and substituting $x=0$

$0+4y+2z=y\Rightarrow 3y=-2z$

Let $z=3$, then $y=-2$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix}0\\-2\\3\end{pmatrix}$.

For $\lambda=3$

$$\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3x\\2x+4y+2z\\-2x+z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}$$

Equating the lowest elements

$-2x+z=3z\Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements and substituting $x=1$ and $z=-1$

$2+4y-2=3y\Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x+4y+2z \\ -2x+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the lowest elements

$-2x+z=3z \Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements and substituting $x=1$ and $z=-1$

$2+4y-2=3y \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x+4y+2z \\ -2x+z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$3x=4x \Rightarrow x=0$

Equating the lowest elements and substituting $x=0$

$0+z=4z \Rightarrow z=0$

As y can take any non-zero value, let $y=1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b $\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & -2 & -4 \\ 2 & 3-\lambda & 0 \\ 2 & -5 & -4-\lambda \end{pmatrix}$

$$\begin{vmatrix} 4-\lambda & -2 & -4 \\ 2 & 3-\lambda & 0 \\ 2 & -5 & -4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ -5 & -4-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 2 & 0 \\ 2 & -4-\lambda \end{vmatrix} + (-4)\begin{vmatrix} 2 & 3-\lambda \\ 2 & -5 \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda)(-4-\lambda)+2(-8-2\lambda)-4-(-10-6+2\lambda)$$

$$= (\lambda^2-16)(3-\lambda)-16-4\lambda+64-8\lambda$$

$$= 3\lambda^2-\lambda^3-48+16\lambda-12\lambda+48$$

$$= -\lambda^3+3\lambda^2+4\lambda = -\lambda(\lambda^2-3\lambda-4) = -\lambda(\lambda-4)(\lambda+1)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0 \Rightarrow -\lambda(\lambda-4)(\lambda+1)=0 \Rightarrow \lambda=0,4,-1$

The eigenvalues are $-1, 0$ and 4

For $\lambda=-1$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

Equating the middle elements

$2x+3y=-y \Rightarrow x=-2y$

Let $y=1$, then $x=-2$

Equating the top elements and substituting $y=1$ and $x=-2$

$-8-2-4z=2 \Rightarrow z=-3$

An eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$.

For $\lambda=0$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the middle elements

$2x+3y=0 \Rightarrow 3y=-2x$

Let $x=3$, then $y=-2$

Equating the top elements and substituting $x=3$ and $y=-2$

$12+4-4z=0 \Rightarrow z=4$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the middle elements

$2x + 3y = 4y \Rightarrow y = 2x$

Let $x = 1$, then $y = 2$

Equating the top elements and substituting $x = 1$ and $y = 2$

$4 - 4 - 4z = 4 \Rightarrow z = -1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ -3 & 2 & 0 \\ 1 & 4 & -3 \end{pmatrix}$.

a Show that -1 is the only real eigenvalue of $\mathbf{A}$.

b Find an eigenvector corresponding to the eigenvalue -1.

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Further matrix algebra
Exercise F, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & 4\\ 0 & 2 & 0\end{pmatrix}$.

a Show that 4 is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues of $\mathbf{A}$.

b Find an eigenvector corresponding to the eigenvalue 4.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & 4\\ 0 & 2 & 0\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}=\begin{pmatrix}2-\lambda & -1 & 3\\ 0 & 2-\lambda & 4\\ 0 & 2 & -\lambda\end{pmatrix}$

$$\begin{vmatrix}2-\lambda & -1 & 3\\ 0 & 2-\lambda & 4\\ 0 & 2 & -\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda & 4\\ 2 & -\lambda\end{vmatrix}-(-1)\begin{vmatrix}0 & 4\\ 0 & -\lambda\end{vmatrix}+3\begin{vmatrix}0 & 2-\lambda\\ 0 & 2\end{vmatrix}$$

$$=(2-\lambda)\left(-2\lambda+\lambda^2-8\right)+0+0$$

$$=(2-\lambda)\left(\lambda^2-2\lambda-8\right)=(2-\lambda)(\lambda-4)(\lambda+2)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(2-\lambda)(\lambda-4)(\lambda+2)=0\Rightarrow\lambda=2,4,-2$

The eigenvalues of $\mathbf{A}$ are 4, as required, 2 and -2.

b For $\lambda=4$

$$\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & 4\\ 0 & 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\mathbf{4}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x-y+3z\\ 2y+4z\\ 2y\end{pmatrix}=\begin{pmatrix}4x\\ 4y\\ 4z\end{pmatrix}$$

Equating the lowest elements

$2y=4z\Rightarrow y=2z$

Let $z=1$, then $y=2$

Equating the top elements and substituting $y=2$ and $z=1$

$2x-2+3=4x\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}\frac{1}{2}\\ 2\\ 1\end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 6

Question:

The matrix $\mathbf{A}=\begin{pmatrix}1 & 1 & 3\\ 2 & 4 & -1\\ 4 & 4 & 3\end{pmatrix}$.

Given that 3 is an eigenvalue of $\mathbf{A}$,

a find the other two eigenvalues of $\mathbf{A}$,

b find eigenvectors corresponding to each of the eigenvalues of $\mathbf{A}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{pmatrix}$

$$\begin{vmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}4-\lambda&-1\\4&3-\lambda\end{vmatrix}-1\begin{vmatrix}2&-1\\4&3-\lambda\end{vmatrix}+3\begin{vmatrix}2&4-\lambda\\4&4\end{vmatrix}$$

$$=(1-\lambda)((4-\lambda)(3-\lambda)+4)-(6-2\lambda+4)+3(8-16+4\lambda)$$

$$=(1-\lambda)(\lambda^2-7\lambda+16)+14\lambda-34$$

$$=-\lambda^3+8\lambda^2-23\lambda+16+14\lambda-34$$

$$=-\lambda^3+8\lambda^2-9\lambda-18$$

Let $\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2+k\lambda-6)$

Equating the coefficients of λ^2

$-8=-3+k\Rightarrow k=-5$

Hence $\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2-5\lambda-6)=(\lambda-3)(\lambda-6)(\lambda+1)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-3)(\lambda-6)(\lambda+1)=0\Rightarrow\lambda=3,6,-1$

The other eigenvalues of $\mathbf{A}$ are -1 and 6.

b For $\lambda=-1$

$$\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+y+3z\\2x+4y-z\\4x+4y+3z\end{pmatrix}=\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$$

Equating the top elements

$x+y+3z=-x$

$2x+y+3z=0$ ①

Equating the middle elements

$2x+4y-z=-y$

$2x+5y-z=0$ ②

② − ①

$4y-4z=0\Rightarrow y=z$

Let $z=1$, then $y=1$

Substituting $y=1$ and $z=1$ into ①

$2x+1+3=0\Rightarrow x=-2$

An eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix}-2\\1\\1\end{pmatrix}$.

For $\lambda = 3$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the lowest elements

$4x+4y+3z = 3z \Rightarrow y = -x$

Let $x = 1$, then $y = -1$

Equating the top elements and substituting $x = 1$ and $y = -1$

$1-1+3z = 3 \Rightarrow z = 1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda = 6$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$x+y+3z = 6x$

$-5x+y+3z = 0$ ①

Equating the lowest elements

$4x+4y+3z = 6z$

$4x+4y-3z = 0$ ②

① + ②

$-x+5y = 0 \Rightarrow x = 5y$

Let $y = 1$, then $x = 5$

Substituting $x = 5$ and $y = 1$ into ①

$-25+1+3z = 0 \Rightarrow z = 8$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}$.

