Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise A, Question 1

Question:

Simplify

a $5\mathbf{j}\times\mathbf{k}$

b $3\mathbf{i}\times\mathbf{k}$

c $\mathbf{k}\times 3\mathbf{i}$

d $3\mathbf{i}\times(9\mathbf{i}-\mathbf{j}+\mathbf{k})$

e $2\mathbf{j}\times(3\mathbf{i}+\mathbf{j}-\mathbf{k})$

f $(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times 2\mathbf{j}$

g $(5\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})$

h $(2\mathbf{i}-\mathbf{j}+6\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

i $(\mathbf{i}+5\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}-\mathbf{j}-\mathbf{k})$

j $(3\mathbf{i}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Solution:

Use the results $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$
$\mathbf{i}\times\mathbf{j}=\mathbf{k}, \mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$
and $\mathbf{j}\times\mathbf{i}=-\mathbf{k}, \mathbf{k}\times\mathbf{j}=-\mathbf{i}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

a $5\mathbf{j}\times\mathbf{k}=5(\mathbf{j}\times\mathbf{k})=5\mathbf{i}$

b $3\mathbf{i}\times\mathbf{k}=3(\mathbf{i}\times\mathbf{k})=-3\mathbf{j}$

c $\mathbf{k}\times3\mathbf{i}=3(\mathbf{k}\times\mathbf{i})=3\mathbf{j}$

d
$$\begin{aligned}3\mathbf{i}\times(9\mathbf{i}-\mathbf{j}+\mathbf{k}) &= 3\mathbf{i}\times9\mathbf{i}-3\mathbf{i}\times\mathbf{j}+3\mathbf{i}\times\mathbf{k}\\ &=27(\mathbf{i}\times\mathbf{i})-3(\mathbf{i}\times\mathbf{j})+3(\mathbf{i}\times\mathbf{k})\\ &=0-3\mathbf{k}-3\mathbf{j}\end{aligned}$$

e
$$\begin{aligned}2\mathbf{j}\times(3\mathbf{i}+\mathbf{j}-\mathbf{k}) &= 2\mathbf{j}\times3\mathbf{i}+2\mathbf{j}\times\mathbf{j}-2\mathbf{j}\times\mathbf{k}\\ &=6(\mathbf{j}\times\mathbf{i})+2(\mathbf{j}\times\mathbf{j})-2(\mathbf{j}\times\mathbf{k})\\ &=-6\mathbf{k}-2\mathbf{i}\end{aligned}$$

f
$$\begin{aligned}(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times2\mathbf{j} &= 3\mathbf{i}\times2\mathbf{j}+\mathbf{j}\times2\mathbf{j}-\mathbf{k}\times2\mathbf{j}\\ &=6(\mathbf{i}\times\mathbf{j})+2(\mathbf{j}\times\mathbf{j})-2(\mathbf{k}\times\mathbf{j})\\ &=6\mathbf{k}+2\mathbf{i}\end{aligned}$$

g
$$(5\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&2&-1\\1&-1&3\end{vmatrix}$$
$$\begin{aligned}&=(2\times3-(-1)\times(-1))\mathbf{i}-(5\times3-(-1)\times1)\mathbf{j}+(5\times(-1)-2\times1)\mathbf{k}\\ &=5\mathbf{i}-16\mathbf{j}-7\mathbf{k}\end{aligned}$$

h
$$(2\mathbf{i}-\mathbf{j}+6\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&6\\1&-2&3\end{vmatrix}$$
$$\begin{aligned}&=((-1)\times3-6\times(-2))\mathbf{i}-(2\times3-6\times1)\mathbf{j}+(2\times-2-(-1)\times1)\mathbf{k}\\ &=9\mathbf{i}-0\mathbf{j}-3\mathbf{k}\\ &=9\mathbf{i}-3\mathbf{k}\end{aligned}$$

i
$$(\mathbf{i}+5\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}-\mathbf{j}-\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&5&-4\\2&-1&-1\end{vmatrix}$$
$$\begin{aligned}&=(5\times(-1)-(-4)\times(-1))\mathbf{i}-(1\times(-1)-(-4)\times2)\mathbf{j}+(1\times-1-5\times2)\mathbf{k}\\ &=-9\mathbf{i}-7\mathbf{j}-11\mathbf{k}\end{aligned}$$

j
$$(3\mathbf{i}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&0&1\\1&-1&2\end{vmatrix}$$
$$\begin{aligned}&=(0\times2-1\times(-1))\mathbf{i}-(3\times2-1\times1)\mathbf{j}+(3\times-1-0\times1)\mathbf{k}\\ &=\mathbf{i}-5\mathbf{j}-3\mathbf{k}\end{aligned}$$

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Vectors
Exercise A, Question 2

Question:

Find the vector product of the vectors **a** and **b**, leaving your answers in terms of λ in each case.

a $\mathbf{a} = (\lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ $\mathbf{b} = (\mathbf{i} - 3\mathbf{k})$

b $\mathbf{a} = (2\mathbf{i} - \mathbf{j} + 7\mathbf{k})$ $\mathbf{b} = (\mathbf{i} - \lambda\mathbf{j} + 3\mathbf{k})$

Solution:

a $\mathbf{a} = (\lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \mathbf{b} = (\mathbf{i} - 3\mathbf{k})$

Use the determinant method to find the vector product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & 2 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$

$$= (2 \times (-3) - 1 \times 0)\mathbf{i} - (\lambda \times (-3) - 1 \times 1)\mathbf{j} + (\lambda \times 0 - 2 \times 1)\mathbf{k}$$

$$= -6\mathbf{i} + (3\lambda + 1)\mathbf{j} - 2\mathbf{k}$$

b $\mathbf{a} = (2\mathbf{i} - \mathbf{j} + 7\mathbf{k}), \mathbf{b} = (\mathbf{i} - \lambda\mathbf{j} + 3\mathbf{k})$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 7 \\ 1 & -\lambda & 3 \end{vmatrix}$$

$$= (-1 \times 3 - 7 \times (-\lambda))\mathbf{i} - (2 \times 3 - 7 \times 1)\mathbf{j} + (2 \times (-\lambda) - (-1) \times 1)\mathbf{k}$$

$$= (7\lambda - 3)\mathbf{i} + \mathbf{j} + (1 - 2\lambda)\mathbf{k}$$

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Vectors

Exercise A, Question 3

Question:

Find a unit vector that is perpendicular to both $2\mathbf{i}-\mathbf{j}$ and to $4\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

Solution:

Let $\mathbf{a}=2\mathbf{i}-\mathbf{j}$ and $\mathbf{b}=(4\mathbf{i}+\mathbf{j}+3\mathbf{k})$

Find the vector product of the two given vectors – then divide by its modulus.

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 0\\ 4 & 1 & 3\end{vmatrix}$

$=-3\mathbf{i}-6\mathbf{j}+6\mathbf{k}$

$\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$|\mathbf{a}\times\mathbf{b}|=\sqrt{(-3)^2+(-6)^2+6^2}$

$=\sqrt{81}$

$=9$

So $\frac{1}{9}(\mathbf{a}\times\mathbf{b})$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$\therefore$ Required vector is $\frac{1}{9}(-3\mathbf{i}-6\mathbf{j}+6\mathbf{k})$

$=\frac{-1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

Another possible answer is $\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$.

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Vectors
Exercise A, Question 4

Question:

Find a unit vector that is perpendicular to both of $4\mathbf{i}+\mathbf{k}$ and $\mathbf{j}-\sqrt{2}\mathbf{k}$.

Solution:

Let $\mathbf{a}=4\mathbf{i}+\mathbf{k}$ and $\mathbf{b}=\mathbf{j}-\sqrt{2}\mathbf{k}$

Find the vector product of the two given vectors, then find its modulus.

Then $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{vmatrix}$

$= -\mathbf{i}+4\sqrt{2}\mathbf{j}+4\mathbf{k}$

Now $|\mathbf{a}\times\mathbf{b}| = \sqrt{(-1)^2+\left(4\sqrt{2}\right)^2+4^2}$

$= \sqrt{1+32+16}$

$= \sqrt{49}$

$= 7$

So $\frac{1}{7}\left(-\mathbf{i}+4\sqrt{2}\mathbf{j}+4\mathbf{k}\right)$ is a unit vector, which is perpendicular to $4\mathbf{i}+\mathbf{k}$ and to $\mathbf{j}-\sqrt{2}\mathbf{k}$.

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Vectors
Exercise A, Question 5

Question:

Find a unit vector that is perpendicular to both $\mathbf{i}-\mathbf{j}$ and $3\mathbf{i}+4\mathbf{j}-6\mathbf{k}$.

Solution:

Let $\mathbf{a}=\mathbf{i}-\mathbf{j}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-6\mathbf{k}$

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 0\\ 3 & 4 & -6\end{vmatrix}$

$= 6\mathbf{i}+6\mathbf{j}+7\mathbf{k}$

Find the vector product of the two given vectors then divide by its modulus.

Also $|\mathbf{a}\times\mathbf{b}|=\sqrt{6^2+6^2+7^2}$

$=\sqrt{36+36+49}$

$=\sqrt{121}$

$=11$

So $\frac{1}{11}(6\mathbf{i}+6\mathbf{j}+7\mathbf{k})$ is the required unit vector.

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Vectors
Exercise A, Question 6

Question:

Find a unit vector that is perpendicular to both $\mathbf{i}+6\mathbf{j}+4\mathbf{k}$ and to $5\mathbf{i}+9\mathbf{j}+8\mathbf{k}$.

Solution:

Let $\mathbf{a}=\mathbf{i}+6\mathbf{j}+4\mathbf{k}$ and $\mathbf{b}=5\mathbf{i}+9\mathbf{j}+8\mathbf{k}$.

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 6 & 4\\ 5 & 9 & 8\end{vmatrix}$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 6 & 4\\ 5 & 9 & 8\end{vmatrix}=(6\times 8-4\times 9)\mathbf{i}-(1\times 8-4\times 5)\mathbf{j}+(1\times 9-6\times 5)\mathbf{k}$$

$=+12\mathbf{i}+12\mathbf{j}-21\mathbf{k}$

Also $|\mathbf{a}\times\mathbf{b}|=\sqrt{12^2+12^2+(-21)^2}$

$=\sqrt{144+144+441}$

$=\sqrt{729}$

$=27$

$\therefore \frac{1}{27}(12\mathbf{i}+12\mathbf{j}-21\mathbf{k})=\frac{1}{9}(4\mathbf{i}+4\mathbf{j}-7\mathbf{k})$ is the required unit vector.

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Vectors
Exercise A, Question 7

Question:

Find a vector of magnitude 5 which is perpendicular to both $4\mathbf{i}+\mathbf{k}$ and $\sqrt{2}\mathbf{j}+\mathbf{k}$.

Solution:

Let $\mathbf{a}=4\mathbf{i}+\mathbf{k}$ and $\mathbf{b}=\sqrt{2}\mathbf{j}+\mathbf{k}$

Find the vector product of the two given vectors and compare its magnitude with 5.

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 0 & 1\\ 0 & \sqrt{2} & 1\end{vmatrix}$

$=-\sqrt{2}\mathbf{i}-4\mathbf{j}+4\sqrt{2}\mathbf{k}$

But $|\mathbf{a}\times\mathbf{b}|=\sqrt{\left[\left(-\sqrt{2}\right)^2+(-4)^2+\left(4\sqrt{2}\right)^2\right]}$

$=\sqrt{(2+16+32)}$

$=\sqrt{50}$

$=5\sqrt{2}$

So $\frac{1}{\sqrt{2}}(\mathbf{a}\times\mathbf{b})$ has magnitude 5

$\therefore \frac{1}{\sqrt{2}}\left(-\sqrt{2}\mathbf{i}-4\mathbf{j}+4\sqrt{2}\mathbf{k}\right)=-\mathbf{i}-2\sqrt{2}\mathbf{j}+4\mathbf{k}$ is the required vector.

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Vectors
Exercise A, Question 8

Question:

Find the magnitude of $(\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})$. **[E]**

Solution:

Let $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}$

$= 0\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

$= -2\mathbf{j}-2\mathbf{k}$

So $|\mathbf{a}\times\mathbf{b}| = \sqrt{(-2)^2+(-2)^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$ or $2\sqrt{2}$ or 2.83 (to 3 s.f.)

Given an exact answer as well as a decimal answer correct to 3 s.f.

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Vectors
Exercise A, Question 9

Question:

Given that $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}, \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ find

a $\mathbf{a} \cdot \mathbf{b}$

b $\mathbf{a} \times \mathbf{b}$

c the unit vector in the direction $\mathbf{a} \times \mathbf{b}$. **[E]**

Solution:

$\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}, \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

a $\mathbf{a} \cdot \mathbf{b} = (-1) \times 5 + 2 \times (-2) + (-5) \times 1$

$= -5 - 4 - 5$

$= -14$

b $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -5 \\ 5 & -2 & 1 \end{vmatrix}$

$= -8\mathbf{i} - 24\mathbf{j} - 8\mathbf{k}$

c $|\mathbf{a} \times \mathbf{b}| = \sqrt{(-8)^2 + (-24)^2 + (-8)^2}$

$= 8\sqrt{(-1)^2 + (-3)^2 + (-1)^2}$

$= 8\sqrt{11}$

$\therefore$ unit vector in direction $\mathbf{a} \times \mathbf{b}$ is

$$\frac{1}{8\sqrt{11}}(-8\mathbf{i} - 24\mathbf{j} - 8\mathbf{k}) = \frac{1}{\sqrt{11}}(-\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

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Vectors
Exercise A, Question 10

Question:

Find the sine of the angle between **a** and **b** in each of the following. You may leave your answers as surds, in their simplest form.

a $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

b $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

c $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

Solution:

a $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}, \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$|\mathbf{a}| = \sqrt{(3)^2 + (-4)^2}, |\mathbf{b}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$= 5 \qquad = 3$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 0 \\ 2 & 2 & 1 \end{vmatrix} = -4\mathbf{i} - 3\mathbf{j} + 14\mathbf{k}$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = \sqrt{(-4)^2 + (-3)^2 + 14^2} = \sqrt{221}$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\sin\theta = \frac{\sqrt{221}}{5 \times 3} = \frac{\sqrt{221}}{15}$$

Use $\sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$

b $\mathbf{a} = \mathbf{j} + 2\mathbf{k}, \mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$$|\mathbf{a}| = \sqrt{1^2 + 2^2}, |\mathbf{b}| = \sqrt{5^2 + 4^2 + (-2)^2}$$
$$= \sqrt{5} \qquad = \sqrt{45}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = \sqrt{(-10)^2 + (10)^2 + (-5)^2} = 5\sqrt{(-2)^2 + 2^2 + (-1)^2}$$
$$= 15$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\sin\theta = \frac{15}{\sqrt{5} \times \sqrt{45}} = \frac{15}{15} = 1$$

c $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

$$|\mathbf{a}| = \sqrt{5^2 + 2^2 + 2^2}, |\mathbf{b}| = \sqrt{4^2 + 4^2 + 1^2}$$
$$= \sqrt{33} \qquad = \sqrt{33}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & 2 \\ 4 & 4 & 1 \end{vmatrix}$$
$$= -6\mathbf{i} + 3\mathbf{j} + 12\mathbf{k} = 3(-2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = 3\sqrt{(-2)^2 + 1^2 + 4^2}$$
$$= 3\sqrt{21}$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\sin\theta = \frac{3\sqrt{21}}{\sqrt{33}\sqrt{33}} = \frac{\sqrt{21}}{11}$$

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Vectors
Exercise A, Question 11

Question:

The line l_1 has equation $\mathbf{r} = (\mathbf{i} - \mathbf{j}) + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and the line l_2 has equation $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. Find a vector that is perpendicular to both l_1 and l_2.

Solution:

The direction of line l_1 is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
The direction of line l_2 is $2\mathbf{i} - \mathbf{j} + \mathbf{k}$

l_1 is in the direction $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and l_2 is in the direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

A vector perpendicular to both l_1 and l_2 is in the direction:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

Any multiple of $(\mathbf{i} + \mathbf{j} - \mathbf{k})$ is perpendicular to lines l_1 and l_2.

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Vectors
Exercise A, Question 12

Question:

It is given that $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + u\mathbf{j} + v\mathbf{k}$ and that $\mathbf{a} \times \mathbf{b} = w\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}$, where u, v and w are scalar constants. Find the values of u, v and w.

