Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further coordinate systems
Exercise A, Question 1

Question:

a Sketch the following ellipses showing clearly where the curve crosses the coordinate axes.

i $x^2 + 4y^2 = 16$

ii $4x^2 + y^2 = 36$

iii $x^2 + 9y^2 = 25$

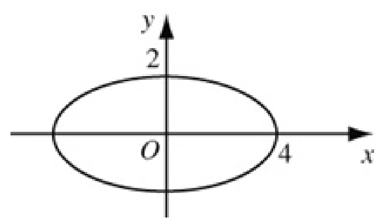
b Find parametric equations for these curves.

Solution:

i $x^2 + 4y^2 = 16$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$

a

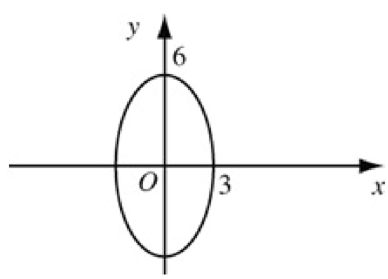


b Parametric equations $x = 4\cos\theta, y = 2\sin\theta$

ii $4x^2 + y^2 = 36$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{36} = 1$

a

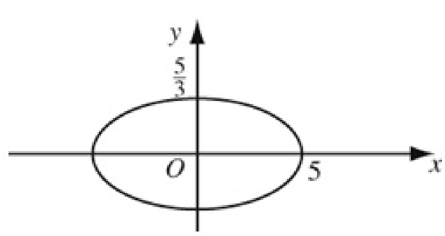


b Parametric equations $x = 3\cos\theta, y = 6\sin\theta$

iii $x^2 + 9y^2 = 25$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{\left(\frac{5}{3}\right)^2} = 1$

a



b Parametric equations $x = 5\cos\theta, y = \frac{5}{3}\sin\theta$

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Further coordinate systems
Exercise A, Question 2

Question:

a Sketch ellipses with the following parametric equations.

b Find a Cartesian equation for each ellipse.

i $x = 2\cos\theta, y = 3\sin\theta$

ii $x = 4\cos\theta, y = 5\sin\theta$

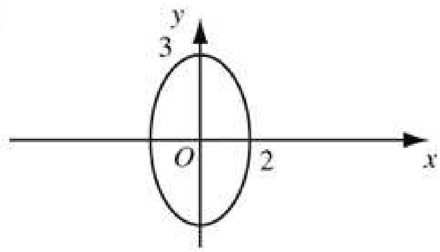
iii $x = \cos\theta, y = 5\sin\theta$

iv $x = 4\cos\theta, y = 3\sin\theta$

Solution:

i $x = 2\cos\theta, y = 3\sin\theta$
$\Rightarrow a = 2, b = 3$

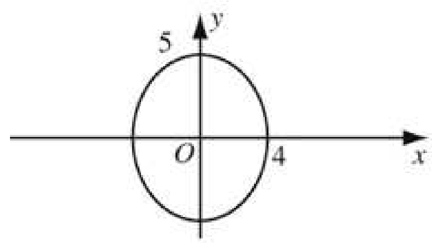
a



b Cartesian equation $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$

ii $x = 4\cos\theta, y = 5\sin\theta$
$\Rightarrow a = 4, b = 5$

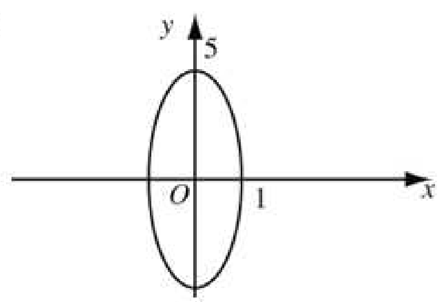
a



b Cartesian equation $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$

iii $x = \cos\theta, y = 5\sin\theta$
$\Rightarrow a = 1, b = 5$

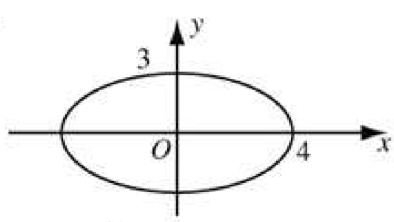
a



b Cartesian equation $x^2 + \frac{y^2}{5^2} = 1$

iv $x = 4\cos\theta, y = 3\sin\theta$
$\Rightarrow a = 4, b = 3$

a



b Cartesian equation $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$

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Further coordinate systems
Exercise B, Question 1