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise F, Question 7

Question:

The matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ -2 & 4 & 0 \\ 4 & 2 & 5 \end{pmatrix}$.

a Show that 2 is an eigenvalue of A.

b Find the other two eigenvalues of A.

c Find a normalised eigenvector of A corresponding to the eigenvalue 2.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&2&1\\-2&4&0\\4&2&5\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}2-\lambda&2&1\\-2&4-\lambda&0\\4&2&5-\lambda\end{pmatrix}$

When $\lambda=2$

$$\mathbf{A}-2\mathbf{I}=\begin{pmatrix}0&2&1\\-2&2&0\\4&2&3\end{pmatrix}$$

$$\det(\mathbf{A}-2\mathbf{I})=\begin{vmatrix}0&2&1\\-2&2&0\\4&2&3\end{vmatrix}=0\begin{vmatrix}2&0\\2&3\end{vmatrix}-2\begin{vmatrix}-2&0\\4&3\end{vmatrix}+1\begin{vmatrix}-2&2\\4&2\end{vmatrix}$$

$$=0-2\times(-6)+1(-4-8)=12-12=0$$

Hence 2 is an eigenvector of $\mathbf{A}$.

b $\begin{vmatrix}2-\lambda&2&1\\-2&4-\lambda&0\\4&2&5-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}4-\lambda&0\\2&5-\lambda\end{vmatrix}-2\begin{vmatrix}-2&0\\4&5-\lambda\end{vmatrix}+1\begin{vmatrix}-2&4-\lambda\\4&2\end{vmatrix}$

$$=(2-\lambda)(4-\lambda)(5-\lambda)+20-4\lambda+(-4-16+4\lambda)$$
$$=(2-\lambda)(4-\lambda)(5-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)(4-\lambda)(5-\lambda)=0\Rightarrow\lambda=2,4,5$$

The other eigenvalues of $\mathbf{A}$ are 4 and 5.

c For $\lambda=2$

$$\begin{pmatrix}2&2&1\\-2&4&0\\4&2&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}2x+2y+z\\-2x+4y\\4x+2y+5z\end{pmatrix}=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

Equating the middle elements

$-2x+4y=2y\Rightarrow y=x$

Let $x=1$, then $y=1$

Equating the top elements and substituting $x=1$ and $y=1$

$2+2+z=2\Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}1\\1\\-2\end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2+1^2+(-2)^2\right)}=\sqrt{6}$

A normalised eigenvector corresponding to the eigenvalue 2 is

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise F, Question 8

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}$.

a Show that -2 is an eigenvalue of $\mathbf{A}$ and that there is only one other distinct eigenvalue.

b Find an eigenvector corresponding to each of the eigenvalues.

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Further matrix algebra
Exercise F, Question 9

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

Given that 2 is an eigenvalue of $\mathbf{A}$,

a find the other two eigenvalues of $\mathbf{A}$,

b find eigenvectors corresponding to each of the eigenvalues of $\mathbf{A}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda&-1&0\\-1&-\lambda&1\\1&2&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}1-\lambda&-1&0\\-1&-\lambda&1\\1&2&1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}-\lambda&1\\2&1-\lambda\end{vmatrix}-(-1)\begin{vmatrix}-1&1\\1&1-\lambda\end{vmatrix}+0\begin{vmatrix}-1&-\lambda\\1&2\end{vmatrix}$$

$$=(1-\lambda)(-\lambda+\lambda^2-2)+1(-1+\lambda-1)+0$$

$$=(1-\lambda)(\lambda-2)(\lambda+1)+1(\lambda-2)$$

$$=(\lambda-2)((1-\lambda)(1+\lambda)+1)=(\lambda-2)(2-\lambda^2)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(2-\lambda)(2-\lambda^2)=0\Rightarrow\lambda=2,\pm\sqrt{2}$

The other eigenvalues of $\mathbf{A}$ are $\pm\sqrt{2}$.

b For $\lambda=\sqrt{2}$

$$\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\sqrt{2}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x-y\\-x+z\\x+2y+z\end{pmatrix}=\begin{pmatrix}\sqrt{2}x\\\sqrt{2}y\\\sqrt{2}z\end{pmatrix}$$

Equating the top elements

$x-y=\sqrt{2}x\Rightarrow y=(1-\sqrt{2})x$

Let $x=1$, then $y=1-\sqrt{2}$

Equating the middle elements and substituting $x=1$ and $y=1-\sqrt{2}$

$-1+z=\sqrt{2}(1-\sqrt{2})=\sqrt{2}-2\Rightarrow z=\sqrt{2}-1$

An eigenvector corresponding to the eigenvalue $\sqrt{2}$ is $\begin{pmatrix}1\\1-\sqrt{2}\\\sqrt{2}-1\end{pmatrix}$.

For $\lambda=-\sqrt{2}$

$$\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-\sqrt{2}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x-y\\-x+z\\x+2y+z\end{pmatrix}=\begin{pmatrix}-\sqrt{2}x\\-\sqrt{2}y\\-\sqrt{2}z\end{pmatrix}$$

Equating the top elements

$x-y=-\sqrt{2}x\Rightarrow y=(\sqrt{2}+1)x$

Equating the middle elements and substituting $x=1$ and $y=1+\sqrt{2}$

$-1+z=-\sqrt{2}(1+\sqrt{2})=-\sqrt{2}-2 \Rightarrow z=-1-\sqrt{2}$

An eigenvector corresponding to the eigenvalue $-\sqrt{2}$ is $\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ -1-\sqrt{2} \end{pmatrix}$.

For $\lambda=2$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix}=\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements

$x-y=2x \Rightarrow y=-x$

Let $x=1$, then $y=-1$

Equating the middle elements and substituting $x=1$ and $y=-1$

$-1+z=-2 \Rightarrow z=-1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise F, Question 10

Question:

Given that $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}$,

a find the eigenvalue of $\mathbf{A}$ corresponding to $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$,

b find the value of a and the value of b,

c show that $\mathbf{A}$ has only one real eigenvalue.

Solution:

a $\begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 8+2-2 \\ 2+2a \\ -2+2-b \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 2\lambda \\ -1\lambda \end{pmatrix}$

Equating the top elements

$8 = 2\lambda \Rightarrow \lambda = 4$

The eigenvalue is 4.

b Equating the middle elements and substituting $\lambda = 4$

$2 + 2a = 8 \Rightarrow a = 3$

Equating the lowest elements and substituting $\lambda = 4$

$-b = -\lambda = -4 \Rightarrow b = 4$

$a = 3$ and $b = 4$

c $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 1 & 2 \\ 1 & 3-\lambda & 0 \\ -1 & 1 & 4-\lambda \end{pmatrix}$

$$\begin{vmatrix} 4-\lambda & 1 & 2 \\ 1 & 3-\lambda & 0 \\ -1 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 3-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (4-\lambda)^2(3-\lambda) - 1(4-\lambda) + 2(1+3-\lambda)$$

$$= (4-\lambda)^2(3-\lambda) + 1(4-\lambda) = (4-\lambda)\big((4-\lambda)(3-\lambda)+1\big)$$

$$= (4-\lambda)(\lambda^2 - 7\lambda + 13)$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (4-\lambda)(\lambda^2 - 7\lambda + 13) = 0 \Rightarrow \lambda = 4$ or $\lambda^2 - 7\lambda + 13 = 0$

The discriminant of $\lambda^2 - 7\lambda + 13 = 0$ is given by

$b^2 - 4ac = 49 - 52 = -3 < 0$

There are no real solutions of $\lambda^2 - 7\lambda + 13 = 0$

4 is the only real eigenvalue of $\mathbf{A}$.