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & u & v \end{vmatrix}$$

Calculate the vector product of **a** and **b**, then equate coefficients of **i**, **j** and **k**.

$$= (3v + u)\mathbf{i} - (v + 2)\mathbf{j} + (u - 6)\mathbf{k}$$

But $\mathbf{a} \times \mathbf{b} = w\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}$

So equating **i**, **j** and **k** components gives

$3v + u = w$ ①

$v + 2 = 6$ ②

$u - 6 = -7$ ③

From ② $v = 4$

From ③ $u = -1$

From ① $w = 12 - 1$ i.e. $w = 11$

So $u = -1, v = 4$ and $w = 11$.

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Vectors
Exercise A, Question 13

Question:

Given that $\mathbf{p} = a\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, that $\mathbf{q} = \mathbf{j} - \mathbf{k}$ and that their vector product $\mathbf{q} \times \mathbf{p} = 3\mathbf{i} - \mathbf{j} + b\mathbf{k}$ where a and b are scalar constants,

a find the values of a and b,

b find the value of the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$.

Solution:

a $\mathbf{q} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ a & -1 & 4 \end{vmatrix}$

$= 3\mathbf{i} - a\mathbf{j} - a\mathbf{k}$

But $\mathbf{q} \times \mathbf{p} = 3\mathbf{i} - \mathbf{j} + b\mathbf{k}$ so equate components of **i**, **j** and **k**.

$\therefore a = 1$ – from **j** component.

$-a = b$ – from **k** component.

$\therefore b = -1$

So $a = 1$ and $b = -1$

b Use $\cos\theta = \dfrac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$

$\mathbf{p} \cdot \mathbf{q} = a \times 0 + (-1) \times 1 + 4 \times (-1) = -5$

$|\mathbf{p}| = \sqrt{a^2 + (-1)^2 + 4^2} = \sqrt{18}$ as $a = 1$

$|\mathbf{q}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\therefore \cos\theta = \dfrac{-5}{\sqrt{18}\sqrt{2}} = -\dfrac{5}{6}$

Use scalar product and the definition $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Note that this gives the obtuse angle between the vectors. The cosine of the corresponding acute angle will be $\dfrac{5}{6}$.

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Vectors
Exercise A, Question 14

Question:

If $\mathbf{a}\times\mathbf{b}=0$, and $\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$, and $\mathbf{b}=3\mathbf{i}+\lambda\mathbf{j}+\mu\mathbf{k}$, where λ and μ are scalar constants, find the values of λ and μ.

Solution:

$\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k},\mathbf{b}=3\mathbf{i}+\lambda\mathbf{j}+\mu\mathbf{k}$

Given $\mathbf{a}\times\mathbf{b}=0$

This implies that $\mathbf{a}$ is parallel to $\mathbf{b}$

i.e. $\mathbf{a}=c\mathbf{b}$ where c is a scalar constant.

If the vector product of two vectors is zero, then one is a multiple of the other.

$$\therefore \begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}3c\\\lambda c\\\mu c\end{pmatrix}$$

So as $3c=2$,

$$c=\frac{2}{3}$$

$$\therefore 1=\frac{2}{3}\lambda\Rightarrow\lambda=\frac{3}{2}$$

$$\text{Also } -1=\frac{2}{3}\mu\Rightarrow\mu=-\frac{3}{2}$$

Alternative method

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-1\\3&\lambda&\mu\end{vmatrix}=(\mu+\lambda)\mathbf{i}-(2\mu+3)\mathbf{j}+(2\lambda-3)\mathbf{k}$$

But $\mathbf{a}\times\mathbf{b}=0\therefore\mu+\lambda=0,2\mu+3=0,2\lambda-3=0$

$$\Rightarrow\lambda=\frac{3}{2}\text{ and }\mu=-\frac{3}{2}.$$

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Vectors
Exercise A, Question 15

Question:

If three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, show that $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

Solution:

Given $\mathbf{a}+\mathbf{b}+\mathbf{c}=0 \quad *$

Take the vector product of this with $\mathbf{a}$

$$\mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{0}$$

Multiply $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, first by $\mathbf{a}$ and then by $\mathbf{b}$.

i.e. $\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$

But $\mathbf{a}\times\mathbf{a}=0$ and $\mathbf{a}\times\mathbf{c}=-\mathbf{c}\times\mathbf{a}$

$\therefore \mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}=\mathbf{0}$

i.e. $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

This time multiply equation $*$ by $\mathbf{b}$, using vector product.

$$\mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{b}\times\mathbf{0}$$

$\therefore \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}=\mathbf{0}$

But $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{b}=0$

$\therefore -\mathbf{a}\times\mathbf{b}+\mathbf{0}+\mathbf{b}\times\mathbf{c}=\mathbf{0}$

$\therefore \mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}$

So $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

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Vectors
Exercise B, Question 1

Question:

Find the area of triangle OAB, where O is the origin, A is the point with position vector $\mathbf{a}$ and B is the point with position vector $\mathbf{b}$, when
$\mathbf{a} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$ $\quad$ $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

Solution:

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix}$$

$$= -6\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = \sqrt{(-6)^2 + (-6)^2 + (-3)^2}$$

$$= \sqrt{81}$$

$$= 9$$

$$\therefore \text{Area of triangle} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

$$= 4.5$$

Find the vector product of $\mathbf{a}$ and $\mathbf{b}$ and use the formula area $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

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Vectors
Exercise B, Question 2

Question:

Find the area of triangle OAB, where O is the origin, A is the point with position vector $\mathbf{a}$ and B is the point with position vector $\mathbf{b}$, when
$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \qquad \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Solution:

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

$$\therefore \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -5 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-3)^2 + (-4)^2 + (-5)^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

$$\therefore \text{Area of triangle} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

$$= \frac{5\sqrt{2}}{2}$$

Use the formula that area of triangle $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

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Vectors
Exercise B, Question 3

Question:

Find the area of triangle OAB, where O is the origin, A is the point with position vector $\mathbf{a}$ and B is the point with position vector $\mathbf{b}$, when

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ -9 \end{pmatrix}$$

Solution:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ -9 \end{pmatrix}$$

$$\text{So } \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 2 & 6 & -9 \end{vmatrix}$$

$$= -27\mathbf{i} + 18\mathbf{j} + 6\mathbf{k} = 3(-9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$$

$$|\mathbf{a}\times\mathbf{b}| = 3\sqrt{(-9)^2 + 6^2 + 2^2}$$

$$= 3\sqrt{121}$$

$$= 33$$

$$\therefore \text{Area of triangle} = \frac{1}{2}\times 33 = 16.5$$

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Vectors
Exercise B, Question 4

Question:

Find the area of the triangle with vertices $A(0,0,0), B(1,-2,1)$ and $C(2,-1,-1)$.

Solution:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

Use area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$\text{Area of triangle } = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\therefore \text{Area of triangle } = \frac{1}{2}\sqrt{3^2 + 3^2 + 3^2}$$

$$= \frac{1}{2}\sqrt{27}$$

$$= \frac{3}{2}\sqrt{3}$$

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Vectors
Exercise B, Question 5

Question:

Find the area of triangle ABC, where the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, in the following cases:

i $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ $\mathbf{b} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

ii $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix}$

Solution:

i $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$= 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$= 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

First find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then calculate their vector product.

Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 3 & -2 & 2 \end{matrix}\right\|$$

$$= \frac{1}{2}|[8\mathbf{i} + 0\mathbf{j} - 12\mathbf{k}]|$$

$$= |4\mathbf{i} - 6\mathbf{k}|$$

$$= \sqrt{4^2 + (-6)^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

ii $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix}$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then use area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -12 \end{pmatrix}$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & -1 & -12 \end{vmatrix}$$

$$= 12\mathbf{i} + 12\mathbf{j} + \mathbf{k}$$

So area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= |6\mathbf{i} + 6\mathbf{j} + \frac{1}{2}\mathbf{k}|$$

$$= \sqrt{6^2 + 6^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{72.25}$$

$$= 8.5$$

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Vectors
Exercise B, Question 6

Question:

Find the area of the triangle with vertices $A(1,0,2)$, $B(2,-2,0)$ and $C(3,-1,1)$.

Solution:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then use area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2}\left|-3\mathbf{j} + 3\mathbf{k}\right|$$

$$= \frac{1}{2}\sqrt{(-3)^2 + (3)^2}$$

$$= \frac{1}{2}\sqrt{18}$$

$$= \frac{3}{2}\sqrt{2}$$

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Vectors
Exercise B, Question 7

Question:

Find the area of the triangle with vertices $A(-1,1,1)$, $B(1,0,2)$ and $C(0,3,4)$.

Solution:

$$\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= -5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2}\left|-5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}\right|$$

$$= \frac{5}{2}\left|-\mathbf{i} - \mathbf{j} + \mathbf{k}\right|$$

$$= \frac{5}{2}\sqrt{(-1)^2 + (-1)^2 + 1^2}$$

$$= \frac{5}{2}\sqrt{3}$$

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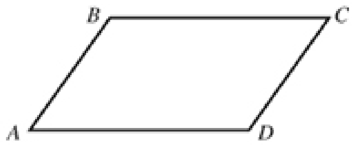
Vectors
Exercise B, Question 8

Question:

Find the area of the parallelogram $ABCD$, shown in the figure, where the position vectors of A, B and D are $\mathbf{i}+\mathbf{j}+\mathbf{k}, -3\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}$ respectively.



Solution:

$$\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}=\begin{pmatrix}-3\\4\\1\end{pmatrix} \text{ and } \mathbf{d}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and calculate $|\overrightarrow{AB}\times\overrightarrow{AD}|$.

$$\therefore \overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}-4\\3\\0\end{pmatrix} \text{ and } \overrightarrow{AD}=\mathbf{d}-\mathbf{a}=\begin{pmatrix}1\\-2\\-1\end{pmatrix}$$

$$\text{So } \overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -4 & 3 & 0\\ 1 & -2 & -1\end{vmatrix}$$

$$=-3\mathbf{i}-4\mathbf{j}+5\mathbf{k}$$

$$\therefore |\overrightarrow{AB}\times\overrightarrow{AD}|=\sqrt{(-3)^2+(-4)^2+5^2}$$

$$=\sqrt{50}$$

$$=5\sqrt{2}$$

So area of parallelogram $ABCD=5\sqrt{2}$.

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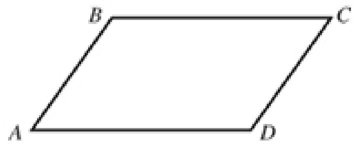
Vectors
Exercise B, Question 9

Question:

Find the area of the parallelogram $ABCD$, shown in the figure, in which the vertices A, B and D have coordinates $(0, 5, 3)$, $(2, 1, -1)$ and $(1, 6, 6)$ respectively.



Solution:

$$\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and use area $= |\overrightarrow{AB} \times \overrightarrow{AD}|$.

$$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} \text{ and } \overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -4 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= -8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$$

$$\therefore |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-8)^2 + (-10)^2 + 6^2}$$

$$= \sqrt{200}$$

$$= 10\sqrt{2}$$

$\therefore$ Area of parallelogram $= 10\sqrt{2}$

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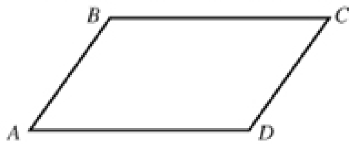
Vectors
Exercise B, Question 10

Question:

Find the area of the parallelogram $ABCD$, shown in the figure, where the position vectors of A, B and D are $\mathbf{j}, \mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+6\mathbf{j}+3\mathbf{k}$ respectively.



Solution:

$$\mathbf{a}=\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{b}=\begin{pmatrix}1\\4\\1\end{pmatrix} \text{ and } \mathbf{d}=\begin{pmatrix}2\\6\\3\end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and then use area of parallelogram $=|\overrightarrow{AB}\times\overrightarrow{AD}|$.

$$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\3\\1\end{pmatrix} \text{ and } \overrightarrow{AD}=\mathbf{d}-\mathbf{a}=\begin{pmatrix}2\\5\\3\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 3 & 1\\2 & 5 & 3\end{vmatrix}=4\mathbf{i}-\mathbf{j}-\mathbf{k}$$

$$\text{The area of } ABCD=|\overrightarrow{AB}\times\overrightarrow{AD}|=\sqrt{4^2+(-1)^2+(-1)^2}$$
$$=\sqrt{18}$$
$$=3\sqrt{2}$$

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Vectors
Exercise B, Question 11

Question:

Relative to an origin O, the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where $\mathbf{p} = a(\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and $\mathbf{q} = a(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$ and $a > 0$. Find the area of triangle OPQ. **[E]**

Solution:

$$\mathbf{p} = a(\mathbf{i}+\mathbf{j}+2\mathbf{k}), \mathbf{q} = a(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

$$\mathbf{p}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & a & 2a \\ 2a & a & 3a \end{vmatrix}$$

$$= a^2\mathbf{i} + a^2\mathbf{j} - a^2\mathbf{k}$$

$$\therefore \text{Area of triangle } OPQ = \frac{1}{2}|a^2\mathbf{i} + a^2\mathbf{j} - a^2\mathbf{k}|$$

$$= \frac{1}{2}a^2\sqrt{1^2+1^2+(-1)^2}$$

$$= \frac{\sqrt{3}}{2}a^2$$

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Vectors
Exercise B, Question 12

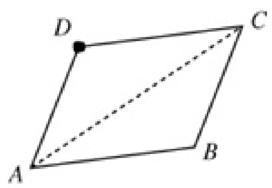
Question:

a Show that the area of the parallelogram $ABCD$ is also given by the formula $|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$.

b Show that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$ implies that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$ and explain the geometrical significance of this vector product.

Solution:

a



Draw a diagram and divide the parallelogram into two triangles by drawing the line AC. Find the area of triangle ABC and deduce the area of the parallelogram.

Area of parallelogram $ABCD = 2\times$area of triangle ABC

$$= 2\times\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$= |\overrightarrow{AB}\times\overrightarrow{AC}|$$

As $\overrightarrow{AB}=(\mathbf{b}-\mathbf{a})$ and $\overrightarrow{AC}=(\mathbf{c}-\mathbf{a})$

Area $=|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$

b $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$

$\therefore (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})-(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})=0$

$\therefore (\mathbf{b}-\mathbf{a})\times\left[(\mathbf{c}-\mathbf{a})-(\mathbf{d}-\mathbf{a})\right]=0$

i.e. $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$

This implies $\overrightarrow{AB}\times\overrightarrow{DC}=0$

i.e. $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

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Vectors
Exercise B, Question 13

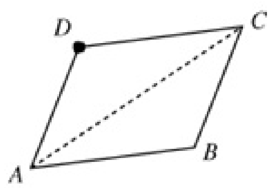
Question:

a Show that the area of the parallelogram $ABCD$ is also given by the formula $|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$.

b Show that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$ implies that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$ and explain the geometrical significance of this vector product.

Solution:

a



Draw a diagram and divide the parallelogram into two triangles by drawing the line AC. Find the area of triangle ABC and deduce the area of the parallelogram.

$$\text{Area of parallelogram } ABCD = 2\times\text{area of triangle } ABC$$
$$= 2\times\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$
$$= |\overrightarrow{AB}\times\overrightarrow{AC}|$$

As $\overrightarrow{AB}=(\mathbf{b}-\mathbf{a})$ and $\overrightarrow{AC}=(\mathbf{c}-\mathbf{a})$

Area $=|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$

b

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$$
$$\therefore (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})-(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})=0$$
$$\therefore (\mathbf{b}-\mathbf{a})\times[(\mathbf{c}-\mathbf{a})-(\mathbf{d}-\mathbf{a})]=0$$
$$\text{i.e. } (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$$

This implies $\overrightarrow{AB}\times\overrightarrow{DC}=0$

i.e. $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

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Vectors
Exercise C, Question 1

Question:

Given that $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{k}$
find

a $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

b $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

c $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

Solution:

a $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{c} = 3\mathbf{i} + 4\mathbf{k}$

Calculate the vector product in the bracket first, then perform the scalar product on the answer.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

$$\therefore \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} - 3\mathbf{k})$$
$$= 20 - 2 + 3$$
$$= 21$$

b

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 4 \\ 5 & 2 & -1 \end{vmatrix} = -8\mathbf{i} + 23\mathbf{j} + 6\mathbf{k}$$

$$\therefore \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (-8\mathbf{i} + 23\mathbf{j} + 6\mathbf{k})$$
$$= -8 + 23 + 6$$
$$= 21$$

c

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

$$\therefore \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (3\mathbf{i} + 4\mathbf{k}) \cdot (3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k})$$
$$= 9 + 12$$
$$= 21$$

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Vectors
Exercise C, Question 2

Question:

Given that $\mathbf{a} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$
find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. What can you deduce about the vectors **a**, **b** and **c**?