Question:

Find the equations of tangents and normals to the following ellipses at the points given.

a $\frac{x^2}{4}+y^2=1$ at $(2\cos\theta, \sin\theta)$

b $\frac{x^2}{25}+\frac{y^2}{9}=1$ at $(5\cos\theta, 3\sin\theta)$

Solution:

$x=a\cos\theta,\ y=b\sin\theta$

$\frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$ $\quad\therefore$ tangent is: $y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

Equation of tangent is: $ay\sin\theta+bx\cos\theta=ab$

Normal gradient is $\frac{a\sin\theta}{b\cos\theta}$ $\quad\therefore$ normal is: $y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$

Equation of normal is: $by\cos\theta-ax\sin\theta=(b^2-a^2)\sin\theta\cos\theta$

a $a=2, b=1$

So equation of tangent is: $2y\sin\theta+x\cos\theta=2$

Equation of normal is: $y\cos\theta-2x\sin\theta=-3\sin\theta\cos\theta$

b $\frac{x^2}{25}+\frac{y^2}{9}=1\Rightarrow a=5, b=3$

So equation of tangent is: $5y\sin\theta+3x\cos\theta=15$

Equation of normal is: $3y\cos\theta-5x\sin\theta=-16\sin\theta\cos\theta$

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Further coordinate systems
Exercise B, Question 2

Question:

Find equations of tangent and normals to the following ellipses at the points given.

a $\frac{x^2}{9}+\frac{y^2}{1}=1$ at $(\sqrt{5}, \frac{2}{3})$

b $\frac{x^2}{16}+\frac{y^2}{4}=1$ at $(-2, \sqrt{3})$

Solution:

a $\frac{x^2}{9}+y^2=1 \Rightarrow \frac{2x}{9}+2y\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=-\frac{x}{9y}$ so at $\left(\sqrt{5}, \frac{2}{3}\right)$ $m=-\frac{\sqrt{5}}{6}$

Tangent at $\left(\sqrt{5}, \frac{2}{3}\right)$ is: $y-\frac{2}{3}=-\frac{\sqrt{5}}{6}(x-\sqrt{5})$

i.e. $6y+\sqrt{5}x=9$

Normal at $\left(\sqrt{5}, \frac{2}{3}\right)$ is: $y-\frac{2}{3}=\frac{6}{\sqrt{5}}(x-\sqrt{5})$

i.e. $3\sqrt{5}y-2\sqrt{5}=18x-18\sqrt{5}$ i.e. $3\sqrt{5}y=18x-16\sqrt{5}$

b $\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow \frac{x}{8}+\frac{y}{2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=-\frac{x}{4y}$ so at $(-2, \sqrt{3})$ $m=\frac{1}{2\sqrt{3}}$

Tangent at $(-2, \sqrt{3})$ is: $y-\sqrt{3}=\frac{1}{2\sqrt{3}}(x--2)$

i.e. $2\sqrt{3}y-x=8$

Normal at $(-2, \sqrt{3})$ is $y-\sqrt{3}=-2\sqrt{3}(x--2)$

i.e. $y+2\sqrt{3}x=-3\sqrt{3}$

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Further coordinate systems
Exercise B, Question 3

Question:

Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos t, b\sin t)$ is $xb\cos t+ya\sin t=ab$

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}\text{, at } (a\cos t, b\sin t) \quad m=\frac{-b^2a\cos t}{a^2b\sin t}$$

$$\therefore m=-\frac{b\cos t}{a\sin t}$$

Equation of tangent at $(a\cos t, b\sin t)$ is:

$$y-b\sin t=-\frac{b\cos t}{a\sin t}(x-a\cos t)$$

i.e. $ay\sin t-ab\sin^2 t=-bx\cos t+ab\cos^2 t$

i.e. $bx\cos t+ay\sin t=ab$.

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Further coordinate systems
Exercise B, Question 4

Question:

a Show that the line $y = x + \sqrt{5}$ is a tangent to the ellipse with equation $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

b Find the point of contact of this tangent.