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Further matrix algebra
Exercise G, Question 1

Question:

Reduce the following matrices to diagonal matrices.

a $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

Solution:

a Using $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\begin{vmatrix}1-\lambda & 3\\ 3 & 1-\lambda\end{vmatrix}=(1-\lambda)^2-9=1-2\lambda+\lambda^2-9$$

$$=\lambda^2-2\lambda-8=(\lambda-4)(\lambda+2)=0$$

$$\lambda=-2,4$$

For $\lambda=-2$

$$\begin{pmatrix}1\ 3\\ 3\ 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-2\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x+3y\\ 3x+y\end{pmatrix}=\begin{pmatrix}-2x\\ -2y\end{pmatrix}$$

Equating the upper elements

$x+3y=-2x\Rightarrow y=-x$

Let $x=1$, then $y=-1$

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix}1\\ -1\end{pmatrix}$.

The magnitude of $\begin{pmatrix}1\\ -1\end{pmatrix}$ is $\sqrt{\left(1^2+(-1)^2\right)}=\sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix}1\ 3\\ 3\ 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=4\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x+3y\\ 3x+y\end{pmatrix}=\begin{pmatrix}4x\\ 4y\end{pmatrix}$$

Equating the upper elements

$x+3y=4x\Rightarrow y=x$

Let $x=1$, then $y=1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}1\\ 1\end{pmatrix}$.

The magnitude of $\begin{pmatrix}1\\ 1\end{pmatrix}$ is $\sqrt{\left(1^2+1^2\right)}=\sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$.

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}, \qquad \mathbf{P}^{\mathrm{T}}\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}-\frac{3}{\sqrt2} & \frac{1}{\sqrt2}+\frac{3}{\sqrt2}\\ \frac{3}{\sqrt2}-\frac{1}{\sqrt2} & \frac{3}{\sqrt2}+\frac{1}{\sqrt2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}-\frac{2}{\sqrt2} & \frac{4}{\sqrt2}\\ \frac{2}{\sqrt2} & \frac{4}{\sqrt2}\end{pmatrix}$$

$$=\begin{pmatrix}-1-1 & 2-2\\ -1+1 & 2+2\end{pmatrix}=\begin{pmatrix}-2 & 0\\ 0 & 4\end{pmatrix}$$

b Using $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\begin{vmatrix}1-\lambda & -2\\ -2 & 4-\lambda\end{vmatrix}=(1-\lambda)(4-\lambda)-4=4-5\lambda+\lambda^2-4$$

$$=\lambda^2-5\lambda=\lambda(\lambda-5)=0$$

$$\lambda=0,5$$

For $\lambda=5$

$$\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x-2y\\ -2x+4y\end{pmatrix}=\begin{pmatrix}5x\\ 5y\end{pmatrix}$$

Equating the upper elements

$x-2y=5x\Rightarrow y=-2x$

Let $x=1$, then $y=-2$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix}1\\ -2\end{pmatrix}$.

The magnitude of $\begin{pmatrix}1\\ -2\end{pmatrix}$ is $\sqrt{\left(1^2+(-2)^2\right)}=\sqrt5$.

A normalised eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix}\frac{1}{\sqrt5}\\ -\frac{2}{\sqrt5}\end{pmatrix}$.

For $\lambda=0$

$$\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x-2y\\ -2x+4y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Equating the upper elements
$x - 2y = 0 \Rightarrow x = 2y$

Let $y = 1$, then $x = 2$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\sqrt{(2^2 + 1^2)} = \sqrt{5}$.

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad \mathbf{P}^{\mathrm{T}} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} & -\frac{4}{\sqrt{5}} + \frac{4}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{5}{\sqrt{5}} & 0 \\ -\frac{10}{\sqrt{5}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4 & 0 \\ 2-2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$$

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Further matrix algebra
Exercise G, Question 2

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$.

b Find normalised eigenvectors of $\mathbf{A}$ corresponding to each of the two eigenvalues of $\mathbf{A}$.

c Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$.

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Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 3

Question:

The matrix $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$.

a Show that $\mathbf{P}$ is an orthogonal matrix.

The matrix $\mathbf{A} = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & \frac{3}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

b Show that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix.

Solution:

a $\mathbf{PP}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{2}{6}-\frac{1}{3} \\ \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{2}{6}-\frac{1}{3} \\ \frac{2}{6}-\frac{1}{3} & \frac{2}{6}-\frac{1}{3} & \frac{4}{6}+\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Hence **P** is an orthogonal matrix.

b $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & \frac{3}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2\sqrt{6}} - \frac{3}{2\sqrt{6}} + \frac{2}{\sqrt{6}} & -\frac{3}{2\sqrt{3}} + \frac{3}{2\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \\ -\frac{3}{2\sqrt{6}} + \frac{3}{2\sqrt{6}} + \frac{2}{\sqrt{6}} & \frac{3}{2\sqrt{3}} - \frac{3}{2\sqrt{3}} + \frac{1}{\sqrt{3}} & -\frac{3}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} \\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{2}} \\ \frac{4}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{6} + \frac{2}{6} + \frac{8}{6} & \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} & \frac{3}{\sqrt{12}} - \frac{3}{\sqrt{12}} \\ -\frac{2}{18} - \frac{2}{\sqrt{18}} + \frac{4}{\sqrt{18}} & -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} & -\frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} \\ \frac{2}{\sqrt{12}} - \frac{2}{\sqrt{12}} & \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{3}{2} + \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \text{ a diagonal matrix.}$$

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Further matrix algebra
Exercise G, Question 4

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}$. Reduce $\mathbf{A}$ to a diagonal matrix.

Solution:

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & 2-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}-0\begin{vmatrix}0 & 0\\ 2 & 2-\lambda\end{vmatrix}+2\begin{vmatrix}0 & 2-\lambda\\ 2 & 0\end{vmatrix}$$

$$=(2-\lambda)^3-4(2-\lambda)=(2-\lambda)\left((2-\lambda)^2-4\right)=(2-\lambda)(-\lambda)(4-\lambda)$$

$$=-\lambda(2-\lambda)(4-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow -\lambda(\lambda-2)(\lambda-4)=0\Rightarrow \lambda=0,2,4$$

For $\lambda=0$

$$\begin{pmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x+2z\\ 2y\\ 2x+2z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Equating the top elements

$2x+2z=0\Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements

$2y=0\Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$.

The magnitude of $\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$ is $\sqrt{\left(1^2+0^2+(-1)^2\right)}=\sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$.

For $\lambda=2$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2z \\ 2y \\ 2x+2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements

$2x+2z=2x \Rightarrow z=0$

Equating the lowest elements

$2x+2z=2z \Rightarrow x=0$

y can take any value

Let $y=1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The magnitude of this vector is 1, so it is already normalised.

For $\lambda=4$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2z \\ 2y \\ 2x+2z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$2x+2z=4x \Rightarrow z=x$

Let $x=1$, then $z=1$

Equating the middle elements

$2y=4y \Rightarrow 2y=0 \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $\sqrt{(1^2+0^2+1^2)}=\sqrt{2}$

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

Then $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ 0 & 2 & 0 \\ \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{4}{\sqrt{2}} \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2-2 \\ 0 & 2 & 0 \\ 0 & 0 & 2+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

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Further matrix algebra
Exercise G, Question 5

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

The eigenvalues of $\mathbf{A}$ are 0, -1 and 8.

a Find a normalised eigenvector corresponding to the eigenvalue 0.