Solution:

$$\mathbf{a} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{c} = 2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & -3 & -5 \end{vmatrix} = -8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$$

$$\therefore \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \cdot (-8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k})$$
$$= -8 - 8 + 16$$
$$= 0$$

If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ then **a** is perpendicular to $\mathbf{b} \times \mathbf{c}$.

a is parallel to the plane containing **b** and **c** (in fact $\mathbf{a} = \frac{1}{8}\mathbf{b} + \frac{3}{8}\mathbf{c}$).

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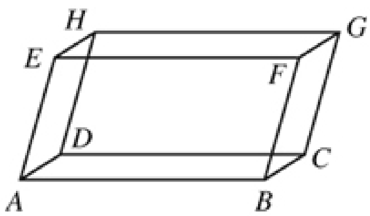
Vectors
Exercise C, Question 3

Question:

Find the volume of the parallelepiped $ABCDEFGH$ where the vertices A, B, D and E have coordinates (0, 0, 0), (3, 0, 1), (1, 2, 0) and (1, 1, 3) respectively.



Solution:

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

Then $\overrightarrow{AE} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AD}\right) = (\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (-2\mathbf{i} + \mathbf{j} + 6\mathbf{k})$

$= -2 + 1 + 18$

$= 17$

Use volume $= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$\therefore$ The volume of the parallelepiped is 17.

Alternative method:

$$\text{Volume} = \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix}$$

$= 1(0-2) - 1(0-1) + 3(6-0)$

$= 17$

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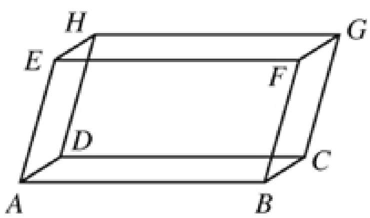
Vectors
Exercise C, Question 4

Question:

Find the volume of the parallelepiped $ABCDEFGH$ where the vertices A, B, D and E have coordinates $(-1,0,1),(3,0,-1),(2,2,0)$ and $(2, 1, 2)$ respectively.



Solution:

$\mathbf{a} = -\mathbf{i}+\mathbf{k}, \mathbf{b} = 3\mathbf{i}-\mathbf{k}, \mathbf{d} = 2\mathbf{i}+2\mathbf{j}$ and $\mathbf{e} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} = 4\mathbf{i}-2\mathbf{k}, \overrightarrow{AD} = \mathbf{d}-\mathbf{a} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$

and $\overrightarrow{AE} = \mathbf{e}-\mathbf{a} = 3\mathbf{i}+\mathbf{j}+\mathbf{k}$

Find the vectors in the directions $\overrightarrow{AE}$, $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and use these in the triple scalar product.

$$\therefore \text{Volume} = \begin{vmatrix} 3 & 1 & 1 \\ 4 & 0 & -2 \\ 3 & 2 & -1 \end{vmatrix}$$

$$= 3(0+4)-1(-4+6)+1(8-0)$$

$$= 12-2+8$$

$$= 18$$

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Vectors
Exercise C, Question 5

Question:

A tetrahedron has vertices at $A(1, 2, 3)$, $B(4, 3, 4)$, $C(1, 3, 1)$ and $D(3, 1, 4)$. Find the volume of the tetrahedron.

Solution:

$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \mathbf{c} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

and $\mathbf{d} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \mathbf{j} - 2\mathbf{k}$

and $\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

Use volume of tetrahedron $= \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

$$\text{Volume of tetrahedron} = \left| \frac{1}{6} \begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix} \right|$$

$$= \left| \frac{1}{6} \{3(1-2) - 1(0+4) + 1(0-2)\} \right|$$

$$= \left| \frac{1}{6} \{-3 - 4 - 2\} \right|$$

$$= \left| -\frac{9}{6} \right|$$

$$= \left| -\frac{3}{2} \right|$$

$$= \frac{3}{2}$$

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Vectors
Exercise C, Question 6

Question:

A tetrahedron has vertices at $A(2,2,1)$, $B(3,-1,2)$, $C(1,1,3)$ and $D(3,1,4)$.

a Find the area of base BCD.

b Find a unit vector normal to the face BCD.

c Find the volume of the tetrahedron.

Solution:

a $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

Find $\overrightarrow{BC} \times \overrightarrow{BD}$ and use this for parts **a** and **b**

$\therefore \overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{BD} = \mathbf{d} - \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

But area of $\Delta BCD = \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{BD}|$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix}$$

$$= 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\therefore \text{Area of } \Delta BCD = \frac{1}{2}\sqrt{2^2 + 4^2 + (-4)^2}$$

$$= 3$$

b The normal to the face BCD is in the direction of $\overrightarrow{BC} \times \overrightarrow{BD}$, i.e. in the direction $2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

As $|2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}| = \sqrt{2^2 + 4^2 + (-4)^2}$

$= 6$

The unit vector normal to the face is $\frac{1}{6}(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$

$= \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

c Given also that $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, the volume of the tetrahedron $ABCD$ is $\frac{1}{6}\left|\overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD})\right|$

Find $\overrightarrow{BA}$ and use $\overrightarrow{BA} \cdot \left(\overrightarrow{BC} \times \overrightarrow{BD}\right)$ to answer part **c**.

$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$

$\therefore \text{Volume} = \frac{1}{6}\{-2 + 12 + 4\} = \frac{14}{6} = 2\frac{1}{3}$

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Vectors
Exercise C, Question 7

Question:

A tetrahedron has vertices at $A(0,0,0)$, $B(2,0,0)$, $C(1,\sqrt{3},0)$ and $D\left(1,\frac{\sqrt{3}}{3},\frac{2\sqrt{6}}{3}\right)$.

a Show that the tetrahedron is regular.
b Find the volume of the tetrahedron.

Solution:

a $|\overrightarrow{AB}| = 2 |\overrightarrow{AC}| = \sqrt{1^2 + (\sqrt{3})^2} = 2$ ← A tetrahedron is regular if all of its edges are the same length.

$|\overrightarrow{AD}| = \sqrt{1^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = \sqrt{1 + \frac{1}{3} + \frac{4\times 6}{9}} = 2$

$\overrightarrow{BC} = \begin{pmatrix} -1 \\ \sqrt{3} \\ 0 \end{pmatrix}$ and $|\overrightarrow{BC}| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\overrightarrow{BD} = \begin{pmatrix} -1 \\ \frac{\sqrt{3}}{3} \\ \frac{2\sqrt{6}}{3} \end{pmatrix}$ and $|\overrightarrow{BD}| = \sqrt{(-1)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = 2$

$\overrightarrow{CD} = \begin{pmatrix} 0 \\ -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{6}}{3} \end{pmatrix}$ and $|\overrightarrow{CD}| = \sqrt{\left(\frac{-2\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2}$

$= \sqrt{\frac{4}{3} + \frac{8}{3}} = 2$

All 6 edges have the same length and the tetrahedron is regular.

b Volume $= \frac{1}{6}\begin{vmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 1 & \frac{\sqrt{3}}{3} & \frac{2\sqrt{6}}{3} \end{vmatrix}$

$= \frac{1}{6} \times 2 \times \left[\frac{2\sqrt{18}}{3}\right]$

$= \frac{4}{18} \times 3\sqrt{2}$

$= \frac{2}{3}\sqrt{2}$

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Vectors
Exercise C, Question 8

Question:

A tetrahedron $OABC$ has its vertices at the points $O(0,0,0), A(1,2,-1), B(-1,1,2)$ and $C(2,-1,1)$.

a Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of **i**, **j** and **k** and find $\overrightarrow{AB} \times \overrightarrow{AC}$.

b Deduce the area of triangle ABC.

c Find the volume of the tetrahedron. **[E]**

Solution:

a $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{c} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 3 \\ 1 & -3 & 2 \end{vmatrix}$$

$$= 7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$$

b Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\sqrt{7^2 + 7^2 + 7^2}$$

$$= \frac{1}{2} \times 7\sqrt{3}$$

$$= \frac{7\sqrt{3}}{2}$$

c Volume of tetrahedron is $|\frac{1}{6}\overrightarrow{AO} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})|$

$$= \frac{1}{6}|(-\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k})|$$

$$= \frac{14}{6}$$

$$= \frac{7}{3}$$

You may use your answer to part **a** and form the triple scalar product $-\mathbf{a} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$ or $-\mathbf{b} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$ or $-\mathbf{c} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$

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Vectors
Exercise C, Question 9

Question:

The points A, B, C and D have position vectors
$\mathbf{a}=(2\mathbf{i}+\mathbf{j})$ $\mathbf{b}=(3\mathbf{i}-\mathbf{j}+\mathbf{k})$ $\mathbf{c}=(-2\mathbf{j}-\mathbf{k})$ $\mathbf{d}=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$ respectively.

a Find $\overrightarrow{AB}\times\overrightarrow{BC}$ and $\overrightarrow{BD}\times\overrightarrow{DC}$.

b Hence find

i the area of triangle ABC

ii the volume of the tetrahedron $ABCD$ **[E]**

Solution:

a $\mathbf{a}=(2\mathbf{i}+\mathbf{j}),\mathbf{b}=(3\mathbf{i}-\mathbf{j}+\mathbf{k}),\mathbf{c}=(-2\mathbf{j}-\mathbf{k}),\mathbf{d}=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$

$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\mathbf{i}-2\mathbf{j}+\mathbf{k},\ \overrightarrow{BC}=\mathbf{c}-\mathbf{b}=(-3\mathbf{i}-\mathbf{j}-2\mathbf{k})$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 1\\ -3 & -1 & -2\end{vmatrix}$$

$$=5\mathbf{i}-\mathbf{j}-7\mathbf{k}$$

Also $\overrightarrow{BD}=\mathbf{d}-\mathbf{b}=-\mathbf{i}+2\mathbf{k},\ \overrightarrow{DC}=\mathbf{c}-\mathbf{d}=(-2\mathbf{i}-\mathbf{j}-4\mathbf{k})$

$$\therefore \overrightarrow{BD}\times\overrightarrow{DC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 0 & 2\\ -2 & -1 & -4\end{vmatrix}$$

$$=2\mathbf{i}-8\mathbf{j}+\mathbf{k}$$

b i Area of $\Delta ABC=\frac{1}{2}|\overrightarrow{BA}\times\overrightarrow{BC}|$

$$=\frac{1}{2}|-5\mathbf{i}+\mathbf{j}+7\mathbf{k}|$$

$$=\frac{1}{2}\sqrt{25+1+49}$$

$$=\frac{1}{2}\sqrt{75}$$

$$=\frac{5}{2}\sqrt{3}$$

($\overrightarrow{AB}\times\overrightarrow{BC}=-\overrightarrow{BA}\times\overrightarrow{BC}$ and $|\overrightarrow{AB}\times\overrightarrow{BC}|=|\overrightarrow{BA}\times\overrightarrow{BC}|$)

ii Volume of tetrahedron $ABCD=\frac{1}{6}\left|\overrightarrow{BD}\cdot\left(\overrightarrow{BA}\times\overrightarrow{BC}\right)\right|$

$$=\frac{1}{6}\left|(-\mathbf{i}+2\mathbf{k})\cdot(-5\mathbf{i}+\mathbf{j}+7\mathbf{k})\right|$$

$$=\frac{19}{6}$$

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Vectors
Exercise C, Question 10

Question:

The edges OP, OQ, OR of a tetrahedron $OPQR$ are the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where

$\mathbf{a} = 2\mathbf{i}+4\mathbf{j}$

$\mathbf{b} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$

$\mathbf{c} = 4\mathbf{i}-2\mathbf{j}+5\mathbf{k}$

a Evaluate $(\mathbf{b}\times\mathbf{c})$ and deduce that OP is perpendicular to the plane OQR.

b Write down the length of OP and the area of triangle OQR and hence the volume of the tetrahedron.

c Verify your result by evaluating $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. **[E]**

Solution:

a $\mathbf{a} = 2\mathbf{i}+4\mathbf{j}, \mathbf{b} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}, \mathbf{c} = 4\mathbf{i}-2\mathbf{j}+5\mathbf{k}$

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= \mathbf{i}+2\mathbf{j}$$

As $\mathbf{a} = 2(\mathbf{b}\times\mathbf{c})$, $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$ and to $\overrightarrow{OR}$, i.e. $\overrightarrow{OP}$ is perpendicular to the plane OQR.

b $|\overrightarrow{OP}| = |\mathbf{a}| = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$

$$\text{Area of } OQR = \frac{1}{2}|\mathbf{b}\times\mathbf{c}|$$

$$= \frac{1}{2}\sqrt{1^2+2^2}$$

$$= \frac{\sqrt{5}}{2}$$

$$\therefore \text{Volume of tetrahedron} = \frac{1}{3}\times\text{base}\times\text{height}$$

Use volume of tetrahedron $= \frac{1}{3}$ base $\times$ height.

$$= \frac{1}{3}\times\frac{\sqrt{5}}{2}\times 2\sqrt{5}$$

$$= \frac{5}{3}$$

c $$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 2 & 4 & 0 \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = 2-(4\times-2) = 10$$

or $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (2\mathbf{i}+4\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) = 2+8 = 10$

This is $6\times$volume of tetrahedron so verified.

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Vectors
Exercise D, Question 1

Question:

Find an equation of the straight line passing through the point with position vector $\mathbf{a}$ which is parallel to the vector $\mathbf{b}$, giving your answer in the form $\mathbf{r}\times\mathbf{b}=\mathbf{c}$, where $\mathbf{c}$ is evaluated:

a $\mathbf{a}=2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ $\quad \mathbf{b}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$

b $\mathbf{a}=2\mathbf{i}-3\mathbf{k}$ $\quad \mathbf{b}=\mathbf{i}+\mathbf{j}+5\mathbf{k}$

c $\mathbf{a}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ $\quad \mathbf{b}=-\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

Solution:

a $[\mathbf{r}-(2\mathbf{i}+\mathbf{j}+2\mathbf{k})]\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=0$

$$\therefore \mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 2\\ 3 & 1 & -2\end{vmatrix}$$

In each case $\mathbf{c}$ is obtained by calculating $\mathbf{a}\times\mathbf{b}$.

i.e. $\mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=-4\mathbf{i}+10\mathbf{j}-\mathbf{k}$

b $[\mathbf{r}-(2\mathbf{i}-3\mathbf{k})]\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=0$

$$\therefore \mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -3\\ 1 & 1 & 5\end{vmatrix}$$

$\therefore \mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=3\mathbf{i}-13\mathbf{j}+2\mathbf{k}$

c $[\mathbf{r}-(4\mathbf{i}-2\mathbf{j}+\mathbf{k})]\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=0$

$$\therefore \mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -2 & 1\\ -1 & -2 & 3\end{vmatrix}$$

$$= -4\mathbf{i}-13\mathbf{j}-10\mathbf{k}$$

i.e. $\mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=-4\mathbf{i}-13\mathbf{j}-10\mathbf{k}$

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Vectors
Exercise D, Question 2

Question:

Find a Cartesian equation for each of the lines given in question 1.

Solution:

a $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{-2}=\lambda$

b $\frac{x-2}{1}=\frac{y}{1}=\frac{z+3}{5}=\lambda$

c $\frac{x-4}{-1}=\frac{y+2}{-2}=\frac{z-1}{3}=\lambda$

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Vectors

Exercise D, Question 3

Question:

Find, in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, an equation of the straight line passing through the points with coordinates

a (1, 3, 5), (6, 4, 2)

b (3, 4, 12), (4, 3, 5)

c $(-2, 2, 6), (3, 7, 11)$

d $(4, 2, -4), (1, 1, 1)$

Solution:

a The line is in the direction

$$\begin{pmatrix}6\\4\\2\end{pmatrix}-\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$$

In each question one solution is given but there are a number of alternatives. Either given point may be substituted for **a** and any multiple of the direction vector may be used as **b**.