Solution:

The line $y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if

$a^2m^2 + b^2 = c^2$

a $m = 1, c = \sqrt{5} \quad (\because y = x + \sqrt{5})$

$a = 2, b = 1 \quad \left(\because \frac{x^2}{4} + \frac{y^2}{1} = 1\right)$

$a^2m^2 + b^2 = 4 \times 1 + 1 = 5$

$= c^2$

$\therefore y = x + \sqrt{5}$ is a tangent.

b Point of contact: $y = x + \sqrt{5}$

$$\frac{x^2}{4} + y^2 = 1 \Rightarrow \frac{x^2}{4} + (x + \sqrt{5})^2 = 1$$

$$\therefore x^2 + 4(x^2 + 2\sqrt{5}x + 5) = 4$$

$$5x^2 + 8\sqrt{5}x + 16 = 0$$

$$(\sqrt{5}x + 4)^2 = 0$$

$$x = -\frac{4}{\sqrt{5}} = -\frac{4}{5}\sqrt{5}$$

$$\therefore y = -\frac{4}{5}\sqrt{5} + \sqrt{5} = \frac{1}{5}\sqrt{5}$$

So point of contact is $\left(-\frac{4}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\right)$

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Further coordinate systems
Exercise B, Question 5

Question:

a Find an equation of the normal to the ellipse with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ at the point $P(3\cos\theta, 2\sin\theta)$.

This normal crosses the x-axis at the point $(-\frac{5}{6},0)$.

b Find the value of θ and the exact coordinates of the possible positions of P.

Solution:

a $x=3\cos\theta, y=2\sin\theta \Rightarrow \frac{dy}{dx}=\frac{2\cos\theta}{-3\sin\theta}$

$\therefore$ Gradient of normal is $\frac{3\sin\theta}{2\cos\theta}$

$\therefore$ Equation of normal is: $y-2\sin\theta=\frac{3\sin\theta}{2\cos\theta}(x-3\cos\theta)$

i.e. $2y\cos\theta-4\cos\theta\sin\theta=3\sin\theta x-9\sin\theta\cos\theta$

$2y\cos\theta-3\sin\theta x=-5\sin\theta\cos\theta$

b $y=0,\ x=-\frac{5}{6}$

$\Rightarrow\quad -3\sin\theta\left(-\frac{5}{6}\right)=-5\sin\theta\cos\theta$

$\frac{5}{2}=-5\cos\theta \text{ or } \sin\theta=0 \text{ or } \sin\theta=0$

i.e. $\cos\theta=-\frac{1}{2}$ i.e. $\theta=0$ or 180° i.e. $\theta=0$ or 180°

$\therefore\quad \theta=120^\circ, 240^\circ$

$\therefore$ P is $\left(-\frac{3}{2},\sqrt{3}\right)$ or $\left(-\frac{3}{2},-\sqrt{3}\right)$ i.e. P is $(3, 0)$ or $(-3, 0)$

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Further coordinate systems
Exercise B, Question 6

Question:

The line $y = 2x + c$ is a tangent to $x^2 + \frac{y^2}{4} = 1$. Find the possible values of c.

Solution:

$y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a^2m^2 + b^2 = c^2$

$y = 2x + c \Rightarrow m = 2, c = ?$

$x^2 + \frac{y^2}{4} = 1 \Rightarrow a = 1, b = 2$

$a^2m^2 + b^2 = c^2 \Rightarrow 1 \times 4 + 4 = c^2$

$\therefore c^2 = 8$

$\therefore c = \pm 2\sqrt{2}$

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Further coordinate systems
Exercise B, Question 7

Question:

The line with equation $y = mx + 3$ is a tangent to $x^2 + \frac{y^2}{5} = 1$.

Find the possible values of m.

Solution:

The $a^2m^2 + b^2 = c^2$ condition could be used as in question 6.

$$\left.\begin{aligned} x^2 + \frac{y^2}{5} &= 1 \\ y &= mx + 3 \end{aligned}\right\} \text{substitution} \Rightarrow x^2 + \frac{(mx+3)^2}{5} = 1$$

i.e. $5x^2 + (mx+3)^2 = 5$

$(5 + m^2 5 + m^2)x^2 + 6mx + 4 = 0$

Since the line is a tangent the discriminant of this equation must equal zero (must have equal roots).