Given that $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1 and that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 8,

b find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

Solution:

a For $\lambda = 0$

$$\begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x+3y+3z \\ 3x+y+z \\ 3x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the top elements

$5x+3y+3z=0$ ①

Equating the middle elements

$3x+y+z=0$ ②

$3\times$② $-$ ①

$x=0$

Substituting $x=0$ into ②

$y+z=0 \Rightarrow z=-y$

Let $y=1$, then $z=-1$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is $\sqrt{\left(0^2+1^2+(-1)^2\right)} = \sqrt{2}$

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

b The magnitude of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is $\sqrt{\left((-1)^2+1^2+1^2\right)} = \sqrt{3}$

A normalised eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is $\sqrt{(2^2+1^2+1^2)} = \sqrt{6}$

A normalised eigenvector corresponding to the eigenvalue 8 is $\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

$$\mathbf{P} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

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Further matrix algebra
Exercise G, Question 6

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 7 & 0 & 2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}$.

a Given that 9 is an eigenvalue of $\mathbf{A}$, find the other two eigenvalues of $\mathbf{A}$.
b Find eigenvectors of $\mathbf{A}$ corresponding to each of the three eigenvalues of $\mathbf{A}$.
c Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{vmatrix}$$

$$=(7-\lambda)\begin{vmatrix}5-\lambda & -2\\ -2 & 6-\lambda\end{vmatrix}-0\begin{vmatrix}0 & -2\\ -2 & 6-\lambda\end{vmatrix}+(-2)\begin{vmatrix}0 & 5-\lambda\\ -2 & -2\end{vmatrix}$$

$$=(7-\lambda)((5-\lambda)(6-\lambda)-4)-2(10-2\lambda)$$

$$=(7-\lambda)(26-11\lambda+\lambda^2)-20+4\lambda$$

$$=182-103\lambda+18\lambda^2-\lambda^3-20+4\lambda=-(\lambda^3-18\lambda^2+99\lambda-162)$$

Let $\lambda^3-18\lambda^2+99\lambda-162=(\lambda-9)(\lambda^2+k\lambda+18)$

Equating coefficients of λ^2

$-18=-9+k\Rightarrow k=-9$

Hence

$\lambda^3-18\lambda^2+99\lambda-162=(\lambda-9)(\lambda^2-9\lambda+18)=(\lambda-9)(\lambda-6)(\lambda-3)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-3)(\lambda-6)(\lambda-9)=0\Rightarrow\lambda=3,6,9$

The other two eigenvalues of **A** are 3 and 6.

b For $\lambda=3$

$$\begin{pmatrix}7 & 0 & -2\\ 0 & 5 & -2\\ -2 & -2 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}7x-2z\\ 5y-2z\\ -2x-2y+6z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the top elements

$7x-2z=3x\Rightarrow z=2x$

Let $x=1$, then $z=2$

Equating the middle elements and substituting $z=2$

$5y-4=3y\Rightarrow y=2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$.

For $\lambda=6$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$7x-2z=6x \Rightarrow x=2z$

Let $z=1$, then $x=2$

Equating the middle elements and substituting $z=1$

$5y-2=6y \Rightarrow y=-2$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

For $\lambda=9$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$$

Equating the top elements

$7x-2z=9x \Rightarrow z=-x$

Let $x=2$, then $z=-2$

Equating the middle elements and substituting $z=-2$

$5y+4=9y \Rightarrow y=1$

An eigenvector corresponding to the eigenvalue 9 is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

c The magnitudes of the vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ are all

$\sqrt{\left(1^2+2^2+2^2\right)}=\sqrt{9}=3$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

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Further matrix algebra
Exercise G, Question 7

Question:

The matrix $\mathbf{A}=\begin{pmatrix}1 & 2 & 0\\ 2 & 1 & \sqrt{5}\\ 0 & \sqrt{5} & 1\end{pmatrix}$.

a Show that 4 is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues of $\mathbf{A}$.

b Find a normalised eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 4.

Given that $\begin{pmatrix}-2\\ 3\\ -\sqrt{5}\end{pmatrix}$ and $\begin{pmatrix}\sqrt{5}\\ 0\\ -2\end{pmatrix}$ are eigenvectors of $\mathbf{A}$,

c find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\mathbf{D}$.

Solution:

a $\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda & 2 & 0\\ 2 & 1-\lambda & \sqrt{5}\\ 0 & \sqrt{5} & 1-\lambda\end{vmatrix}$

Substituting $\lambda=4$,

$$\begin{vmatrix}1-4 & 2 & 0\\ 2 & 1-4 & \sqrt{5}\\ 0 & \sqrt{5} & 1-4\end{vmatrix}=\begin{vmatrix}-3 & 2 & 0\\ 2 & -3 & \sqrt{5}\\ 0 & \sqrt{5} & -3\end{vmatrix}=(-3)\begin{vmatrix}-3 & \sqrt{5}\\ \sqrt{5} & -3\end{vmatrix}-2\begin{vmatrix}2 & \sqrt{5}\\ 0 & -3\end{vmatrix}+0\begin{vmatrix}2 & -3\\ 0 & \sqrt{5}\end{vmatrix}$$

$$=\quad(-3)(9-5)-2(-6-0)=-12+12=0$$

Hence, by the factor theorem, 4 is an eigenvalue of $\mathbf{A}$.

$$\begin{vmatrix}1-\lambda & 2 & 0\\ 2 & 1-\lambda & \sqrt{5}\\ 0 & \sqrt{5} & 1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}1-\lambda & \sqrt{5}\\ \sqrt{5} & 1-\lambda\end{vmatrix}-2\begin{vmatrix}2 & \sqrt{5}\\ 0 & 1-\lambda\end{vmatrix}+0\begin{vmatrix}2 & 1-\lambda\\ 0 & \sqrt{5}\end{vmatrix}$$

$=(1-\lambda)\left((1-\lambda)^2-5\right)-4+4\lambda$

$=(1-\lambda)(\lambda^2-2\lambda-4)-4+4\lambda=-\lambda^3+3\lambda^2+6\lambda-8$

$=-\lambda^3+4\lambda^2-\lambda^2+4\lambda+2\lambda-8=-\lambda^2(\lambda-4)-\lambda(\lambda-4)+2(\lambda-4)$

$=-(\lambda-4)(\lambda^2+\lambda-2)=-(\lambda-4)(\lambda+2)(\lambda-1)$

$\det(\mathbf{A}-\lambda\mathbf{I})=-(\lambda-4)(\lambda+2)(\lambda-1)=0\Rightarrow\lambda=4,-2,1$

The other two eigenvalues of $\mathbf{A}$ are -2 and 1.

b For $\lambda=4$

$$\begin{pmatrix}1 & 2 & 0\\ 2 & 1 & \sqrt{5}\\ 0 & \sqrt{5} & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=4\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}x+2y\\ 2x+y+\sqrt{5}z\\ \sqrt{5}y+z\end{pmatrix}=\begin{pmatrix}4x\\ 4y\\ 4z\end{pmatrix}$$

Equating the top elements

$x+2y=4x\Rightarrow 2y=3x$

Let $x=2$, then $y=3$

Equating the lowest elements and substituting $y=3$

$3\sqrt{5}+z=4z\Rightarrow z=\sqrt{5}$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}2\\ 3\\ \sqrt{5}\end{pmatrix}$