The equation is $\left[\mathbf{r}-\begin{pmatrix}1\\3\\5\end{pmatrix}\right]\times\begin{pmatrix}5\\1\\-3\end{pmatrix}=0$

b The line is in the direction

$$\begin{pmatrix}4\\3\\5\end{pmatrix}-\begin{pmatrix}3\\4\\12\end{pmatrix}=\begin{pmatrix}1\\-1\\-7\end{pmatrix}$$

The equation is $\left[\mathbf{r}-\begin{pmatrix}3\\4\\12\end{pmatrix}\right]\times\begin{pmatrix}1\\-1\\-7\end{pmatrix}=0$

c The line is in the direction

$$\begin{pmatrix}3\\7\\11\end{pmatrix}-\begin{pmatrix}-2\\2\\6\end{pmatrix}=\begin{pmatrix}5\\5\\5\end{pmatrix}$$

The equation is $\left[\mathbf{r}-\begin{pmatrix}-2\\2\\6\end{pmatrix}\right]\times\begin{pmatrix}5\\5\\5\end{pmatrix}=0$

d The line is in the direction

$$\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}4\\2\\-4\end{pmatrix}=\begin{pmatrix}-3\\-1\\5\end{pmatrix}$$

The equation is $\left[\mathbf{r}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\times\begin{pmatrix}-3\\-1\\5\end{pmatrix}=0$

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Vectors
Exercise D, Question 4

Question:

Find a Cartesian equation for each of the lines given in question 3.

Solution:

a $\frac{x-1}{5}=\frac{y-3}{1}=\frac{z-5}{-3}=\lambda$

b $\frac{x-3}{1}=\frac{y-4}{-1}=\frac{z-12}{-7}=\lambda$

c $\frac{x+2}{5}=\frac{y-2}{5}=\frac{z-6}{5}=\lambda$ *or* as $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is also in the direction of the line

$x+2=y-2=z-6=\mu$

d $\frac{x-4}{3}=\frac{y-2}{1}=\frac{z+4}{-5}=\lambda$

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Vectors
Exercise D, Question 5

Question:

Find, in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, an equation of the straight line given by the equation, where λ is scalar

a $\mathbf{r}=\mathbf{i}+\mathbf{j}-2\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{k})$

b $\mathbf{r}=\mathbf{i}+4\mathbf{j}+\lambda(3\mathbf{i}+\mathbf{j}-5\mathbf{k})$

c $\mathbf{r}=3\mathbf{i}+4\mathbf{j}-4\mathbf{k}+\lambda(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})$

Solution:

a $\left[\mathbf{r}-(\mathbf{i}+\mathbf{j}-2\mathbf{k})\right]\times(2\mathbf{i}-\mathbf{k})=0$

b $\left[\mathbf{r}-(\mathbf{i}+4\mathbf{j})\right]\times(3\mathbf{i}+\mathbf{j}-5\mathbf{k})=0$

c $\left[\mathbf{r}-(3\mathbf{i}+4\mathbf{j}-4\mathbf{k})\right]\times(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})=0$

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Vectors
Exercise D, Question 6

Question:

Find, in the form

i $\mathbf{r}\times\mathbf{b}=\mathbf{c}$, and also in the form

ii $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, where t is a scalar parameter, the equation of the straight line with Cartesian equation $\frac{(x-3)}{2}=\frac{(y+1)}{5}=\frac{(2z-3)}{3}=\lambda$.

Solution:

When $\frac{x-3}{2}=\frac{y+1}{5}=\frac{2z-3}{3}=\lambda$

then $\frac{x-3}{2}=\frac{y+1}{5}=\frac{z-\frac{3}{2}}{\frac{3}{2}}=\lambda$

The direction of the line, $\mathbf{b}=2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}$

A point on the line has position vector

$\mathbf{a}=3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}$.

a $\therefore \mathbf{r}\times\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)=\left(3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}\right)\times\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & \frac{3}{2}\\ 2 & 5 & \frac{3}{2}\end{vmatrix}$$

i.e. $\mathbf{r}\times\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)=-9\mathbf{i}-\frac{3}{2}\mathbf{j}+17\mathbf{k}$

b $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}+t\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)$

or $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}+s(4\mathbf{i}+10\mathbf{j}+3\mathbf{k})$

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Vectors
Exercise D, Question 7

Question:

Given that the point with coordinates $(p, q, 1)$ lies on the line with equation

$$\mathbf{r}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$$, find the values of p and q.

Solution:

As $(p, q, 1)$ lies an the line with equation

$$\mathbf{r}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix} \text{ then } \begin{pmatrix}p\\q\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$$

Find the vector product of $\begin{pmatrix}p\\q\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and equate to $\begin{pmatrix}8\\-7\\-3\end{pmatrix}$.

But $$\begin{pmatrix}p\\q\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}3q-1\\2-3p\\p-2q\end{pmatrix}$$

$$\therefore \begin{pmatrix}3q-1\\2-3p\\p-2q\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$$

i.e. $3q-1=8 \Rightarrow q=3$

$2-3p=-7 \Rightarrow p=3$

i.e: $p=3$ and $q=3$

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Vectors
Exercise D, Question 8

Question:

Given that the equation of a straight line is $\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$,

find an equation for the line in the form $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, where t is a scalar parameter.

Hint: Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and set up simultaneous equations.

Solution:

The line with equation

$$\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

has direction $\mathbf{i}+\mathbf{j}-\mathbf{k}$, i.e. $\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

It passes through a point (a_1,a_2,a_3) where

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

i.e. $\begin{pmatrix}-a_2-a_3\\a_1+a_3\\a_1-a_2\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and set up simultaneous equations. These equations have an infinite number of solutions so let $a_1=0$ and find a_2 and a_3.

Let $a_1=0$, then as $a_1+a_3=2$ and $a_1-a_2=1$ this implies that $a_3=2$ and $a_2=-1$

$\therefore (0,-1,2)$ lies on the line.

So the line equation may be written as

$\mathbf{r}=-\mathbf{j}+2\mathbf{k}+t(\mathbf{i}+\mathbf{j}-\mathbf{k})$

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Vectors
Exercise E, Question 1

Question:

Find, in the form $\mathbf{r}\cdot\mathbf{n}=p$, an equation of the plane that passes through the point with position vector $\mathbf{a}$ and is perpendicular to the vector $\mathbf{n}$ where

a $\mathbf{a}=\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{n}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$

b $\mathbf{a}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{n}=5\mathbf{i}-\mathbf{j}-3\mathbf{k}$

c $\mathbf{a}=2\mathbf{i}-3\mathbf{k}$ and $\mathbf{n}=\mathbf{i}+3\mathbf{j}+4\mathbf{k}$

d $\mathbf{a}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{n}=4\mathbf{i}+\mathbf{j}-5\mathbf{k}$

Solution:

a
$$\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})=(\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$=2-1-1$$
i.e. $\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})=0$

b
$$\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}-3\mathbf{k})=(\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(5\mathbf{i}-\mathbf{j}-3\mathbf{k})$$
$$=5-2-3$$
i.e. $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}-3\mathbf{k})=0$

c
$$\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+4\mathbf{k})=(2\mathbf{i}-3\mathbf{k})\cdot(\mathbf{i}+3\mathbf{j}+4\mathbf{k})$$
$$=2-12$$
i.e. $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+4\mathbf{k})=-10$

d
$$\mathbf{r}\cdot(4\mathbf{i}+\mathbf{j}-5\mathbf{k})=(4\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}+\mathbf{j}-5\mathbf{k})$$
$$=16-2-5$$
i.e. $\mathbf{r}\cdot(4\mathbf{i}+\mathbf{j}-5\mathbf{k})=9$

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Vectors
Exercise E, Question 2

Question:

Find a Cartesian equation for each of the planes in question 1.

Solution:

a $2x + y + z = 0$
b $5x - y - 3z = 0$
c $x + 3y + 4z = -10$
d $4x + y - 5z = 9$

Replace $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in each equation.

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Vectors
Exercise E, Question 3

Question:

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ an equation of the plane that passes through the points

a $(1,2,0),(3,1,-1)$ and $(4, 3, 2)$
b $(3,4,1),(-1,-2,0)$ and $(2, 1, 4)$
c $(2,-1,-1),(3,1,2)$ and $(4, 0, 1)$
d $(-1,1,3),(-1,2,5)$ and $(0, 4, 4)$.

Solution:

a Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$
and $\mathbf{c} = (4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j}) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Choose one of the points to have position vector **a** then let the other two points have position vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$ respectively.

b Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \mathbf{b} = -\mathbf{i} - 2\mathbf{j} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$
and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
$\therefore \mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$
or $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda'(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

c Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k})$
$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
and $\mathbf{c} = 4\mathbf{i} + \mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\therefore \mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d Let $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
$= \mathbf{j} + 2\mathbf{k}$
and $\mathbf{c} = 4\mathbf{j} + 4\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$
$\therefore \mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

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Vectors
Exercise E, Question 4

Question:

Find a Cartesian equation for each of the planes in question 3.

Solution:

a Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{vmatrix} = -\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$

Find the equation in the form $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ using $\mathbf{a}$ from question 3 and finding $\mathbf{n} = \mathbf{b}\times\mathbf{c}$ from question 3.

$\therefore$ Equation is $\mathbf{r}\cdot(-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}) = (\mathbf{i} + 2\mathbf{j})\cdot(-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$

In Cartesian form: $-x - 7y + 5z = -15$

or $x + 7y - 5z = 15$

b Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & 1 \\ -1 & -3 & 3 \end{vmatrix} = 21\mathbf{i} - 13\mathbf{j} - 6\mathbf{k}$

$\therefore$ Equation is $\mathbf{r}\cdot(21\mathbf{i} - 13\mathbf{j} - 6\mathbf{k}) = (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})\cdot(21\mathbf{i} - 13\mathbf{j} - 6\mathbf{k})$

i.e. $21x - 13y - 6z = 63 - 52 - 6$

i.e. $21x - 13y - 6z = 5$

c Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

$\therefore$ Equation is $\mathbf{r}\cdot(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) = (2\mathbf{i} - \mathbf{j} - \mathbf{k})\cdot(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

$= 2 - 4 + 3$

i.e. $x + 4y - 3z = 1$

d Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = -5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\therefore$ Equation is $\mathbf{r}\cdot(-5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})\cdot(-5\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$= 5 + 2 - 3$

i.e. $-5x + 2y - z = 4$

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Vectors
Exercise E, Question 5

Question:

Find a Cartesian equation of the plane that passes through the points

a $(0, 4, 2)$, $(1, 1, 2)$ and $(-1, 5, 0)$

b $(1, 1, 0)$, $(2, 3, -3)$ and $(3, 7, -2)$

c $(3, 0, 0)$, $(2, 0, -1)$ and $(4, 1, 3)$

d $(1, -1, 6)$, $(3, 1, -2)$ and $(4, 1, 0)$.

Solution:

a $\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}1\\-3\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\5\\0\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\-2\end{pmatrix}$

are two directions in the plane.

> Find two directions in the plane and take their vector product to give a normal to the plane.

The normal to the plane is $\mathbf{n}$ where $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -3 & 0\\ -1 & 1 & -2\end{vmatrix}$

i.e: $\mathbf{n}=+6\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ is also normal to plane.

$\therefore$ Equation of plane is $\mathbf{r}\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k})=(4\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k})$

i.e. $3x+y-z=2$

b $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}-(\mathbf{i}+\mathbf{j})=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+7\mathbf{j}-2\mathbf{k}-(\mathbf{i}+\mathbf{j})=2\mathbf{i}+6\mathbf{j}-2\mathbf{k}$ are two directions in the plane.

The normal to the plane is $\mathbf{n}$ where $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -3\\ 2 & 6 & -2\end{vmatrix}$

i.e: $\mathbf{n}=14\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $7\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is also a normal.

$\therefore$ Equation of plane is

$\mathbf{r}\cdot(7\mathbf{i}-2\mathbf{j}+\mathbf{k})=(\mathbf{i}+\mathbf{j})\cdot(7\mathbf{i}-2\mathbf{j}+\mathbf{k})$

i.e. $7x-2y+z=5$

c $(2\mathbf{i}-\mathbf{k})-(3\mathbf{i})=-\mathbf{i}-\mathbf{k}$ and $(4\mathbf{i}+\mathbf{j}+3\mathbf{k})-3\mathbf{i}=\mathbf{i}+\mathbf{j}+3\mathbf{k}$ are two directions in the plane.

The normal to the plane is $\mathbf{n}$ where $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 0 & -1\\ 1 & 1 & 3\end{vmatrix}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$

The equation an of the plane is

$\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3\mathbf{i}\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$\therefore x+2y-z\quad=3$

d Two directions in the plane are:

$3\mathbf{i}+\mathbf{j}-2\mathbf{k}-(\mathbf{i}-\mathbf{j}+6\mathbf{k})=2\mathbf{i}+2\mathbf{j}-8\mathbf{k}$ and

$4\mathbf{i}+\mathbf{j}-(\mathbf{i}-\mathbf{j}+6\mathbf{k})=3\mathbf{i}+2\mathbf{j}-6\mathbf{k}$

The normal to the plane is $\mathbf{n}$ where

$$\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & -8\\ 3 & 2 & -6\end{vmatrix}=4\mathbf{i}-12\mathbf{j}-2\mathbf{k}$$

The equation of the plane is

$\mathbf{r}\cdot(4\mathbf{i}-12\mathbf{j}-2\mathbf{k})=(\mathbf{i}-\mathbf{j}+6\mathbf{k})\cdot(4\mathbf{i}-12\mathbf{j}-2\mathbf{k})$

$=4$

i.e. $4x-12y-2z=4$ *or* $2x-6y-z=2$

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Vectors
Exercise E, Question 6

Question:

Find, in the form $\mathbf{r} \cdot \mathbf{n} = p$, an equation of the plane which contains the line l and the point with position vector $\mathbf{a}$ where

a l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{k})$ and $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

b l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ and $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$

c l has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{a} = 7\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

Solution:

a The line has direction $2\mathbf{i}-\mathbf{k}$, and this is a direction in the plane.

Another vector in the plane is $4\mathbf{i}+3\mathbf{j}+\mathbf{k}-(\mathbf{i}+\mathbf{j}-2\mathbf{k})$

i.e. $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

The equation of the line includes the position vector of another point on the plane and includes a direction vector in the plane.

The normal to the plane is in direction

$(2\mathbf{i}-\mathbf{k})\times(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

i.e. $\mathbf{n}=2\mathbf{i}-9\mathbf{j}+4\mathbf{k}$

$\therefore$ The plane has equation

$$\mathbf{r}\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=(4\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})$$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=8-27+4$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=-15$

b The line has direction $(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$

Another vector in the plane is $3\mathbf{i}+5\mathbf{j}+\mathbf{k}-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

i.e: $(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$

$\therefore$ the normal to the plane is $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -3\\ 2 & 3 & -1\end{vmatrix}=8\mathbf{i}-4\mathbf{j}+4\mathbf{k}$

$\therefore$ Equation of the plane is

$$\mathbf{r}\cdot(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})=(3\mathbf{i}+5\mathbf{j}+\mathbf{k})\cdot(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})$$
$$=24-20+4$$
$$=8$$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=2$

c $7\mathbf{i}+8\mathbf{j}+6\mathbf{k}$ is the position vector of a point on the plane.

You need 2 directions in the plane. One is the direction of the line. The other is the vector joining the two points that are given in the plane i.e. (7, 8, 6) and $(2,-1,1)$.

$2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is the position vector of another point on the plane.

The vector joining these points is $5\mathbf{i}+9\mathbf{j}+5\mathbf{k}$

This lies in the plane.

A second vector which lies in the plane is $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

The normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 9 & 5\\ 1 & 2 & 2\end{vmatrix}$

i.e: $\mathbf{n}=8\mathbf{i}-5\mathbf{j}+\mathbf{k}$

$\therefore$ The equation of the plane is

$$\mathbf{r}\cdot(8\mathbf{i}-5\mathbf{j}+\mathbf{k})=(7\mathbf{i}+8\mathbf{j}+6\mathbf{k})\cdot(8\mathbf{i}-5\mathbf{j}+\mathbf{k})$$
$$=56-40+6$$

$\therefore \mathbf{r}\cdot(8\mathbf{i}-5\mathbf{j}+\mathbf{k})=22$

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Vectors
Exercise E, Question 7

Question:

Find a Cartesian equation of the plane which passes through the point (1, 1, 1) and contains the line with equation $\frac{x-2}{3}=\frac{y+4}{1}=\frac{z-1}{2}$.

Solution:

The line is in the direction $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. This lies in the plane.