So $36m^2 = 16(5 + m^2)$

$20m^2 = 80$

$m^2 = 4$

$\therefore m = \pm 2$

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Further coordinate systems
Exercise B, Question 8

Question:

The line $y = mx + 4$ $(m > 0)$ is a tangent to the ellipse E with equation $\frac{x^2}{3} + \frac{y^2}{4} = 1$ at the point P.

a Find the value of m.
b Find the coordinates of the point P.
The normal to E at P crosses the y-axis at the point A.
c Find the coordinates of A.
The tangent to E at P crosses the y-axis at the point B.
d Find the area of triangle APB.

Solution:

a $y = mx + 4, \quad \frac{x^2}{3} + \frac{y^2}{4} = 1 \Rightarrow c = 4, a^2 = 3, b^2 = 4$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 4 + 3m^2 = 16$

$3m^2 = 12$

$m = \pm 2$ but $m > 0$

$\therefore m = 2$

b $y = 2x + 4, \frac{x^2}{3} + \frac{y^2}{4} = 1$ substitute $\frac{x^2}{3} + \frac{(4x^2 + 16x + 16)}{4} = 1$

$\Rightarrow x^2 + 3x^2 + 12x + 12 = 3$

$4x^2 + 12x + 9 = 0$

$(2x + 3)^2 = 0$

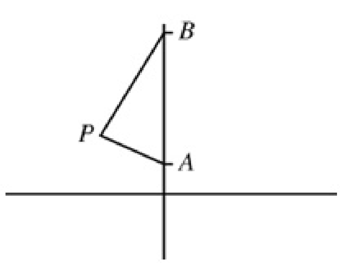
$x = -\frac{3}{2}, y = 2x + 4 = 1 \therefore P$ is $\left(-\frac{3}{2}, 1\right)$

c Gradient of normal $= -\frac{1}{2}$

Equation of normal: $y - 1 = -\frac{1}{2}\left(x - -\frac{3}{2}\right)$

$x = 0 \Rightarrow y = 1 - \frac{3}{4} = \frac{1}{4} \therefore A\left(0, \frac{1}{4}\right)$

d Tangent is $y = 2x + 4, x = 0 \Rightarrow y = 4 \therefore B(0, 4)$



Area of $\Delta APB = \frac{1}{2}\left(4 - \frac{1}{4}\right) \times \frac{3}{2}$

$= \frac{1}{2} \times \frac{15}{4} \times \frac{3}{2} = \frac{45}{16}$

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Further coordinate systems
Exercise B, Question 9

Question:

The ellipse E has equation $\frac{x^2}{9}+\frac{y^2}{4}=1$.

a Show that the gradient of the tangent to E at the point $P(3\cos\theta, 2\sin\theta)$ is $-\frac{2}{3}\cot\theta$.

b Show that the point $Q(\frac{9}{5},-\frac{8}{5})$ lies on E.

c Find the gradient of the tangent to E at Q.

The tangents to E at the points P and Q are perpendicular.

d Find the value of $\tan\theta$ and hence the exact coordinates of P.

Solution:

a $\frac{\mathrm{d}y}{\mathrm{d}\theta}=2\cos\theta, \frac{\mathrm{d}x}{\mathrm{d}\theta}=-3\sin\theta \therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{3}\cot\theta$

b $\frac{\left(\frac{9}{5}\right)^2}{9}+\frac{\left(\frac{-8}{5}\right)^2}{4}=\frac{9}{25}+\frac{16}{25}=1=\text{RHS}$

$\therefore \left(\frac{9}{5},-\frac{8}{5}\right)$ lies on E

c $\frac{9}{5}=3\cos\phi \Rightarrow \cos\phi=\frac{3}{5}$

$-\frac{8}{5}=2\sin\phi \Rightarrow \sin\phi=-\frac{4}{5}$

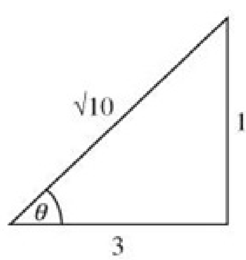
$\therefore \cot\phi=-\frac{3}{4}$ where Q is $(3\cos\phi, 2\sin\phi)$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$