The magnitude of $\begin{pmatrix}2\\ 3\\ \sqrt{5}\end{pmatrix}$ is $\sqrt{\left(2^2+3^2+\left(\sqrt{5}\right)^2\right)}=\sqrt{18}$

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{2}{\sqrt{18}} \\ \frac{3}{\sqrt{18}} \\ \frac{\sqrt{5}}{\sqrt{18}} \end{pmatrix}$.

c $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2+6 \\ -4+3-5 \\ 3\sqrt{5}-\sqrt{5} \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2\sqrt{5} \end{pmatrix} = (-2)\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$

The magnitude of $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ is $\sqrt{\left((-2)^2 + 3^2 + (-\sqrt{5})^2\right)} = \sqrt{18}$

A normalised eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} -\frac{2}{\sqrt{18}} \\ \frac{3}{\sqrt{18}} \\ -\frac{5}{\sqrt{18}} \end{pmatrix}$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ is $\sqrt{\left((\sqrt{5})^2 + 0^2 + 2^2\right)} = \sqrt{9} = 3$

A normalised eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \frac{\sqrt{5}}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}$.

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{18}} & -\frac{2}{\sqrt{18}} & \frac{\sqrt{5}}{3} \\ \frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} & 0 \\ \frac{\sqrt{5}}{\sqrt{18}} & -\frac{\sqrt{5}}{\sqrt{18}} & -\frac{2}{3} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Further matrix algebra
Exercise G, Question 8

Question:

The eigenvalue of the matrix $\mathbf{A}=\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}$ are $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 > \lambda_2 > \lambda_3$.

a Show that $\lambda_1 = 6$ and find the other two eigenvalues λ_2 and λ_3.

b Verify that $\det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3$.

c Find an eigenvector corresponding to the value $\lambda_1 = 6$.

Given that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are eigenvectors corresponding to λ_2 and λ_3,

d write down a matrix **P** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2-\lambda & 2 & -3\\ 2 & 2-\lambda & 3\\ -3 & 3 & 3-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 2 & -3\\ 2 & 2-\lambda & 3\\ -3 & 3 & 3-\lambda\end{vmatrix}$

$=(2-\lambda)\begin{vmatrix}2-\lambda & 3\\ 3 & 3-\lambda\end{vmatrix}-2\begin{vmatrix}2 & 3\\ -3 & 3-\lambda\end{vmatrix}+(-3)\begin{vmatrix}2 & 2-\lambda\\ -3 & 3\end{vmatrix}$

$=(2-\lambda)\left((2-\lambda)(3-\lambda)-9\right)-2(6-2\lambda+9)-3(6+6-3\lambda)$

$=(2-\lambda)(\lambda^2-5\lambda-3)-30+4\lambda-36+9\lambda$

$=-\lambda^3+7\lambda^2-7\lambda-6-66+13\lambda=-\lambda^3+7\lambda^2+6\lambda-72$

$=-\lambda^3+6\lambda^2+\lambda^2-6\lambda+12\lambda-72$

$=-\lambda^2(\lambda-6)+\lambda(\lambda-6)+12(\lambda-6)=-(\lambda-6)(\lambda^2-\lambda-12)$

$=-(\lambda-6)(\lambda-4)(\lambda+3)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-6)(\lambda-4)(\lambda+3)=0\Rightarrow\lambda=6,4,-3$

As $\lambda_1>\lambda_2>\lambda_3, \lambda_1=6$, as required, $\lambda_2=4$ and $\lambda_3=-3$.

b $\det(\mathbf{A})=\begin{vmatrix}2 & 2 & -3\\ 2 & 2 & 3\\ -3 & 3 & 3\end{vmatrix}=2\begin{vmatrix}2 & 3\\ 3 & 3\end{vmatrix}-2\begin{vmatrix}2 & 3\\ -3 & 3\end{vmatrix}+(-3)\begin{vmatrix}2 & 2\\ -3 & 3\end{vmatrix}$

$=2(6-9)-2(6+9)-3(6+6)=-6-30-36$

$=-72=6\times4\times(-3)=\lambda_1\lambda_2\lambda_3$, as required.

c For $\lambda_1=6$

$\begin{pmatrix}2 & 2 & -3\\ 2 & 2 & 3\\ -3 & 3 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=6\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

$\begin{pmatrix}2x+2y-3z\\ 2x+2y+3z\\ -3x+3y+3z\end{pmatrix}=\begin{pmatrix}6x\\ 6y\\ 6z\end{pmatrix}$

Equating the top elements

$2x+2y-3z=6x\Rightarrow-4x+2y-3z=0$ ①

Equating the middle elements

$2x+2y+3z=6y\Rightarrow 2x-4y+3z=0$ ②

① + ②

$-2x-2y=0\Rightarrow y=-x$

Let $x=1$, then $y=-1$

Substitute $x=1$ and $y=-1$ into ①

$-4-2-3z=0\Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

d The magnitude of $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2+(-1)^2+(-2)^2\right)}=\sqrt{6}$

The magnitude of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is $\sqrt{\left(1^2+1^2+0^2\right)}=\sqrt{2}$

The magnitude of $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(1^2+(-1)^2+1^2\right)}=\sqrt{3}$

Hence $\mathbf{P}=\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$

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Further matrix algebra
Exercise H, Question 1

Question:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

Given that $\mathbf{A}$ is singular, find the value of t. **[E]**

Solution:

$$\begin{vmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} t & 1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} t & 3 \\ -2 & -1 \end{vmatrix}$$

$$= 1(3+1) + 2(-t+6) = 16 - 2t$$

As $\mathbf{A}$ is singular

$\det(\mathbf{A}) = 16 - 2t = 0 \Rightarrow t = 8$

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Further matrix algebra
Exercise H, Question 2

Question:

$$\mathbf{M}=\begin{pmatrix}1 & 0 & 0\\ x & 2 & 0\\ 3 & 1 & 1\end{pmatrix}$$

Find $\mathbf{M}^{-1}$ in terms of x. **[E]**

Solution:

$$\det(\mathbf{M})=\begin{vmatrix}1 & 0 & 0\\ x & 2 & 0\\ 3 & 1 & 1\end{vmatrix}=1\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix}-0\begin{vmatrix}x & 0\\ 3 & 1\end{vmatrix}+0\begin{vmatrix}x & 2\\ 3 & 1\end{vmatrix}$$

$$=2-0+0=2$$

The matrix of minors is

$$\begin{pmatrix}\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix} & \begin{vmatrix}x & 0\\ 3 & 1\end{vmatrix} & \begin{vmatrix}x & 2\\ 3 & 1\end{vmatrix}\\ \begin{vmatrix}0 & 0\\ 1 & 1\end{vmatrix} & \begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix} & \begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix}\\ \begin{vmatrix}0 & 0\\ 2 & 0\end{vmatrix} & \begin{vmatrix}1 & 0\\ x & 0\end{vmatrix} & \begin{vmatrix}1 & 0\\ x & 2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}2 & x & x-6\\ 0 & 1 & 1\\ 0 & 0 & 2\end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C}=\begin{pmatrix}2 & -x & x-6\\ 0 & 1 & -1\\ 0 & 0 & 2\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}2 & 0 & 0\\ -x & 1 & 0\\ x-6 & -1 & 2\end{pmatrix}$$

$$\mathbf{M}^{-1}=\frac{1}{\det(\mathbf{M})}\mathbf{C}^{\mathrm{T}}=\frac{1}{2}\begin{pmatrix}2 & 0 & 0\\ -x & 1 & 0\\ x-6 & -1 & 2\end{pmatrix}$$