$(2,-4,1)$ is a point on the line. This also lies in the plane, as does the point (1, 1, 1).

$\therefore \begin{pmatrix}2\\-4\\1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-5\\0\end{pmatrix}$ is a direction in the plane.

First obtain the equation of the plane in the form $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, then convert to Cartesian form.

The normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-5&0\\3&1&2\end{vmatrix}$

$=-10\mathbf{i}-2\mathbf{j}+16\mathbf{k}$

$\therefore$ The equation of the plane is

$\mathbf{r}\cdot(-10\mathbf{i}-2\mathbf{j}+16\mathbf{k})=(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(-10\mathbf{i}-2\mathbf{j}+16\mathbf{k})$

i.e: $-10x-2y+16z=4$

This is a Cartesian equation of the plane.

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Vectors
Exercise F, Question 1

Question:

In each case establish whether lines l_1 and l_2 meet and if they meet find the coordinates of their point of intersection:

a l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b l_1 has equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mu(-\mathbf{i} + \mathbf{j} - \mathbf{k})$

c l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and l_2 has equation $\mathbf{r} = \mathbf{i} + 2\frac{1}{2}\mathbf{j} + 2\frac{1}{2}\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

(In each of the above cases λ and μ are scalars.)

Solution:

a The line l_1 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$$

Use column vector form for clarity. Put the two equations equal and compare x, y and z components. Then solve simultaneous equations.

and the line l_2 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

These lines meet when

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

i.e. $1 + \lambda = -1 + \mu$ ①

$3 - \lambda = -3 + \mu$ ②

$5\lambda = 2 + 2\mu$ ③

Add equations ① and ②

$4 = -4 + 2\mu$

$\therefore 2\mu = 8$

i.e: $\mu = 4$

Substitute into equation ①

$\therefore 1 + \lambda = -1 + 4$

i.e: $\lambda = 2$

Substitute $\lambda = 2$ into equation for line l_1

$\therefore (x, y, z) = (3, 1, 10)$

Substitute $\mu = 4$ into equation for line l_2

$\therefore (x, y, z) = (3, 1, 10)$

So the two lines do meet at the point (3, 1, 10)

b l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

These lines meet when $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

i.e. $3 + \lambda = 4 - \mu$ ①

$2 + \lambda = 3 + \mu$ ②

$1 + 2\lambda = -\mu$ ③

Add equations ① and ②

$\therefore 5 + 2\lambda = 7$

i.e: $\lambda = 1$

Substitute into equation ①

$\therefore 3 + 1 = 4 - \mu$

i.e: $\mu = 0$

Substitute $\lambda = 1$ into equation for line l_1:

$\therefore (x, y, z) = (4, 3, 3)$

Substitute $\mu = 0$ into line l_2:

$\therefore (x, y, z) = (4, 3, 0)$

This is a contradiction and the lines do not meet.

[N.B. $\lambda = 1$ and $\mu = 0$ do not satisfy equation ③ above.]

c l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

l_1 meets l_2 when $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

i.e. $1 + 2\lambda = 1 + \mu$ ①

$3 + 3\lambda = 2\frac{1}{2} + \mu$ ②

$5 + \lambda = 2\frac{1}{2} - 2\mu$ ③

Subtract equation ① from equation ②

$\therefore 2 + \lambda = 1\frac{1}{2}$

i.e. $\lambda = -\frac{1}{2}$

Substitute into equation ①

$\therefore 1 - 1 = 1 + \mu$

i.e. $\mu = -1$

Substitute $\lambda = -\frac{1}{2}$ into equation for line l_1:

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

Substitute $\mu = -1$ into equation for line l_2::

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

So the two lines do meet at the point $\left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$.

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Vectors
Exercise F, Question 2

Question:

In each case establish whether the line l meets the plane Π and, if they meet, find the coordinates of their point of intersection.

a $l : \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
$\Pi : \mathbf{r} \cdot (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 16$

b $l : \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\Pi : \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1$

c $l : \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - 2\mathbf{k})$
$\Pi : \mathbf{r} \cdot (3\mathbf{i} - \mathbf{j} - 6\mathbf{k}) = 1$

(In each of the above cases λ is a scalar.)

Solution:

a The line meets the plane when

$$\left[(1-2\lambda)\mathbf{i}+(1+\lambda)\mathbf{j}+(1-4\lambda)\mathbf{k}\right]\cdot(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=16$$

i.e. $3(1-2\lambda)-4(1+\lambda)+2(1-4\lambda)=16$

$\therefore 3-6\lambda-4-4\lambda+2-8\lambda=16$

$\therefore 1-18\lambda=16$

i.e. $-18\lambda=15$

$\therefore \lambda=-\frac{15}{18}$

i.e. $\lambda=\frac{-5}{6}$

Assume that the line meets the plane and perform the scalar product. Solve the resulting equation to find the value of λ. If there is no value for λ, then the line does not meet the plane.

Substitute into the equation of the line

$$\therefore (x,y,z)=\left(1+\frac{10}{6},1-\frac{5}{6},1+\frac{20}{6}\right)$$

$$=\left(2\frac{2}{3},\frac{1}{6},4\frac{1}{3}\right)$$

b The line meets the plane when

$$\left[(2+\lambda)\mathbf{i}+(3+\lambda)\mathbf{j}+(-2+\lambda)\mathbf{k}\right]\cdot(\mathbf{i}+\mathbf{j}-2\mathbf{k})=1$$

i.e. $(2+\lambda)+(3+\lambda)-2(-2+\lambda)=1$

$\therefore 2+\lambda+3+\lambda+4-2\lambda=1$

$\therefore 9=1$

This is a contradiction.
There are no values of λ for which the line meets the plane.
The line is parallel to the plane.

c The line meets the plane when

$$\left[\mathbf{i}+(1+2\lambda)\mathbf{j}+(1-2\lambda)\mathbf{k}\right]\cdot(3\mathbf{i}-\mathbf{j}-6\mathbf{k})=1$$

i.e. $3-(1+2\lambda)-6(1-2\lambda)=1$

i.e. $3-1-2\lambda-6+12\lambda=1$

$\therefore 10\lambda-4=1$

$\therefore \lambda=\frac{1}{2}$

Substitute into the equation of the line

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise F, Question 3

Question:

Find the equation of the line of intersection of the planes Π_1 and Π_2 where

a Π_1 has equation $\mathbf{r}\cdot(3\mathbf{i}-2\mathbf{j}-\mathbf{k})=5$ and Π_2 has equation $\mathbf{r}\cdot(4\mathbf{i}-\mathbf{j}-2\mathbf{k})=5$

b Π_1 has equation $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}-2\mathbf{k})=16$ and Π_2 has equation $\mathbf{r}\cdot(16\mathbf{i}-5\mathbf{j}-4\mathbf{k})=53$

c Π_1 has equation $\mathbf{r}\cdot(\mathbf{i}-3\mathbf{j}+\mathbf{k})=10$ and Π_2 has equation $\mathbf{r}\cdot(4\mathbf{i}-3\mathbf{j}-2\mathbf{k})=1$.

Solution:

a The planes have equations

$3x - 2y - z = 5$ and ①

$4x - y - 2z = 5$ ②

Express the equations of the planes in Cartesian form then eliminate one of the variables (x, y or z) from the equations.

Multiply ① by 2 then subtract ②

$\therefore 2x - 3y = 5$

$\therefore x = \dfrac{5+3y}{2}$

Substitute this into ①

$\therefore 3\dfrac{(5+3y)}{2} - 2y - z = 5$

$\therefore z = 3\dfrac{(5+3y)}{2} - 2y - 5$

$= \dfrac{5+5y}{2}$

Let $y = \lambda$.

Then $x = \dfrac{5+3\lambda}{2}$ and $z = \dfrac{5+5\lambda}{2}$

i.e. $\lambda = \dfrac{x - \frac{5}{2}}{\frac{3}{2}}$ and $\lambda = \dfrac{z - \frac{5}{2}}{\frac{5}{2}}$

$\therefore$ Equation of the line of intersection is

$$\frac{x - \frac{5}{2}}{\frac{3}{2}} = y = \frac{z - \frac{5}{2}}{\frac{5}{2}} = \lambda$$

or $\mathbf{r} = \left(\frac{5}{2}\mathbf{i} + \frac{5}{2}\mathbf{k}\right) + \lambda\left(\frac{3}{2}\mathbf{i} + \mathbf{j} + \frac{5}{2}\mathbf{k}\right)$

b The planes have equations

$5x - y - 2z = 16$ ①

and $16x - 5y - 4z = 53$ ②

Express the equations of the planes in Cartesian form then eliminate one of the variables (x, y or z) from the equations.

Multiply equation ① by 5 then subtract equation ②

$\therefore 9x - 6z = 27$

$\therefore x = \dfrac{27+6z}{9} = \dfrac{9+2z}{3}$

Substitute into equation ①

Then $5\dfrac{(9+2z)}{3} - y - 2z = 16$

$\therefore y = 5\dfrac{(9+2z)}{3} - 2z - 16$

$= \dfrac{4z-3}{3}$

Let $z = \lambda$

Then $x = \dfrac{9+2\lambda}{3}$ and $y = \dfrac{4\lambda - 3}{3}$ and $z = \lambda$

$$\therefore \frac{x-3}{\frac{2}{3}} = \frac{y+1}{\frac{4}{3}} = z = \lambda$$

This is the equation of the line of intersection.
In vector form:

$$\mathbf{r} = (3\mathbf{i} - \mathbf{j}) + \lambda\left(\frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \mathbf{k}\right)$$

c The planes have equations

$$x - 3y + z = 10 \quad ①$$

and $4x - 3y - 2z = 1 \quad ②$

Express the equations of the planes in Cartesian form then eliminate one of the variables (x, y or z) from the equations.

Subtract equation ① from equation ②

$\therefore 3x - 3z = -9$

$\therefore x = z - 3$

Substitute into equation ①

$\therefore z - 3 - 3y + z = 10$

i.e. $3y = 2z - 13$

$$\therefore y = \frac{2z-13}{3}$$

Let $z = \lambda$

Then $x = \lambda - 3$ and $y = \dfrac{2\lambda - 13}{3}$ and $z = \lambda$

$$\therefore \frac{x+3}{1} = \frac{y + \frac{13}{3}}{\frac{2}{3}} = z = \lambda$$

This is the Cartesian form of the equation of the line of intersection.
The vector form is

$$\mathbf{r} = \left(-3\mathbf{i} - \frac{13}{3}\mathbf{j}\right) + \lambda\left(\mathbf{i} + \frac{2}{3}\mathbf{j} + \mathbf{k}\right)$$

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Vectors
Exercise F, Question 4

Question:

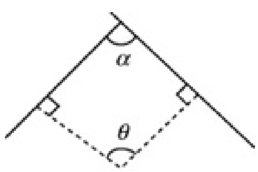
Find the acute angle between the planes with equations $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=1$ and $\mathbf{r}\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})=7$ respectively.

Solution:

The angle θ between the two normal vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ is given by

$$\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})}{|\mathbf{i}+2\mathbf{j}-2\mathbf{k}||-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}|}$$
$$= \frac{-4+8-14}{\sqrt{1^2+2^2+(-2)^2}\sqrt{(-4)^2+4^2+7^2}}$$
$$= \frac{-10}{\sqrt{9}\sqrt{81}}$$
$$= -\frac{10}{27}$$

First find the angle between the two normal vectors.



The acute angle, α, between the two planes is such that

$\alpha+\theta=180°$

So $\cos\alpha = -\cos\theta$

$$= \frac{10}{27}$$

$\therefore \alpha = 68.3°$ (3 s.f.)

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Vectors
Exercise F, Question 5

Question:

Find the acute angle between the planes with equations $\mathbf{r}\cdot(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=9$ and $\mathbf{r}\cdot(5\mathbf{i}-12\mathbf{k})=7$ respectively.

Solution:

The angle θ between the two normal vectors $3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ and $5\mathbf{i}-12\mathbf{k}$ is given by

$$\cos\theta=\frac{(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot(5\mathbf{i}-12\mathbf{k})}{|3\mathbf{i}-4\mathbf{j}+12\mathbf{k}||5\mathbf{i}-12\mathbf{k}|}$$

First find the angle between the two normal vectors.

$$=\frac{15-144}{\sqrt{3^2+(-4)^2+12^2}\sqrt{5^2+(-12)^2}}$$

$$=\frac{-129}{\sqrt{169}\sqrt{169}}$$

$$=\frac{-129}{169}$$

The acute angle α between the planes is such that $\alpha+\theta=180°$

So $\cos\alpha=-\cos\theta=\frac{129}{169}$

$\therefore \alpha=40.2°\quad$ (3 s.f.)

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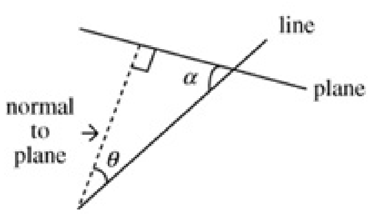
Vectors
Exercise F, Question 6

Question:

Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k})$ and the plane with equation $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 13$.

Solution:



Find the acute angle between the given line and the normal to the plane, then subtract from $90°$.

Let θ be the acute angle between the line and the normal to the plane.

$$\text{Then } \cos\theta = \left|\frac{(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{4^2+4^2+7^2}\sqrt{2^2+1^2+(-2)^2}}\right|$$

$$= \left|\frac{8+4-14}{\sqrt{81}\sqrt{9}}\right|$$

$$= \left|\frac{-2}{27}\right| = \frac{2}{27}$$

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

So $\sin\alpha = \cos\theta = \frac{2}{27}$

$\therefore \alpha = 4.25°$ (3 s.f.)

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Vectors
Exercise F, Question 7

Question:

Find the acute angle between the line with equation $\mathbf{r} = -\mathbf{i} - 7\mathbf{j} + 13\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and the plane with equation $\mathbf{r} \cdot (4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}) = 9$.

Solution:

Let θ be the acute angle between the line and the normal to the plane.

Then $\cos\theta = \dfrac{(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) \cdot (4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k})}{\sqrt{3^2 + 4^2 + (-12)^2}\sqrt{4^2 + (-4)^2 + (-7)^2}}$

$= \dfrac{12 - 16 + 84}{\sqrt{169}\sqrt{81}}$

$= \dfrac{80}{13 \times 9}$

$= \dfrac{80}{117}$

Find the acute angle between the given line and the normal to the plane, then subtract from $90°$.

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

So $\sin\alpha = \cos\theta = \dfrac{80}{117}$

$\therefore \alpha = 43.1°$ (3 s.f.)

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Vectors
Exercise F, Question 8

Question:

Find the acute angle between the line with equation $(\mathbf{r}-3\mathbf{j})\times(-4\mathbf{i}-7\mathbf{j}+4\mathbf{k})=0$ and the plane with equation $\mathbf{r}=\lambda(4\mathbf{i}-\mathbf{j}-\mathbf{k})+\mu(4\mathbf{i}-5\mathbf{j}+3\mathbf{k})$.

Solution:

First find a normal $\mathbf{n}$ to the plane

$$\mathbf{n}=(4\mathbf{i}-\mathbf{j}-\mathbf{k})\times(4\mathbf{i}-5\mathbf{j}-3\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -1 & -1\\ 4 & -5 & 3\end{vmatrix}$$

$$=-8\mathbf{i}-16\mathbf{j}-16\mathbf{k}$$

So a simple normal to the plane is $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

Let θ be the acute angle between the line and the normal to the plane,

Then $\cos\theta=\left|\dfrac{(-4\mathbf{i}-7\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\sqrt{(-4)^2+(-7)^2+4^2}\sqrt{1^2+2^2+2^2}}\right|=\left|\dfrac{-4-14+8}{9\times3}\right|$

Let α be the angle between the line and the plane.

Then $\theta+\alpha=90°$, so $\sin\alpha=\cos\theta=\dfrac{10}{27}$.

$\therefore \alpha=21.7°$ (3 s.f.)

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Vectors
Exercise F, Question 9

Question:

The plane Π has equation $\mathbf{r}\cdot(10\mathbf{j}+10\mathbf{j}+23\mathbf{k})=81$.

a Find the perpendicular distance from the origin to plane Π.
b Find the perpendicular distance from the point $(-1,-1,4)$ to the plane Π.
c Find the perpendicular distance from the point (2, 1, 3) to the plane Π.
d Find the perpendicular distance from the point $(6,12,-9)$ to the plane Π.