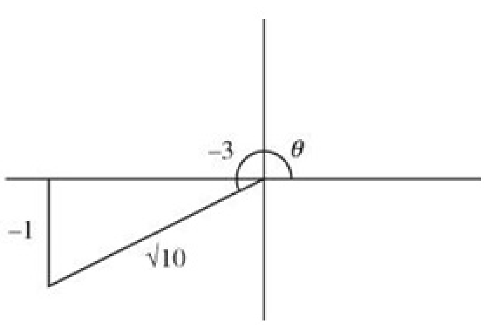
d Gradient of tangent at $P=-2$

$\therefore -2=-\frac{2}{3}\cot\theta \Rightarrow \tan\theta=\frac{1}{3}$

$\therefore$ P is $\left(3\times\frac{3}{\sqrt{10}}, 2\frac{1}{\sqrt{10}}\right)$



or



P is $\left(\frac{9}{10}\sqrt{10}, \frac{2}{10}\sqrt{10}\right)$

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Further coordinate systems
Exercise B, Question 10

Question:

The line $y = mx + c$ is a tangent to both the ellipses $\frac{x^2}{9} + \frac{y^2}{46} = 1$ and $\frac{x^2}{25} + \frac{y^2}{14} = 1$.

Find the possible values of m and c.

Solution:

$y = mx + c$ and $\frac{x^2}{9} + \frac{y^2}{46} = 1 \Rightarrow a^2 = 9, b^2 = 46$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 46 + 9m^2 = c^2$ ①

$y = mx + c$ and $\frac{x^2}{25} + \frac{y^2}{14} = 1 \Rightarrow a^2 = 25, b^2 = 14$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 14 + 25m^2 = c^2$ ②

① − ② $\Rightarrow 32 - 16m^2 = 0$

$\Rightarrow m^2 = 2$

$\therefore m = \pm\sqrt{2}$

$m^2 = 2$ and $14 + 25m^2 = c^2 \Rightarrow c^2 = 64$

$\therefore c = \pm 8$

$\therefore m = \pm 2, c = \pm 8$

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Further coordinate systems
Exercise C, Question 1

Question:

Sketch the following hyperbolae showing clearly the intersections with the x-axis and the equations of the asymptotes.

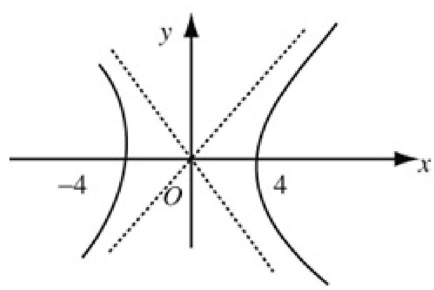
a $x^2-4y^2=16$

b $4x^2-25y^2=100$

c $\frac{x^2}{8}-\frac{y^2}{2}=1$

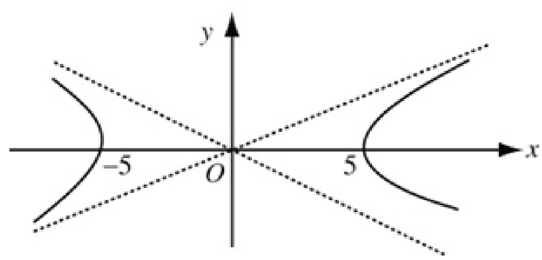
Solution:

a $\frac{x^2}{16}-\frac{y^2}{4}=1$
$a=4, b=2$



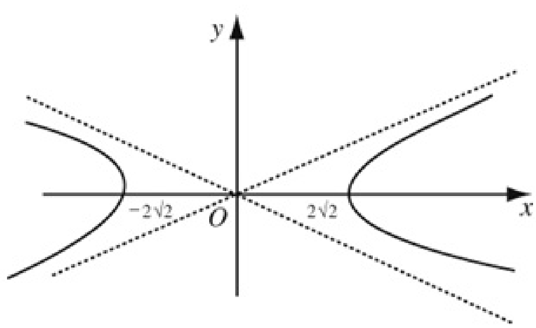
Asymptotes $y=\pm\frac{1}{2}x$

b $4x^2-25y^2=100$
$\Rightarrow \frac{x^2}{25}-\frac{y^2}{4}=1$
$a=5, b=2$



Asymptotes $y=\pm\frac{2}{5}x$

c $\frac{x^2}{8}-\frac{y^2}{2}=1$
$a=2\sqrt{2}, b=\sqrt{2}$



Asymptotes $y=\pm\frac{1}{2}x$

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Further coordinate systems
Exercise C, Question 2