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Further matrix algebra
Exercise H, Question 3

Question:

The matrix $\mathbf{M}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -15$ and $\mathbf{M} = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}$.

a For each eigenvalue, find a corresponding eigenvector.

b Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{pmatrix} 5 & 0 \\ 0 & -15 \end{pmatrix}$. **[E]**

Solution:

a For $\lambda_1 = 5$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$x + 8y = 5x \Rightarrow x = 2y$

Let $y = 1$, then $x = 2$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -15$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -15\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} -15x \\ -15y \end{pmatrix}$$

Equating the upper elements

$x + 8y = -15x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

An eigenvector corresponding to the eigenvalue -15 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

b The magnitude of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\sqrt{(2^2 + 1^2)} = \sqrt{5}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $\sqrt{(1^2 + (-2)^2)} = \sqrt{5}$

Hence $\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$

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Further matrix algebra
Exercise H, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$.

a Find $\mathbf{AB}$.

b Verify that $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = (\mathbf{AB})^{\mathrm{T}}$.

Solution:

a $\mathbf{AB} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 10-8 & -5+4 \\ 4-4 & -2+2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$

b $(\mathbf{AB})^{\mathrm{T}} = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix}$

$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10-8 & 4-4 \\ -5+4 & -2+2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix}$

$= (\mathbf{AB})^{\mathrm{T}}$, as required.

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Further matrix algebra
Exercise H, Question 5

Question:

A transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A} = \begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

Solution:

a $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -5-\lambda & 8 \\ 3 & -7-\lambda \end{pmatrix}$

$\begin{vmatrix} -5-\lambda & 8 \\ 3 & -7-\lambda \end{vmatrix} = (5+\lambda)(7+\lambda) - 24 = \lambda^2 + 12\lambda + 11 = (\lambda+1)(\lambda+11)$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (\lambda+1)(\lambda+11) = 0 \Rightarrow \lambda = -1, -11$

The eigenvalues of $\mathbf{A}$ are -1 and -11.

b For $\lambda = -1$

$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

Equating the upper elements

$-5x + 8y = -x \Rightarrow y = \frac{1}{2}x$

For $\lambda = -11$

$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -11\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -11x \\ -11y \end{pmatrix}$

Equating the upper elements

$-5x + 8y = -11x \Rightarrow y = -\frac{3}{4}x$

Cartesian equations of the lines through the origin which are invariant under T are

$y = \frac{1}{2}x$ and $y = -\frac{3}{4}x$.

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Further matrix algebra
Exercise H, Question 6

Question:

Given that 1 is an eigenvalue of the matrix $\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$,

a find a corresponding eigenvector,

b find the other eigenvalues of the matrix. **[E]**

Solution:

a For $\lambda = 1$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x+y \\ 2x+4y \\ x+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements

$3x + y = x \Rightarrow 2x + y = 0$ ①

Equating the middle elements

$2x + 4y = y \Rightarrow 2x + 3y = 0$ ②

② − ①

$2y = 0 \Rightarrow y = 0$

Substituting $y = 0$ into ①

$2x = 0 \Rightarrow x = 0$

z can take any non-zero value

Let $z = 1$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

b Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 2 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & 1 & 0 \\ 2 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 1 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 4-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(1-\lambda) - 2(1-\lambda)$$

$$= (1-\lambda)((3-\lambda)(4-\lambda)-2) = (1-\lambda)(\lambda^2 - 7\lambda + 10)$$

$$= (1-\lambda)(\lambda-2)(\lambda-5)$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (1-\lambda)(\lambda-2)(\lambda-5) = 0 \Rightarrow \lambda = 1, 2, 5$$

The other eigenvalues are 2 and 5.

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Further matrix algebra
Exercise H, Question 7

Question:

The transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 4 & 3 & 0 \\ 0 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}.$$

The line l_1 is transformed by T to the line l_2. The line l_1 has vector equation

$$r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \text{ where } t \text{ is a real parameter.}$$

Find Cartesian equations of l_2.

Solution:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2t \\ -3t \\ 2 \end{pmatrix}$$

$$\mathbf{Tr} = \begin{pmatrix} 4 & 3 & 0 \\ 0 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1+2t \\ -3t \\ 2 \end{pmatrix} = \begin{pmatrix} 4+8t-9t \\ 6t+2 \\ 3+6t-3t-4 \end{pmatrix} = \begin{pmatrix} 4-t \\ 2+6t \\ -1+3t \end{pmatrix}$$

Equations of l_2 are given by

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-t \\ 2+6t \\ -1+3t \end{pmatrix}$$

Equating elements

$x = 4-t, y = 2+6t, z = -1+3t$

$$\frac{x-4}{-1} = \frac{y-2}{6} = \frac{z+1}{3} = t$$

Cartesian equations of l_2 are

$$\frac{x-4}{-1} = \frac{y-2}{6} = \frac{z+1}{3}$$

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Further matrix algebra
Exercise H, Question 8

Question:

$$A = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

a Show that 3 is an eigenvalue of **A** and find the other two eigenvalues.
b Find an eigenvector corresponding to the eigenvalue 3.

Given that the vectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors corresponding to the other two eigenvalues,

c find a matrix **P** such that $\mathbf{P}^T\mathbf{AP}$ is a diagonal matrix. **[E]**

Solution:

a $$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 4 & -4\\ 4 & 5-\lambda & 0\\ -4 & 0 & 1-\lambda\end{pmatrix}$$

$$\begin{vmatrix}3-\lambda & 4 & -4\\ 4 & 5-\lambda & 0\\ -4 & 0 & 1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}5-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix}-4\begin{vmatrix}4 & 0\\ -4 & 1-\lambda\end{vmatrix}+(-4)\begin{vmatrix}4 & 5-\lambda\\ -4 & 0\end{vmatrix}$$

$$=(3-\lambda)(5-\lambda)(1-\lambda)-16+16\lambda-80+16\lambda$$

$$=(3-\lambda)(5-\lambda)(1-\lambda)-96+32\lambda$$

$$=(3-\lambda)(5-\lambda)(1-\lambda)-32(3-\lambda)$$

$$=(3-\lambda)((5-\lambda)(1-\lambda)-32)=(3-\lambda)(\lambda^2-6\lambda-27)$$

$$=(3-\lambda)(\lambda+3)(\lambda-9)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(\lambda+3)(\lambda-9)=0\Rightarrow\lambda=3,-3,9$

3 is an eigenvalue of $\mathbf{A}$ and the other eigenvalues are -3 and 9.

b $$\begin{pmatrix}3 & 4 & -4\\ 4 & 5 & 0\\ -4 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x+4y-4z\\ 4x+5y\\ -4x+z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the middle elements

$4x+5y=3y\Rightarrow y=-2x$

Let $x=1$, then $y=-2$

Equating the lowest elements and substituting $x=1$

$-4+z=3z\Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ -2\\ -2\end{pmatrix}$.

c The magnitudes of $\begin{pmatrix}1\\ -2\\ -2\end{pmatrix}$, $\begin{pmatrix}2\\ 2\\ -1\end{pmatrix}$ and $\begin{pmatrix}2\\ -1\\ 2\end{pmatrix}$ are all

$\sqrt{(1^2+2^2+2^2)}=\sqrt{9}=3$

Hence

$$\mathbf{P}=\begin{pmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

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Further matrix algebra
Exercise H, Question 9

Question:

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

a Show that $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$, giving their corresponding eigenvalues.

b Given that 6 is the third eigenvalue of $\mathbf{A}$, find a corresponding eigenvector.

c Hence write down a matrix such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. **[E]**

Solution:

a $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-6+0 \\ -4+3-2 \\ 0+6-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is an eigenvalue of $\mathbf{A}$ corresponding to the eigenvalue -1.