Solution:

a The length of the normal vector $10\mathbf{i}+10\mathbf{j}+23\mathbf{k}$ is $\sqrt{10^2+10^2+23^2}=\sqrt{729}=27$

$\therefore \frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$ is a unit vector normal to the plane.

The plane has equation

$$\mathbf{r}\cdot(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=81$$

or $\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=\frac{81}{27}=3$

$\therefore$ The perpendicular distance from the origin to the plane is 3.

b A plane parallel to π through the point $(-1,-1,4)$ has equation

$$\begin{aligned}\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) &= (-\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})\\ &= \frac{-10}{27}-\frac{10}{27}+\frac{92}{27}\\ &= \frac{72}{27}\\ &= \frac{8}{3}\end{aligned}$$

$\therefore$ The perpendicular distance from the origin to this new plane is $2\frac{2}{3}$.

The distance between the planes is $3-2\frac{2}{3}=\frac{1}{3}$

$\therefore$ The perpendicular distance from the point $(-1,-1,4)$ to the plane π is $\frac{1}{3}$.

c A plane parallel to π through the point (2, 1, 3) has equation

$$\begin{aligned}\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) &= (2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})\\ &= \frac{20}{27}+\frac{10}{27}+\frac{69}{27}\\ &= \frac{99}{27}\\ &= \frac{11}{3}\end{aligned}$$

$\therefore$ The perpendicular distance from the origin to this new plane is $3\frac{2}{3}$

$\therefore$ The distances between this plane and π is $3\frac{2}{3}-3=\frac{2}{3}$

$\therefore$ The perpendicular distance from (2, 1, 3) to π is $\frac{2}{3}$.

d A plane parallel to π through the point $(6, 12, -9)$ has equation

$$\mathbf{r} \cdot \frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (6\mathbf{i}+12\mathbf{j}-9\mathbf{k}) \cdot \frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$

$$= \frac{60}{27}+\frac{120}{27}-\frac{207}{27}$$

$$= -\frac{27}{27}$$

$$= -1$$

$\therefore$ The perpendicular distance from the origin to this new plane is 1, in the opposite direction.

$\therefore$ The distance between this plane and π is $3-(-1)=4$

$\therefore$ The perpendicular distance from (2, 1, 3) to π is 4.

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Vectors
Exercise F, Question 10

Question:

Find the shortest distance between the parallel planes.

a $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=55$ and $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=22$.

b $\mathbf{r}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{k})+\mu(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$ and
$\mathbf{r}=14\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{j}+\mathbf{k})+\mu(8\mathbf{i}-9\mathbf{j}-\mathbf{k})$

Solution:

a The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=55$ is $\dfrac{55}{\sqrt{6^2+6^2+(-7)^2}}$

$$=\frac{55}{\sqrt{121}}$$
$$=\frac{55}{11}$$
$$=5$$

First find the distance from the origin to each plane, then subtract.

The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=22$ is $\dfrac{22}{\sqrt{6^2+6^2+(-7)^2}}$

$$=\frac{22}{11}$$
$$=2$$

$\therefore$ The distance between the planes is $5-2=3$

b $\mathbf{r}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{k})+\mu(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$

The normal to the plane is $\mathbf{n}$ where

Express the equations of the planes in the form $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

$$\mathbf{n}=(4\mathbf{i}+\mathbf{k})\times(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&0&1\\8&3&3\end{vmatrix}=-3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$$

$\therefore$ Equation of plane may be written

$$\mathbf{r}\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=(3\mathbf{i}+4\mathbf{j}+\mathbf{k})\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$$

i.e. $\mathbf{r}\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=-13$

The distance from the origin to this plane is $\dfrac{-13}{\sqrt{(-3)^2+(-4)^2+12^2}}=-1$

The second plane

$\mathbf{r}=14\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{j}+\mathbf{k})+\mu(8\mathbf{i}-9\mathbf{j}-\mathbf{k})$ has normal $\mathbf{n}$ where

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&3&1\\8&-9&-1\end{vmatrix}=6\mathbf{i}+8\mathbf{j}-24\mathbf{k}$$

This shows it is parallel to the first plane as the normal vectors are parallel.

$\therefore$ Equation of second plane may be written

$$\mathbf{r}\cdot(6\mathbf{i}+8\mathbf{j}-24\mathbf{k})=(14\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(6\mathbf{i}+8\mathbf{j}-24\mathbf{k})$$

i.e. $\mathbf{r}\cdot(6\mathbf{i}+8\mathbf{j}-24\mathbf{k})=52$ or $\mathbf{r}\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=-26$

The distance from the origin to this plane is $\dfrac{-26}{\sqrt{3^2+4^2+(-12)^2}}=-2$

$\therefore$ The distance between the two planes is $-1-(-2)=1$.

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Vectors
Exercise F, Question 11

Question:

Find the shortest distance between the two skew lines with equations $\mathbf{r} = \mathbf{i} + \lambda(-3\mathbf{i} - 12\mathbf{j} + 11\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k})$, where λ and μ are scalars.

Solution:

The shortest distance is found by using the formula $\left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$.

Use the formula for shortest distance between skew lines.

$$\mathbf{a} - \mathbf{c} = \mathbf{i} - (3\mathbf{i} - \mathbf{j} + \mathbf{k}) = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\mathbf{b} \times \mathbf{d} = (-3\mathbf{i} - 12\mathbf{j} + 11\mathbf{k}) \times (2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -12 & 11 \\ 2 & 6 & -5 \end{vmatrix}$$

$$= -6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$$\therefore \text{shortest distance} = \left|\frac{(-2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(-6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})}{\sqrt{(-6)^2 + 7^2 + 6^2}}\right|$$

$$= \left|\frac{12 + 7 - 6}{\sqrt{121}}\right|$$

$$= \frac{13}{11}$$

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Vectors
Exercise F, Question 12

Question:

Find the shortest distance between the parallel lines with equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$, where λ and μ are scalars.

Solution:

Let A be a general point on the first line and B be a general point on the second line,

then $\overrightarrow{AB} = \begin{pmatrix} -2 \\ +2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$, where $t = \mu - \lambda$.

Find the minimum value of the quadratic by using calculus, or completion of the square.

Let the distance $AB = x$ then

$$x^2 = (-2-3t)^2 + (2-4t)^2 + (5t)^2$$
$$= 8 - 4t + 50t^2$$

The minimum value of x^2 occurs when $t = \frac{1}{25}$.

So $x^2 = 8 - \frac{4}{25} + \frac{50}{625}$

$$= \frac{198}{25}$$

$$\therefore x = \frac{\sqrt{198}}{5} \text{ or } 2.81 \quad (3\text{ s.f.})$$

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Vectors
Exercise F, Question 13

Question:

Determine whether the lines l_1 and l_2 meet. If they do, find their point of intersection. If they do not, find the shortest distance between them. (In each of the following cases λ and μ are scalars.)

a l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})$

b l_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

c l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Solution:

a Assume that l_1 and l_2 meet.

Then $\begin{pmatrix} 1+2\lambda \\ 1-\lambda \\ 5\lambda \end{pmatrix} = \begin{pmatrix} -1+2\mu \\ 1-5\mu \\ 2+\mu \end{pmatrix}$

i.e.

$1+2\lambda = -1+2\mu$ ①

$1-\lambda = 1-5\mu$ ②

$5\lambda = 2+\mu$ ③

Add ① $+2\times$ ②

$\therefore 3 = 1-8\mu$

i.e. $\mu = -\frac{1}{4}$

Substitute into equation ①

$\therefore 1+2\lambda = -1\frac{1}{2}$

$\therefore \lambda = -1\frac{1}{4}$

But for these values of λ and μ equation ③ does not hold true. There is a contradiction.

$\therefore$ The lines do not meet.

They must be skew so the shortest distance between them is calculated from the formula

$\left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$ where $\mathbf{a}-\mathbf{c} = 2\mathbf{i}-2\mathbf{k}$ and

$$\mathbf{b}\times\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ 2 & -5 & 1 \end{vmatrix}$$

$$= 24\mathbf{i}+8\mathbf{j}-8\mathbf{k}$$

$$\therefore \text{Distance} = \left|\frac{2\mathbf{i}-2\mathbf{k}\cdot(24\mathbf{i}+8\mathbf{j}-8\mathbf{k})}{8\sqrt{3^2+1^2+(-1)^2}}\right| = \frac{32}{8\sqrt{11}} = \frac{4\sqrt{11}}{11} \text{ or } 1.21$$

b Assume that l_1 and l_2 meet:

$\begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ -1-\mu \\ 3+\mu \end{pmatrix}$

i.e. $2+2\lambda = 1+\mu$ ①

$1-2\lambda = -1-\mu$ ②

$-2+2\lambda = 3+\mu$ ③

Adding equations ① and ② gives $3=0$

This is a contradiction.

$\therefore$ Lines do not meet.

The lines are in fact parallel as $2\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ is a multiple of $\mathbf{i}-\mathbf{j}+\mathbf{k}$.

The distance between them is found by considering A on line l_1 and B on line l_2.

Then $\overrightarrow{AB} = -\mathbf{i}-2\mathbf{j}+5\mathbf{k}+t(\mathbf{i}-\mathbf{j}+\mathbf{k})$

$$|\overrightarrow{AB}|^2 = x^2 = (-1+t)^2+(-2-t)^2+(5+t)^2$$
$$= 1-2t+t^2+4+4t+t^2+25+10t+t^2$$
$$= 30+12t+3t^2$$

The minimum value of x^2 occurs when $\dfrac{\mathrm{d}(x^2)}{\mathrm{d}t}=0$

$$\frac{\mathrm{d}(x)^2}{\mathrm{d}t}=12+6t$$

When $\dfrac{\mathrm{d}(x)^2}{\mathrm{d}t}=0, t=-2$

$$\therefore x^2 = 30-24+12$$
$$= 18$$
$$\therefore x = \sqrt{18} = 3\sqrt{2} \text{ or } 4.24 \text{ (3 s.f.)}$$

c Let l_1 meet l_2, then

$$\begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 5-2\lambda \end{pmatrix} = \begin{pmatrix} -1+\mu \\ -1+\mu \\ 2+\mu \end{pmatrix} \quad \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

Subtract ① − ②

Then $\lambda = 0$

Substitute into equation ①

Then $\mu = 2$

But $\lambda=0, \mu=2$ does not satisfy equation ③

So the lines do not meet.

They are skew.

Using the formula distance $= \left|\dfrac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

$$\mathbf{a}-\mathbf{c} = 2\mathbf{i}+2\mathbf{j}+3\mathbf{k}$$
$$\mathbf{b}\times\mathbf{d} = (2\mathbf{i}+\mathbf{j}-2\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$= 3\mathbf{i}-4\mathbf{j}+\mathbf{k}$$

$$\therefore \text{shortest distance} = \left|\frac{(2\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\sqrt{3^2+(-4)^2+1^2}}\right|$$
$$= \frac{6-8+3}{\sqrt{26}}$$
$$= \frac{1}{\sqrt{26}}$$
$$= 0.196 \text{ (3 s.f.)}$$

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Vectors
Exercise F, Question 14

Question:

Find the shortest distance between the point with coordinates $(4,1,-1)$ and the line with equation
$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, where μ is a scalar.

Solution:

Let A be the point $(4,1,-1)$ and B be the point $(3+2t,-1-t,2-t)$ which lies on the line.

Find the distance between $(4,1,-1)$ and $(3+2t,-1-t,2-t)$ at a point on the line.

Then $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

$$= [4-(3+2t), 1-(-1-t), -1-(2-t)]$$

$$= [1-2t, 2+t, -3+t]$$

$$\therefore |\overrightarrow{BA}|^2 = (1-2t)^2 + (2+t)^2 + (-3+t)^2$$

$$= 6t^2 - 6t + 14$$

$|\overrightarrow{BA}|$ is a minimum when $|\overrightarrow{BA}|^2$ is minimum

This minimum value can be found by calculus or completion of the square.

$$|\overrightarrow{BA}|^2 = 6(t^2 - t) + 14$$

$$= 6\left(t - \frac{1}{2}\right)^2 + 14 - \frac{6}{4}$$

This is a minimum when $t = \frac{1}{2}$ and

$$|\overrightarrow{BA}|^2 = 14 - 1\frac{1}{2} = 12\frac{1}{2}$$

$$\therefore |\overrightarrow{BA}| = \sqrt{12\frac{1}{2}} = 3.54 \text{ (3 s.f.)}$$

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Vectors
Exercise F, Question 15

Question:

The plane Π has equation $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=4$.

a Show that the line with equation $\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ lies in the plane Π.

b Show that the line with equation $\mathbf{r}=-\mathbf{i}+2\mathbf{j}+4\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ is parallel to the plane Π and find the shortest distance from the line to the plane.

Solution:

a The line $\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ passes through the point (2, 3, 1).
The point (2, 3, 1) also lies on the plane $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=4$ as $2\times1+3\times1-1=4$.
So the line and plane have a point in common.
The line is in the direction $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$.
This direction is parallel to the plane as it is perpendicular to the normal $\mathbf{i}+\mathbf{j}-\mathbf{k}$, as $-1\times1+2\times1+1\times-1=0$.
As the line also has a common point with the plane it lies in the plane.

Check that the line is perpendicular to the normal to the plane and check that the line and plane have a common point.

b The line $\mathbf{r}=-\mathbf{i}+2\mathbf{j}+4\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ is also parallel to the plane as its direction is $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$ which is perpendicular to the normal to the plane (see a).
The point $(-1,2,4)$ lies on the line. It does not lie on the plane as

$$(-\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k}) = -1+2-4 = -3 \neq 4$$

Show that there is a point on the line which does not lie on the plane.

$\therefore$ This line is parallel to the plane π but lies on the plane $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=-3$

The distance between the two planes is $\dfrac{4-(-3)}{|\mathbf{i}+\mathbf{j}-\mathbf{k}|}=\dfrac{7}{\sqrt{3}}$

$\therefore$ The shortest distance from the line to the plane is $\dfrac{7\sqrt{3}}{3}=4.04$ (3 s.f.)

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Vectors
Exercise G, Question 1

Question:

Find the shortest distance between the lines with vector equations
$\mathbf{r} = 3\mathbf{i} + s\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
where s, t are scalars. **[E]**

Solution:

Use the formula $\left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

Write the first equation in the form $\mathbf{r} = 3\mathbf{i} - \mathbf{k} + s\mathbf{j}$

with $\mathbf{a} = 3\mathbf{i} - \mathbf{k}, \mathbf{b} = \mathbf{j}, \mathbf{c} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Then $\mathbf{a} - \mathbf{c} = -6\mathbf{i} + 2\mathbf{j}$

and $\mathbf{b} \times \mathbf{d} = \mathbf{i} - \mathbf{k}$

$\therefore$ The shortest distance is $\left|\frac{(-6\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}-\mathbf{k})}{\sqrt{1^2+(-1)^2}}\right|$

$= \left|\frac{-6}{\sqrt{2}}\right|$

$= 3\sqrt{2}$ or 4.24

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Vectors
Exercise G, Question 2

Question:

Obtain the shortest distance between the lines with equations
$\mathbf{r} = (3s-3)\mathbf{i} - s\mathbf{j} + (s+1)\mathbf{k}$
and $\mathbf{r} = (3+t)\mathbf{i} + (2t-2)\mathbf{j} + \mathbf{k}$
where s, t are parameters. **[E]**

Solution:

Use the formula $\left|\dfrac{(\mathbf{a}-\mathbf{c})\cdot\mathbf{b}\times\mathbf{d}}{|\mathbf{b}\times\mathbf{d}|}\right|$

with $\mathbf{a} = -3\mathbf{i}+\mathbf{k}$, $\mathbf{b} = 3\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{c} = 3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{d} = \mathbf{i}+2\mathbf{j}$

Then $\mathbf{a}-\mathbf{c} = -6\mathbf{i}+2\mathbf{j}$ and $\mathbf{b}\times\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$

$$= -2\mathbf{i}+\mathbf{j}+7\mathbf{k}$$

So shortest distance $= \left|\dfrac{(-6\mathbf{i}+2\mathbf{j})\cdot(-2\mathbf{i}+\mathbf{j}+7\mathbf{k})}{\sqrt{(-2)^2+1^2+7^2}}\right|$

$$= \frac{12+2}{\sqrt{54}}$$

$$= \frac{14}{\sqrt{54}}$$

$$= \frac{14}{3\sqrt{6}}$$

$$= \frac{14\sqrt{6}}{18}$$

$$= \frac{7\sqrt{6}}{9} \text{ or } 1.91 \quad (3 \text{ s.f.})$$

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Vectors
Exercise G, Question 3

Question:

The position vectors of the points A, B, C and D relative to a fixed origin O, are $(-\mathbf{j}+2\mathbf{k}), (\mathbf{i}-3\mathbf{j}+5\mathbf{k}), (2\mathbf{i}-2\mathbf{j}+7\mathbf{k})$ and $(\mathbf{j}+2\mathbf{k})$ respectively.

a Find $\mathbf{p} = \overrightarrow{AB} \times \overrightarrow{CD}$.

b Calculate $\overrightarrow{AC} \cdot \mathbf{p}$.