Question:

a Sketch the hyperbolae with the following parametric equations. Give the equations of the asymptotes and show points of intersection with the x-axis.

b Find the Cartesian equation for each hyperbola.

i $x = 2\sec\theta$
$y = 3\tan\theta$

ii $x = 4\cosh t$
$y = 3\sinh t$

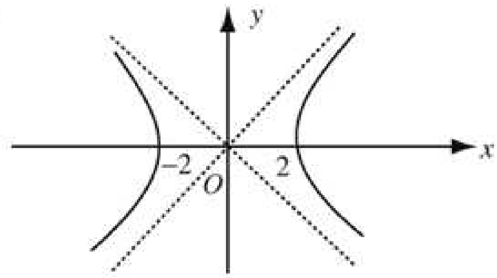
iii $x = \cosh t$
$y = 2\sinh t$

iv $x = 5\sec\theta$
$y = 7\tan\theta$

Solution:

i $x = 2\sec\theta, y = 3\tan\theta$
$a = 2, b = 3$

a

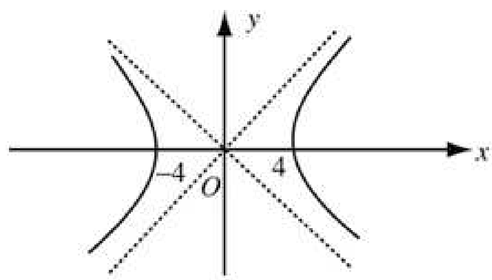


Asymptotes $y = \pm\frac{3}{2}x$

b $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$

ii $x = 4\cosh t, y = 3\sinh t$
$a = 4, b = 3$

a

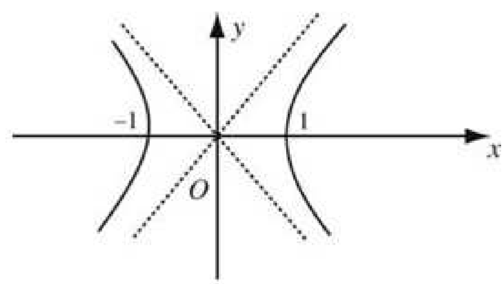


Asymptotes $y = \pm\frac{3}{4}x$

b Equation: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

iii $x = \cosh t, y = 2\sinh t$
$a = 1, b = 2$

a

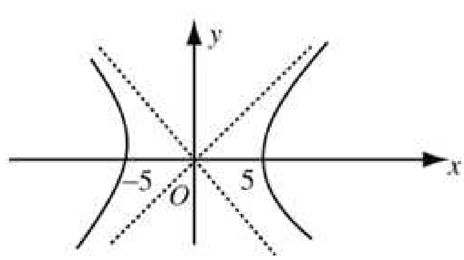


b Equation: $x^2 - \frac{y^2}{4} = 1$

Asymptotes $y = \pm 2x$

iv $x = 5\sec\theta, y = 7\tan\theta$
$a = 5, b = 7$

a



b Equation:
$\frac{x^2}{25} - \frac{y^2}{49} = 1$

Asymptotes $y = \pm\frac{7}{5}x$

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Further coordinate systems
Exercise D, Question 1

Question:

Find the equations of the tangents and normals to the hyperbolae with the following equations at the points indicated.

a $\frac{x^2}{16} - \frac{y^2}{2} = 1$ at the point (12, 4)

b $\frac{x^2}{36} - \frac{y^2}{12} = 1$ at the point (12, 6)

c $\frac{x^2}{25} - \frac{y^2}{3} = 1$ at the point (10, 3)

Solution:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2 x}{a^2 y}$$

a $a^2 = 16, b^2 = 2 \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{8y}$ At (12, 4) $y' = \frac{3}{8}$

At (12, 4) equation of tangent is: $y - 4 = \frac{3}{8}(x - 12)$

$$8y = 3x - 4$$

Equation of normal is: $y - 4 = -\frac{8}{3}(x - 12)$

$$3y + 8x = 108$$

b $a^2 = 36, b^2 = 12 \therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{3y}$ At (12, 6) $y' = \frac{2}{3}$