$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+2+0 \\ -4-1+2 \\ 0-2+5 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, is an eigenvalue of $\mathbf{A}$ corresponding to the eigenvalue 3.

b For $\lambda = 6$

$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 2x-2y \\ -2x+y+2z \\ 2y+5z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$

Equating the top elements

$2x-2y = 6x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

Equating the lowest elements and substituting $y = -2$

$-4+5z = 6z \Rightarrow z = -4$

$\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ is an eigenvalue of $\mathbf{A}$ corresponding to the eigenvalue 6.

c The magnitude of $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is $\sqrt{\left(2^2+3^2+(-1)^2\right)} = \sqrt{14}$

The magnitude of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(2^2+(-1)^2+1^2\right)} = \sqrt{6}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ is $\sqrt{\left(1^2+(-2)^2+(-4)^2\right)} = \sqrt{21}$

Hence

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \\ -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{21}} \end{pmatrix}$$

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Further matrix algebra
Exercise H, Question 10

Question:

a Calculate the inverse of the matrix $A(x) = \begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}, x \neq \frac{5}{2}$.

The image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when it is transformed by the matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ is the vector $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$.

b Find the values of a, b and c. **[E]**

Solution:

a $\det(\mathbf{A}(x)) = \begin{vmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - x\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}$

$= -2 + 2x - 3 = 2x - 5$

The matrix of the minors is given by

$$\mathbf{M} = \begin{Vmatrix} \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} x & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} x & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & x \\ 3 & 0 \end{vmatrix} \end{Vmatrix} = \begin{pmatrix} -2 & -2 & 3 \\ 1 & 1 & 1-x \\ 2x & 5 & -3x \end{pmatrix}$$

The matrix of the cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -2 & 2 & 3 \\ -1 & 1 & x-1 \\ 2x & -5 & -3x \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$$

$$(\mathbf{A}(x))^{-1} = \frac{1}{\det(\mathbf{A}(x))}\mathbf{C}^{\mathrm{T}} = \frac{1}{2x-5}\begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$$

b Substituting $x = 3$

$$(\mathbf{A}(3))^{-1} = \begin{pmatrix} -2 & -1 & 6 \\ 2 & 1 & -5 \\ 3 & 2 & -9 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 & -1 & 6 \\ 2 & 1 & -5 \\ 3 & 2 & -9 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -8-3+30 \\ 8+3-25 \\ 12+6-45 \end{pmatrix} = \begin{pmatrix} 19 \\ -14 \\ -27 \end{pmatrix}$$

Equating elements

$a = 19, b = -14, c = -27$

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Further matrix algebra
Exercise H, Question 11

Question:

a Show that for all values of the constant α, an eigenvalue of the matrix $\mathbf{A}$ is 1,

where $\mathbf{A} = \begin{pmatrix} \alpha & 0 & 2 \\ 4 & 3 & 0 \\ -2 & -1 & 1 \end{pmatrix}$.

An eigenvector of the matrix $\mathbf{A}$ is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and the corresponding eigenvalue is $\beta (\beta \neq 1)$.

b Find the value of α and the value of β.

c For your value of α, find the third eigenvalue of $\mathbf{A}$. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}\alpha-\lambda & 0 & 2\\ 4 & 3-\lambda & 0\\ -2 & -1 & 1-\lambda\end{pmatrix}$

$$\begin{vmatrix}\alpha-\lambda & 0 & 2\\ 4 & 3-\lambda & 0\\ -2 & -1 & 1-\lambda\end{vmatrix}=(\alpha-\lambda)\begin{vmatrix}3-\lambda & 0\\ -1 & 1-\lambda\end{vmatrix}-0\begin{vmatrix}4 & 0\\ -2 & 1-\lambda\end{vmatrix}+2\begin{vmatrix}4 & 3-\lambda\\ -2 & -1\end{vmatrix}$$

$$=(\alpha-\lambda)(3-\lambda)(1-\lambda)+2(-4+6-2\lambda)$$

$$=(\alpha-\lambda)(3-\lambda)(1-\lambda)+4(1-\lambda)$$

$$=(1-\lambda)\left((\alpha-\lambda)(3-\lambda)+4\right) \quad *$$

Hence, for all α, $\lambda=1$ is a solution of $\det(\mathbf{A}-\lambda\mathbf{I})=0$, and, for all α, an eigenvalue of $\mathbf{A}$ is 1.

b

$$\begin{pmatrix}\alpha & 0 & 2\\ 4 & 3 & 0\\ -2 & -1 & 1\end{pmatrix}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}=\beta\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}$$

$$\begin{pmatrix}2\alpha+2\\ 8-6\\ -4+2+1\end{pmatrix}=\begin{pmatrix}2\beta\\ -2\beta\\ \beta\end{pmatrix}=\begin{pmatrix}2\alpha+2\\ 2\\ -1\end{pmatrix}$$

Equating the lowest elements

$\beta=-1$

Equating the top elements and substituting $\beta=-1$

$2\alpha+2=-2\Rightarrow\alpha=-2$

$$\alpha=-2, \beta=-1$$

c Substituting $\alpha=-2$ into $*$ in part **a** and equating to 0

$(1-\lambda)\left((-2-\lambda)(3-\lambda)+4\right)=0$

$$(1-\lambda)\left(\lambda^2-\lambda-2\right)=(1-\lambda)(\lambda-2)(\lambda+1)$$

$$\lambda=1,2,-1$$

The third eigenvalue is 2.

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Further matrix algebra
Exercise H, Question 12

Question:

The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & u\\ 0 & 1 & 1\end{pmatrix}$.

a Find $\mathbf{A}^{-1}$ in terms of u, stating the condition for which $\mathbf{A}$ is non-singular.

The image vector of $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$ when transformed by the matrix $\mathbf{A}=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 4\\ 0 & 1 & 1\end{pmatrix}$ is $\begin{pmatrix}-2.8\\ 5.3\\ 2.3\end{pmatrix}$.

b Find the values of a, b and c. **[E]**

Solution:

a

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & u \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & u \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & u \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1 - u + 2 + 6 = 9 - u$$

A is singular if $\det(\mathbf{A}) = 0 \Rightarrow 9 - u = 0 \Rightarrow u = 9$

The condition for which A is non-singular is $u \neq 9$.