Hence determine the shortest distance between the line containing AB and the line containing CD. **[E]**

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Vectors
Exercise G, Question 4

Question:

Relative to a fixed origin O, the point M has position vector $-4\mathbf{i}+\mathbf{j}-2\mathbf{k}$

The straight line l has equation $\mathbf{r}\times\overrightarrow{OM}=5\mathbf{i}-10\mathbf{k}$.

a Express the equation of the line l in the form $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter.

b Verify that the point N with coordinates $(2,-3,1)$ lies on l and find the area of ΔOMN . **[E]**

Solution:

a $\mathbf{b}=\overrightarrow{OM}=-4\mathbf{i}+\mathbf{j}-2\mathbf{k}$

Let $\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Then as $\mathbf{a}$ represents a point on the line

$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\times(-4\mathbf{i}+\mathbf{j}-2\mathbf{k})=5\mathbf{i}-10\mathbf{k}$

i.e. $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ -4 & 1 & -2\end{vmatrix}=5\mathbf{i}-10\mathbf{k}$

$\therefore(-2y-z)\mathbf{i}+(2x-4z)\mathbf{j}+(x+4y)\mathbf{k}=5\mathbf{i}-10\mathbf{k}$

Compare coefficients

$-2y-z=5$ ①

$2x-4z=0$ ②

$x+4y=-10$ ③

Let $x=2$ say

Then from equation ③ $4y=-12$ $\therefore y=-3$

Also from equation ② $4-4z=0$ $\therefore z=1$

$\therefore(2,-3,1)$ is one point on the line.

[Any value that you take for x will give a point on the line.]

So equation of line may be written

$\mathbf{r}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}+\lambda(-4\mathbf{i}+\mathbf{j}-2\mathbf{k})$

b It has already been shown that $(2,-3,1)$ lies on the line.

$$\text{Area }\Delta OMN=\frac{1}{2}|\overrightarrow{OM}\times\overrightarrow{ON}|=\frac{1}{2}|5\mathbf{i}-10\mathbf{k}|$$

$$=\frac{1}{2}\sqrt{5^2+(-10)^2}$$

$$=\frac{5}{2}\sqrt{5}\text{ or }5.59\text{ (3 s.f.)}$$

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Vectors
Exercise G, Question 5

Question:

The line l_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and the line l_2 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

a Find a vector which is perpendicular to both l_1 and l_2.

The point A lies on l_1 and the point B lies on l_2. Given that AB is also perpendicular to l_1 and l_2,

b find the coordinates of A and B. **[E]**

Solution:

a A vector perpendicular to l_1 and l_2 is

$$(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

b $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$= \begin{pmatrix} 2+2\mu \\ 1-\mu \\ 1+\mu \end{pmatrix} - \begin{pmatrix} 1+\lambda \\ -1+2\lambda \\ 3\lambda \end{pmatrix}$$
$$= \begin{pmatrix} 1+2\mu-\lambda \\ 2-\mu-2\lambda \\ 1+\mu-3\lambda \end{pmatrix}$$

As this is perpendicular to l_1 and to l_2 it is a multiple of $(\mathbf{i} + \mathbf{j} - \mathbf{k})$

$\therefore 1 + 2\mu - \lambda = 2 - \mu - 2\lambda \Rightarrow 3\mu + \lambda = 1$ ①

and $1 + 2\mu - \lambda = -(1 + \mu - 3\lambda) \Rightarrow 3\mu - 4\lambda = -2$ ②

Subtract ① − ②

Then $5\lambda = 3 \Rightarrow \lambda = \frac{3}{5}$

Substitute into equation ①.

Then $3\mu = 1 - \frac{3}{5}$

$\therefore \mu = \frac{2}{15}$

$\therefore$ A is the point with coordinates $\left(1\frac{3}{5}, \frac{1}{5}, 1\frac{4}{5}\right)$ and B is the point with coordinates $\left(2\frac{4}{15}, \frac{13}{15}, 1\frac{2}{15}\right)$

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Vectors
Exercise G, Question 6

Question:

A plane passes through the three points A, B, C, whose position vectors, referred to an origin O, are $(\mathbf{i}+3\mathbf{j}+3\mathbf{k}), (3\mathbf{i}+\mathbf{j}+4\mathbf{k}), (2\mathbf{i}+4\mathbf{j}+\mathbf{k})$ respectively.

a Find, in the form $(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$, a unit vector normal to this plane.

b Find also a Cartesian equation of the plane.

c Find the perpendicular distance from the origin to this plane. **[E]**

Solution:

a $\overrightarrow{AB} = (3\mathbf{i}+\mathbf{j}+4\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$
$= 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\overrightarrow{AC} = (2\mathbf{i}+4\mathbf{j}+\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$
$= \mathbf{i}+\mathbf{j}-2\mathbf{k}$

A vector normal to this plane ABC is in the direction $\overrightarrow{AB}\times\overrightarrow{AC}$.

i.e. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$

A unit vector normal to the plane is $\frac{1}{\sqrt{3^2+5^2+4^2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

$= \frac{1}{\sqrt{50}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

b The equation of the plane may be written as

$\mathbf{r}\cdot(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}) = (\mathbf{i}+3\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$
$= 3+15+12$
$= 30$

i.e. $3x+5y+4z=30$

c The perpendicular distance from the origin to the plane is

$\frac{30}{\sqrt{3^2+5^2+4^2}} = \frac{30}{\sqrt{50}} = \frac{30\sqrt{50}}{50} = 3\sqrt{2}$.

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Vectors
Exercise G, Question 7

Question:

a Show that the vector $\mathbf{i}+\mathbf{k}$ is perpendicular to the plane with vector equation $\mathbf{r}=\mathbf{i}+s\mathbf{j}+t(\mathbf{i}-\mathbf{k})$.

b Find the perpendicular distance from the origin to this plane.

c Hence or otherwise obtain a Cartesian equation of the plane. **[E]**

Solution:

The plane with vector equation

$\mathbf{r}=\mathbf{i}+s\mathbf{j}+t(\mathbf{i}-\mathbf{k})$

is perpendicular to $\mathbf{i}+\mathbf{k}$, as $(\mathbf{i}+\mathbf{k})\cdot\mathbf{j}=0$ and $(\mathbf{i}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{k})=1-1=0$

The plane also has equation

$\mathbf{r}\cdot(\mathbf{i}+\mathbf{k})=\mathbf{i}\cdot(\mathbf{i}+\mathbf{k})$, as $\mathbf{i}$ is the position vector of a point on the plane.

i.e. $\mathbf{r}\cdot(\mathbf{i}+\mathbf{k})=1$

The perpendicular distance from the origin to this plane is $\frac{1}{\sqrt{1^2+1^2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$ or

0.707 (3 s.f.)

The Cartesian form of the equation of the plane is $x+z=1$

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Vectors
Exercise G, Question 8

Question:

The points A, B and C have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}, 5\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ respectively, referred to an origin O.

a Find a vector perpendicular to the plane containing the points A, B and C.

b Hence, or otherwise, find an equation for the plane which contains the points A, B and C, in the form $ax+by+cz+d=0$.

The point D has coordinates (1, 5, 6).

c Find the volume of the tetrahedron $ABCD$. **[E]**

Solution:

a $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(5\mathbf{i}-2\mathbf{j}+\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=4\mathbf{i}-3\mathbf{j}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=2\mathbf{i}+\mathbf{j}+5\mathbf{k}$

Perpendicular vector to the plane is in direction

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -3 & 0\\ 2 & 1 & 5\end{vmatrix}=-15\mathbf{i}-20\mathbf{j}+10\mathbf{k}$$

b The equation on of plane containing A, B and C is

$\mathbf{r}\cdot(-15\mathbf{i}-20\mathbf{j}+10\mathbf{k})=(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(-15\mathbf{i}-20\mathbf{j}+10\mathbf{k})$

i.e. $-15x-20y+10z=-25$

or $3x+4y-2z-5 \quad =0$

c Volume of tetrahedron $ABCD=\left|\frac{1}{6}\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(\mathbf{i}+5\mathbf{j}+6\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=4\mathbf{j}+5\mathbf{k}$

$\therefore$ Volume $=\frac{1}{6}\left|(4\mathbf{j}+5\mathbf{k})\cdot(-15\mathbf{i}-20\mathbf{j}+10\mathbf{k})\right|$
$=\frac{1}{6}\left|(-80+50)\right|$
$=5$

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Vectors

Exercise G, Question 9

Question:

The plane Π passes through $A(3,-5,-1), B(-1,5,7)$ and $C(2,-3,0)$.

a Find $\overrightarrow{AC} \times \overrightarrow{BC}$.

b Hence, or otherwise, find the equation, in the form $\mathbf{r} \cdot \mathbf{n} = p$, of the plane Π.

c The perpendicular from the point $(2,3,-2)$ to Π meets the plane at P. Find the coordinates of P. **[E]**

Solution:

a
$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = (2\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} - 5\mathbf{j} - \mathbf{k})$$
$$= -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (2\mathbf{i} - 3\mathbf{j}) - (-\mathbf{i} + 5\mathbf{j} + 7\mathbf{k})$$
$$= 3\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$$
$$\therefore \overrightarrow{AC} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & -8 & -7 \end{vmatrix} = -6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

b Equation of the plane π is
$$\mathbf{r} \cdot (-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = (3\mathbf{i} - 5\mathbf{j} - \mathbf{k}) \cdot (-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$$
$$= -18 + 20 - 2$$
$$= 0$$
or $\mathbf{r} \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$

c The perpendicular from $(2,3,-2)$ to π has equation
$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$
This meets the plane π when
$$((2+3\lambda)\mathbf{i} + (3+2\lambda)\mathbf{j} + (-2-\lambda)\mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$$
$$\text{i.e. } 3(2+3\lambda) + 2(3+2\lambda) - 1(-2-\lambda) = 0$$
$$\text{i.e. } 14\lambda + 14 = 0$$
$$\therefore \lambda = -1$$
$\therefore$ Substitute into equation of line
$$\mathbf{r} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$$
$\therefore$ Foot of perpendicular is at $(-1,1,-1)$

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Vectors
Exercise G, Question 10

Question:

Given that P and Q are the points with position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, relative to an origin O, and that

$\mathbf{p} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$\mathbf{q} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$,

a find $\mathbf{p} \times \mathbf{q}$.

b Hence, or otherwise, find an equation of the plane containing O, P and Q in the form $ax + by + cz = d$.

The line with equation $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = 0$ meets the plane with equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ at the point T.

c Find the coordinates of the point T. **[E]**

Solution:

a $\mathbf{p} \times \mathbf{q} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - \mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

b The equation of the plane is

$\mathbf{r} \cdot (-\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 7\mathbf{j} + 5\mathbf{k})$

i.e. $-x + 7y + 5z = 0$

c The line equation may be written in the form

$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

This meets the plane $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ when $(3 + 2\lambda) + (-1 + \lambda) + (2 - \lambda) = 2$

i.e. $2\lambda + 4 = 2$

$\therefore \lambda = -1$

Substitute into the line equation

Then $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

The coordinates of point T are $(1, -2, 3)$

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Vectors
Exercise G, Question 11

Question:

The planes Π_1 and Π_2 are defined by the equations $2x+2y-z=9$ and $x-2y=7$ respectively.

a Find the acute angle between Π_1 and Π_2, giving your answer to the nearest degree.

b Find in the form $\mathbf{r}\times\mathbf{u}=\mathbf{v}$ an equation of the line of intersection of Π_1 and Π_2. **[E]**

Solution:

a The normals to the planes are

$\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j}$

The angle between the normals is θ where

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2\times1 - 2\times2}{\sqrt{2^2+2^2+(-1)^2}\sqrt{1^2+(-2)^2}}$$

$$= \frac{-2}{\sqrt{9}\sqrt{5}}$$

$$= \frac{-2\sqrt{5}}{15}$$

$\therefore$ The acute angle α between the planes is given by $\cos\alpha = \frac{2\sqrt{5}}{15}$,

i.e. $\alpha = 72.7° = 73°$ (nearest degree)

b The planes have equations $2x + 2y - z = 9$ ①

and $x - 2y = 7$ ②

Add ① + ②

Then $3x - z = 16$

$\therefore x = \frac{z+16}{3}$.

Also from equation ②

$x = \frac{7+2y}{1}$

Let $x = \lambda$

Then $\frac{x-0}{1} = \frac{y+\frac{7}{2}}{\frac{1}{2}} = \frac{z+16}{3} = \lambda$

This may be written

$$\mathbf{r}\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right) = \left(\frac{-7}{2}\mathbf{j}-16\mathbf{k}\right)\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right)$$

$$\text{i.e } \mathbf{r}\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{7}{2} & -16 \\ 1 & \frac{1}{2} & 3 \end{vmatrix}$$

$$= \left(\frac{-21}{2}+8\right)\mathbf{i} - 16\mathbf{j} + \frac{7}{2}\mathbf{k}$$

$$= -\frac{5}{2}\mathbf{i} - 16\mathbf{j} + \frac{7}{2}\mathbf{k}$$

$$\therefore \mathbf{r}\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right) = \left(-\frac{5}{2}\mathbf{i}-16\mathbf{j}+\frac{7}{2}\mathbf{k}\right)$$

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Vectors
Exercise G, Question 12

Question:

A pyramid has a square base $OPQR$ and vertex S. Referred to O, the points P, Q, R and S have position vectors $\overrightarrow{OP} = 2\mathbf{i}, \overrightarrow{OQ} = 2\mathbf{i} + 2\mathbf{j}, \overrightarrow{OR} = 2\mathbf{j}, \overrightarrow{OS} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

a Express PS in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

b Show that the vector $-4\mathbf{j} + \mathbf{k}$ is perpendicular to OS and PS.

c Find to the nearest degree the acute angle between the line SQ and the plane OSP.

[E]

Solution:

a $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$

$= \mathbf{i} + \mathbf{j} + 4\mathbf{k} - 2\mathbf{i}$

$= -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

b $(-4\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = -4 + 4 = 0$

$\therefore -4\mathbf{j} + \mathbf{k}$ is perpendicular to $\overrightarrow{OS}$.

Also $(-4\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = -4 + 4 = 0$

$\therefore -4\mathbf{j} + \mathbf{k}$ is perpendicular to $\overrightarrow{PS}$.

c $-4\mathbf{j} + \mathbf{k}$ is normal to the plane OSP.

$\overrightarrow{SQ} = \overrightarrow{OQ} - \overrightarrow{OS}$

$= 2\mathbf{i} + 2\mathbf{j} - (\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= \mathbf{i} + \mathbf{j} - 4\mathbf{k}$

The acute angle θ between $\overrightarrow{SQ}$ and the normal to the plane is given by

$$\cos\theta = \left| \frac{(-4\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 4\mathbf{k})}{\sqrt{(-4)^2 + 1^2}\sqrt{1^2 + 1^2 + (-4)^2}} \right|$$

$$= \left| \frac{-8}{\sqrt{17}\sqrt{18}} \right| = \frac{8}{\sqrt{17}\sqrt{18}}$$

The angle α between the line SQ and the plane OSP is such that $\alpha + \theta = 90^\circ$ and so $\sin\alpha = \dfrac{8}{\sqrt{17}\sqrt{18}}$ and $\alpha = 27^\circ$ (nearest degree)

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Vectors
Exercise G, Question 13

Question:

The plane Π has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + u\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + v\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$, where u and v are parameters.

The line L has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, where t is a parameter.

a Show that L is parallel to Π.

b Find the shortest distance between L and Π. **[E]**

Solution:

a The normal to the plane Π is in the direction $(4\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$

i.e. $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 1 & 2\\ 3 & 2 & -1\end{vmatrix} = -5\mathbf{i}+10\mathbf{j}+5\mathbf{k}$

The line L is in the direction $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$
As $(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-4\mathbf{k}) = 0$
the line L is perpendicular to the normal to the plane.

Thus L is parallel to the plane Π.

b The line L passes through point $(2, 1, -3)$

The perpendicular to plane π through $(2, 1, -3)$ has equation
$\mathbf{r} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}+\lambda(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k})$

The equation of the plane may be written
$\mathbf{r}\cdot(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k}) = (\mathbf{i}+3\mathbf{j}+4\mathbf{k})\cdot(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k})$
$= 45$

This perpendicular meets plane Π when
$((2-5\lambda)\mathbf{i}+(1+10\lambda)\mathbf{j}+(-3+5\lambda)\mathbf{k})\cdot(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k}) = 45$
i.e. $-10+25\lambda+10+100\lambda-15+25\lambda = 45$
i.e. $150\lambda = 60 \Rightarrow \lambda = \frac{2}{5}$
Substitute $\lambda = \frac{2}{5}$ into the equation of the perpendicular.
Then $\mathbf{r} = 5\mathbf{j}-\mathbf{k}$
i.e. The perpendicular to Π from $(2,1,-3)$ meets the plane at $(0,5,-1)$
$\therefore$ Shortest distance from L to Π is

$\sqrt{(2-0)^2+(1-5)^2+(-3-(-1))^2}$
$= \sqrt{4+16+4}$
$= \sqrt{24} = 2\sqrt{6}$ or 4.90
or

Take point A on Π $\begin{pmatrix}1\\3\\4\end{pmatrix}$ and B on L $\begin{pmatrix}2\\1\\-3\end{pmatrix}$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -7 \end{pmatrix}$$

$$\text{Distance} = |\overrightarrow{AB} \cdot \hat{\mathbf{n}}|$$

$$|\mathbf{n}| = \sqrt{(-5)^2 + 10^2 + 5^2} = \sqrt{150} = 5\sqrt{6}$$

$$\therefore \hat{\mathbf{n}} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore \text{Distance} = \frac{1}{\sqrt{6}} \left| \begin{pmatrix} 1 \\ -2 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right| = \frac{1}{\sqrt{6}} \times 12 = \frac{12}{\sqrt{6}} = 2\sqrt{6} = 4.90$$

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Vectors
Exercise G, Question 14

Question:

Planes Π_1 and Π_2 have equations given by

$\Pi_1 : \mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$,

$\Pi_2 : \mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$.

a Show that the point $A(2, -2, 3)$ lies in Π_2.

b Show that Π_1 is perpendicular to Π_2.

c Find, in vector form, an equation of the straight line through A which is perpendicular to Π_1.

d Determine the coordinates of the point where this line meets Π_1.

e Find the perpendicular distance of A from Π_1.

f Find a vector equation of the plane through A parallel to Π_1. **[E]**

Solution:

a $(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = 2-10+9$

$= 1$

$\therefore (2,-2,3)$ lies on the plane Π_2

b $(2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = 2-5+3$

$= 0$

$\therefore$ the normal to plane Π_1 is perpendicular to the normal to plane Π_2.

$\therefore \Pi_1$ is perpendicular to Π_2.

c $\mathbf{r} = 2\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

d This line meets the plane Π_1 when

$[(2+2\lambda)\mathbf{i}+(-2-\lambda)\mathbf{j}+(3+\lambda)\mathbf{k}]\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=0$

i.e. $4+4\lambda+2+\lambda+3+\lambda=0$

i.e. $6\lambda+9=0$

$\therefore \lambda = -\frac{3}{2}$

Substitute $\lambda = -\frac{3}{2}$ into the equation of the line: then $\mathbf{r} = -\mathbf{i}-\frac{1}{2}\mathbf{j}+\frac{3}{2}\mathbf{k}$

i.e. The line meets Π_1 at the point $\left(-1,-\frac{1}{2},\frac{3}{2}\right)$

e The distance required is

$$\sqrt{(2-(-1))^2+\left(-2-\left(-\frac{1}{2}\right)\right)^2+\left(3-\frac{3}{2}\right)^2} = \sqrt{9+2\frac{1}{4}+2\frac{1}{4}} = \sqrt{13\frac{1}{2}}$$

$= 3.67$ (3 s.f.)

f $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = (2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

$= 4+2+3$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 9$

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Vectors
Exercise G, Question 15

Question:

The plane Π has equation $2x + y + 3z = 21$ and the origin is O. The line l passes through the point $P(1, 2, 1)$ and is perpendicular to Π.

a Find a vector equation of l.

The line l meets the plane Π at the point M.

b Find the coordinates of M.

c Find $\overrightarrow{OP} \times \overrightarrow{OM}$.

d Hence, or otherwise, find the distance from P to the line OM, giving your answer in surd form.

The point Q is the reflection of P in Π.

e Find the coordinates of Q. **[E]**

Solution:

a $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

b This line meets plane Π when

$(1+2\lambda)\cdot 2 + (2+\lambda)\cdot 1 + (1+3\lambda)\cdot 3 = 21$

i.e. $14\lambda + 7 = 21$

i.e. $\lambda = 1$

Substitute $\lambda = 1$ into the equation of the line l.

Then $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

So M has coordinates (3, 3, 4)

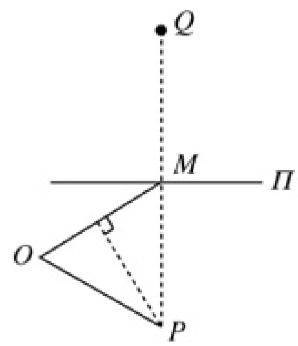
c $\overrightarrow{OP} \times \overrightarrow{OM} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$= 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

d Area of $\Delta\, OPM = \frac{1}{2}|5\mathbf{i} - \mathbf{j} - 3\mathbf{k}|$

$= \frac{1}{2}\sqrt{5^2 + (-1)^2 + (-3)^2}$

$= \frac{1}{2}\sqrt{35}$



$\therefore$ Distance from P to line $OM = \dfrac{\frac{1}{2}\sqrt{35}}{\frac{1}{2}|\text{OM}|}$

$= \dfrac{\frac{1}{2}\sqrt{35}}{\frac{1}{2}\sqrt{3^2 + 3^2 + 4^2}}$

$= \dfrac{\sqrt{35}}{\sqrt{34}}$

e $\overrightarrow{PM} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} - (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\therefore \overrightarrow{MQ} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

And $\overrightarrow{OQ} = \overrightarrow{OM} + \overrightarrow{MQ} = 5\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$

Q has coordinates (5, 4, 7)

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Vectors
Exercise G, Question 16

Question:

With respect to a fixed origin O, the straight lines l_1 and l_2 are given by

$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$,

$l_2 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(-3\mathbf{i} + 4\mathbf{k})$,

where λ and μ are scalar parameters.

a Show that the lines intersect.

b Find the position vector of their point of intersection.

c Find the cosine of the acute angle contained between the lines.

d Find a vector equation of the plane containing the lines. **[E]**

Solution:

a The lines l_1 and l_2 intersect if

$$\begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ -2\lambda \end{pmatrix} = \begin{pmatrix} 1-3\mu \\ 2 \\ 2+4\mu \end{pmatrix}$$

have consistent solutions.

i.e. $2\lambda = -3\mu$ ①

$\lambda = 3$ ②

and $-2\lambda = 4\mu + 2$ ③

Substitute $\lambda = 3$ from ② into ①, then $\mu = -2$

Check in equation ③ $\lambda = 3$ and $\mu = -2$ satisfy equation ③

$\therefore$ the lines intersect

b Substitute $\lambda = 3$ into equation of l_1

Then $\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$

This is the position vector of the point of intersection.

c Let θ be the acute angle between the lines.

Then $\cos\theta = \left| \dfrac{(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (-3\mathbf{i} + 4\mathbf{k})}{\sqrt{2^2 + 1^2 + (-2)^2}\sqrt{(-3)^2 + 4^2}} \right|$

$= \left| \dfrac{-6-8}{\sqrt{9}\sqrt{25}} \right|$

$= \dfrac{14}{15}$

d $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \mu(-3\mathbf{i} + 4\mathbf{k})$ is a vector equation for the plane.

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Vectors
Exercise G, Question 17

Question:

Relative to an origin O, the points A and B have position vectors $\mathbf{a}$ metres and $\mathbf{b}$ metres respectively, where

$\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

The point C moves such that the volume of the tetrahedron $OABC$ is always $5\,\text{m}^3$.

Determine Cartesian equations of the locus of the point C. **[E]**

Solution:

Let C be the point with position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The volume of the tetrahedron $OABC$ is given by

$$\frac{1}{6}\begin{vmatrix} x & y & z \\ 5 & 2 & 0 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= \frac{1}{6}(-6x + 15y - 9z)$$

As the volume is $5\,\text{m}^3$,

$$\therefore \frac{1}{6}(-6x + 15y - 9z) = 5$$

i.e. $-6x + 15y - 9z = 30$

or $2x - 5y + 3z + 10 = 0$, which is the locus of the point C.

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Vectors
Exercise G, Question 18

Question:

The lines L_1 and L_2 have equations $\mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + t\mathbf{b}_2$ respectively, where

$\mathbf{a}_1 = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b}_1 = \mathbf{j} + 2\mathbf{k}$,

$\mathbf{a}_2 = 8\mathbf{i} + 3\mathbf{j}$, $\mathbf{b}_2 = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

a Verify that the point P with position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ lies on both L_1 and L_2.

b Find $\mathbf{b}_1 \times \mathbf{b}_2$.

c Find a Cartesian equation of the plane containing L_1 and L_2.

The points with position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are A_1 and A_2 respectively.

d By expressing $\overrightarrow{A_1P}$ and $\overrightarrow{A_2P}$ as multiples of $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively, or otherwise, find the area of the triangle PA_1A_2. **[E]**

Solution:

a Equation of l_1 is

$\mathbf{r} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k})$

When $\lambda = 2, \mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ So P lies on l_1.

Equation of l_2 is

$\mathbf{r} = 8\mathbf{i} + 3\mathbf{j} + \mu(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

When $\mu = -1, \mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. So P lies on l_2.

b $$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}$$

c The normal to the plane is in direction of $\mathbf{b}_1 \times \mathbf{b}_2$. So $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is a normal.

$\therefore$ Equation of plane is

$$\mathbf{r} \cdot (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) \cdot (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$
$$= -6 - 6 + 2$$

$\therefore -2x + 2y - z = -10$

$\therefore +2x - 2y + z = 10$ is a Cartesian equation of the plane.

d $\overrightarrow{A_1P} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 2\mathbf{j} + 4\mathbf{k} = 2\mathbf{b}_1$

$\overrightarrow{A_2P} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (8\mathbf{i} + 3\mathbf{j}) = (-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = -\mathbf{b}_2$

$$\text{Area of } PA_1A_2 = \frac{1}{2}\left|\overrightarrow{A_1P} \times \overrightarrow{A_2P}\right| = \frac{1}{2}\left|2\mathbf{b}_1 \times -\mathbf{b}_2\right|$$
$$= |\mathbf{b}_1 \times \mathbf{b}_2|$$
$$= \sqrt{(-10)^2 + (10)^2 + (-5)^2}$$
$$= \sqrt{225}$$
$$= 15$$

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Vectors
Exercise G, Question 19

Question:

With respect to the origin O the points A, B, C have position vectors $a(5\mathbf{i}-\mathbf{j}-3\mathbf{k}), a(-4\mathbf{i}+4\mathbf{j}-\mathbf{k}), a(5\mathbf{i}-2\mathbf{j}+11\mathbf{k})$ respectively, where a is a non-zero constant.

Find

a a vector equation for the line BC,

b a vector equation for the plane OAB,

c the cosine of the acute angle between the lines OA and OB.

Obtain, in the form $\mathbf{r}\cdot\mathbf{n}=p$, a vector equation for Π, the plane which passes through A and is perpendicular to BC.

Find Cartesian equations for

d the plane Π,

e the line BC.

Solution:

a $\overrightarrow{BC} = a(5\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}) - a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

$= a(9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$

$\therefore$ vector equation for the line BC is

$\mathbf{r} = a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + \lambda a(9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$

b A vector equation for the plane OAB is

$\mathbf{r} = a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + \lambda a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + \mu a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

c Let the acute angle between OA and OB be θ

Then $\cos\theta = \left|\dfrac{a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \cdot a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})}{a\sqrt{25+1+9}\ a\sqrt{16+16+1}}\right|$

$= \left|\dfrac{-12}{\sqrt{35}\sqrt{33}}\right|$

$= \dfrac{12}{\sqrt{35}\sqrt{33}} = 0.353$ (3 s.f.)

The plane through A, perpendicular to BC has equation

$\mathbf{r} \cdot (9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \cdot (9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$

i.e. $\mathbf{r} \cdot (9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = 15a$

or $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 5a$

d The Cartesian equation for this plane Π is $3x - 2y + 4z = 5a$

e The Cartesian equation for the line BC comes from

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4a \\ 4a \\ -a \end{pmatrix} + \lambda \begin{pmatrix} 9a \\ -6a \\ 12a \end{pmatrix}$$

$\therefore \dfrac{x+4a}{9} = \dfrac{y-4a}{-6} = \dfrac{z+a}{12} = \lambda a$

or $\dfrac{x+4a}{3} = \dfrac{y-4a}{-2} = \dfrac{z+a}{4} = \lambda'$

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Vectors
Exercise G, Question 20

Question:

In a tetrahedron $ABCD$ the coordinates of the vertices B, C, D are respectively (1, 2, 3), (2, 3, 3), (3, 2, 4). Find

a the equation of the plane BCD.

b the sine of the angle between BC and the plane $x+2y+3z=4$.

If AC and AD are perpendicular to BD and BC respectively and if $AB=\sqrt{26}$, find the coordinates of the two possible positions of A.

Solution:

a $\overrightarrow{BC} = (2\mathbf{i}+3\mathbf{j}+3\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \mathbf{i}+\mathbf{j}$

$\overrightarrow{BD} = (3\mathbf{i}+2\mathbf{j}+4\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 2\mathbf{i}+\mathbf{k}$

$$\therefore \overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

This is normal to the plane BCD.

$\therefore$ The equation of the plane BCD is

$$\mathbf{r} \cdot (\mathbf{i}-\mathbf{j}-2\mathbf{k}) = (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) \cdot (\mathbf{i}-\mathbf{j}-2\mathbf{k})$$

$$= 1-2-6$$

$$= -7$$

This may be written $x - y - 2z + 7 = 0$

b Let the required angle be α. Then $\sin\alpha = \cos\theta$ where θ is the acute angle between BC and the normal vector $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

$$\therefore \cos\theta = \frac{(\mathbf{i}+\mathbf{j}) \cdot (\mathbf{i}+2\mathbf{j}+3\mathbf{k})}{\sqrt{1^2+1^2}\sqrt{1^2+2^2+3^2}}$$

$$= \frac{3}{\sqrt{2}\sqrt{14}} = 0.567 \text{ (3 s.f.)}$$

c Let A have coordinates (x, y, z).

Then $\overrightarrow{AC} = (2-x)\mathbf{i} + (3-y)\mathbf{j} + (3-z)\mathbf{k}$

Also $\overrightarrow{AD} = (3-x)\mathbf{i} + (2-y)\mathbf{j} + (4-z)\mathbf{k}$

As AC is perpendicular to BD, $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$\therefore 2(2-x) + 0(3-y) + 1(3-z) = 0$

$\therefore 2x + z = 7$ ①

As AD is perpendicular to BC, $\overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$\therefore 1(3-x) + 1(2-y) + 0(4-z) = 0$

$\therefore x + y = 5$ ②

Also $AB = \sqrt{26}$.

$\therefore (x-1)^2 + (y-2)^2 + (z-3)^2 = 26$ ③

Substitute $z = 7-2x$ and $y = 5-x$ from equations ① and ② into equation ③

Then $(x-1)^2 + (3-x)^2 + (4-2x)^2 = 26$

$\therefore 6x^2 - 24x + 26 = 26$

$\therefore 6x(x-4) = 0$

$\therefore x = 0$ or 4

When $x = 0, y = 5$ and $z = 7$

When $x = 4, y = 1$ and $z = -1$

$\therefore$ The two possible positions are $(0, 5, 7)$ and $(4, 1, -1)$