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & u \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & u \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & u \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & u \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1-u & 2 & 2 \\ -4 & 1 & 1 \\ -u-3 & u-6 & 3 \end{pmatrix}$$

The matrix of the cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1-u & -2 & 2 \\ 4 & 1 & -1 \\ -u-3 & 6-u & 3 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1-u & 4 & -3-u \\ -2 & 1 & 6-u \\ 2 & -1 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{9-u}\begin{pmatrix} 1-u & 4 & -3-u \\ -2 & 1 & 6-u \\ 2 & -1 & 3 \end{pmatrix}$$

b Substituting $u = 4$

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 8.4+21.2-16.1 \\ 5.6+5.3+4.6 \\ -5.6-5.3+6.9 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 13.5 \\ 15.5 \\ -4 \end{pmatrix} = \begin{pmatrix} 2.7 \\ 3.1 \\ -0.8 \end{pmatrix}$$

Equating elements

$a = 2.7, b = 3.1, c = -0.8$

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Further matrix algebra
Exercise H, Question 13

Question:

$$\mathbf{M}=\begin{pmatrix}3 & 0 & 0\\1 & 1 & 1\\4 & -1 & 3\end{pmatrix}$$

a Show that the matrix $\mathbf{M}$ has only two distinct eigenvalues.
b Find an eigenvector corresponding to each of these eigenvalues. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 0 & 0\\ 1 & 1-\lambda & 1\\ 4 & -1 & 3-\lambda\end{pmatrix}$

$$=\begin{vmatrix}3-\lambda & 0 & 0\\ 1 & 1-\lambda & 1\\ 4 & -1 & 3-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}1-\lambda & 1\\ -1 & 3-\lambda\end{vmatrix}-0\begin{vmatrix}1 & 1\\ 4 & 3-\lambda\end{vmatrix}+0\begin{vmatrix}1 & 1-\lambda\\ 4 & -1\end{vmatrix}$$

$$=(3-\lambda)\left((1-\lambda)(3-\lambda)+1\right)=(3-\lambda)\left(\lambda^2-4\lambda+4\right)$$

$$=(3-\lambda)(\lambda-2)^2$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(\lambda-2)^2=0\Rightarrow\lambda=3,2$ repeated.

There are only two distinct eigenvalues of **A**, 2 and 3.

b For $\lambda=2$

$$\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x\\ x+y+z\\ 4x-y+3z\end{pmatrix}=\begin{pmatrix}2x\\ 2y\\ 2z\end{pmatrix}$$

Equating the top elements

$3x=2x\Rightarrow x=0$

Equating the middle elements and substituting $x=0$

$0+y+z=2y\Rightarrow y=z$

Let $z=1$, then $y=1$

An eigenvalue corresponding to the eigenvalue 2 is $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$.

For $\lambda=3$

$$\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x\\ x+y+z\\ 4x-y+3z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the lower elements

$4x-y+3z=3z\Rightarrow y=4x$

Let $x=1$, then $y=4$

Equating the middle elements and substituting $x=1$ and $y=4$

$1+4+z=12\Rightarrow z=7$

An eigenvalue corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ 4\\ 7\end{pmatrix}$.

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Further matrix algebra
Exercise H, Question 14

Question:

The matrix $\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$.

a Show that the matrix **P** is orthogonal.

The transformation $P:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix **P**.
The plane Π_1 is transformed by A to the plane Π_2. The plane Π_2 has Cartesian equation $x + y - \sqrt{2}z = 0$.

b Find a Cartesian equation of the plane Π_1.

Solution:

a $$\mathbf{PP}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}+\frac{1}{4}+\frac{1}{2} & \frac{1}{4}+\frac{1}{4}-\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 \\ \frac{1}{4}+\frac{1}{4}-\frac{1}{2} & \frac{1}{4}+\frac{1}{4}+\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 \\ \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 & \frac{1}{2}+\frac{1}{2}+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Hence **P** is orthogonal.

b As **P** is orthogonal, $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$

$x + y - \sqrt{2}z = 0$

Let $x = s$ and $y = t$, then $z = \frac{1}{\sqrt{2}}(s+t)$

A parametric form of the general point on Π_2 is $\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix}$

A parametric form for the general point of Π_1 is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix} = \mathbf{P}^{\mathrm{T}}\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}s+\frac{1}{2}t+\frac{1}{2}(s+t) \\ -\frac{1}{2}s-\frac{1}{2}t+\frac{1}{2}(s+t) \\ \frac{1}{\sqrt{2}}s-\frac{1}{\sqrt{2}}t+0 \end{pmatrix} = \begin{pmatrix} s+t \\ 0 \\ \frac{1}{\sqrt{2}}(s-t) \end{pmatrix}$$

Equating elements

$x = s+t, y = 0, z = \frac{1}{\sqrt{2}}(s-t)$

x and z can take any values

A Cartesian equation of Π_1 is $y = 0$.

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Further matrix algebra
Exercise H, Question 15

Question:

a Determine the eigenvalues of the matrix $\mathbf{A}=\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}$

b Show that $\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{A}$.

$\mathbf{B}=\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}$

c Show that $\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{B}$ and write down the corresponding eigenvalue.

d Hence, or otherwise, write down an eigenvector of the matrix $\mathbf{AB}$, and state the corresponding eigenvalue. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}2-\lambda & -8\\ 0 & -2-\lambda\end{vmatrix}-(-3)\begin{vmatrix}0 & -8\\ 0 & -2-\lambda\end{vmatrix}+6\begin{vmatrix}0 & 2-\lambda\\ 0 & 0\end{vmatrix}$$

$$=(3-\lambda)(2-\lambda)(-2-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(2-\lambda)(-2-\lambda)=0\Rightarrow\lambda=-2,2,3$$

The eigenvalues are $-2, 2$ and 3.

b $\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=\begin{pmatrix}9-3\\2\\0\end{pmatrix}=\begin{pmatrix}6\\2\\0\end{pmatrix}=2\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

c $\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=\begin{pmatrix}21-6\\3+2\\3-3\end{pmatrix}=\begin{pmatrix}15\\5\\0\end{pmatrix}=5\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue 5.

d $\mathbf{AB}\begin{pmatrix}3\\1\\0\end{pmatrix}=\mathbf{A}.\left[\mathbf{B}\begin{pmatrix}3\\1\\0\end{pmatrix}\right]=\mathbf{A}.5\begin{pmatrix}3\\1\\0\end{pmatrix}=5\mathbf{A}\begin{pmatrix}3\\1\\0\end{pmatrix}=5\times2\begin{pmatrix}3\\1\\0\end{pmatrix}=10\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{AB}$ corresponding to the eigenvalue 10.

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Further matrix algebra
Exercise H, Question 16

Question:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 4 & 2 & 7 \end{pmatrix}$$

a Showing your working, find A^{-1}.

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix A.

b Find Cartesian equations of the line which is mapped by T onto the line $x = \frac{y}{4} = \frac{z}{3}$.

[E]

Solution:

a $\det(\mathbf{A}) = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 4 & 2 & 7 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}$

$= 7 - 2 + 6 - 4 = 7$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 5 & 17 & 2 \\ -2 & 3 & 2 \\ -1 & -2 & 1 \end{pmatrix}$$

The matrix of the cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 5 & -17 & 2 \\ 2 & 3 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 5 & 2 & -1 \\ -17 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{7}\begin{pmatrix} 5 & 2 & -1 \\ -17 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

b Let $x = \frac{y}{4} = \frac{z}{3} = t$, then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 4t \\ 3t \end{pmatrix}$

Equations of the original line are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} t \\ 4t \\ 3t \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5 & 2 & -1 \\ -17 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} t \\ 4t \\ 3t \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 5t + 8t - 3t \\ -17t + 12t + 6t \\ 2t - 8t + 3t \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 10t \\ t \\ -3t \end{pmatrix}$$

Equating elements

$x = \frac{10t}{7}, y = \frac{t}{7}, z = -\frac{3t}{7}$

Hence

$\frac{x}{10} = \frac{y}{1} = \frac{z}{-3} = \frac{t}{7}$

Cartesian equations of the line are

$\frac{x}{10} = \frac{y}{1} = \frac{z}{-3}$