At (12, 6) equation of tangent is: $y - 6 = \frac{2}{3}(x - 12)$

$$3y = 2x - 6$$

Equation of normal is $y - 6 = -\frac{3}{2}(x - 12)$

$$2y + 3x = 48$$

c $a^2 = 25, b^2 = 3 \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{25y}$ at (10, 3) $y' = \frac{2}{5}$

At (10, 3) equation of tangent is: $y - 3 = \frac{2}{5}(x - 10)$

$$5y = 2x - 5$$

Equation of normal is: $y - 3 = -\frac{5}{2}(x - 10)$

$$2y + 5x = 56$$

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Further coordinate systems
Exercise D, Question 2

Question:

Find the equations of the tangents and normals to the hyperbolae with the following equations at the points indicated.

a $\frac{x^2}{25}-\frac{y^2}{4}=1$ at the point $(5\cosh t, 2\sinh t)$

b $\frac{x^2}{1}-\frac{y^2}{9}=1$ at the point $(\sec t, 3\tan t)$

Solution:

a $x=5\cosh t\ y=2\sinh t \quad \therefore \frac{dy}{dx}=\frac{2\cosh t}{5\sinh t}$

$\therefore$ Equation of tangent: $y-2\sinh t=\frac{2\cosh t}{5\sinh t}(x-5\cosh t)$

$$5y\sinh t+10=2x\cosh t$$

Equation of normal:

$$y-2\sinh t=-\frac{5\sinh t}{2\cosh t}(x-5\cosh t)$$

$$2y\cosh t+5x\sinh t=29\cosh t\sinh t$$

b $x=\sec t, y=3\tan t \quad \therefore \frac{dy}{dx}=\frac{3\sec^2 t}{\sec t\tan t}=\frac{3\sec t}{\tan t}$

$\therefore$ Equation of tangent: $y-3\tan t=\frac{3\sec t}{\tan t}(x-\sec t)$

$$y\tan t+3=3\sec t x$$

Equation of normal: $y-3\tan t=-\frac{\tan t}{3\sec t}(x-\sec t)$

$$3y\sec t+x\tan t=10\sec t\tan t$$

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Further coordinate systems
Exercise D, Question 3

Question:

Show that an equation of the tangent to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec t, b\tan t)$ is $bx\sec t - ay\tan t = ab$.

Solution:

$x = a\sec t \quad y = b\tan t$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t}$

Equation of tangent is:

$$y - b\tan t = \frac{b\sec t}{a\tan t}(x - a\sec t)$$

$$ya\tan t - ab\tan^2 t = b\sec tx - ab\sec^2 t$$

$$ab = bx\sec t - ay\tan t$$

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Further coordinate systems
Exercise D, Question 4

Question:

Show that an equation of the normal to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\cosh t, b\sinh t)$ is $b\cosh ty + a\sinh tx = (a^2 + b^2)\sinh t\cosh t$.

Solution:

$x = a\cosh t \quad y = b\sinh t$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b\cosh t}{a\sinh t}$

$\therefore$ gradient of normal $= -\frac{a\sinh t}{b\cosh t}$

$\therefore$ Equation of normal is:

$$y - b\sinh t = -\frac{a\sinh t}{b\cosh t}(x - a\cosh t)$$

$$yb\cosh t - b^2\sinh t\cosh t = -a\sinh tx + a^2\cosh t\sinh t$$

$$b\cosh ty + a\sinh tx = (a^2 + b^2)\cosh t\sinh t$$

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Further coordinate systems
Exercise D, Question 5

Question:

The point $P(4\cosh t, 3\sinh t)$ lies on the hyperbola with equation $\frac{x^2}{16}-\frac{y^2}{9}=1$.

The tangent at P crosses the y-axis at the point A.

a Find, in terms of t, the coordinates of A.

The normal to the hyperbola at P crosses the y-axis at B.

b Find, in terms of t, the coordinates of B.

c Find, in terms of t, the area of triangle APB.

Solution:

$x=4\cosh t \quad y=3\sinh t \Rightarrow \frac{dy}{dx}=\frac{3\cosh t}{4\sinh t}$

$\therefore$ Equation of tangent is: $y-3\sinh t=\frac{3\cosh t}{4\sinh t}(x-4\cosh t)$

a $x=0 \Rightarrow y=3\sinh t-\frac{3\cosh^2 t}{\sinh t}=-\frac{3}{\sinh t}$

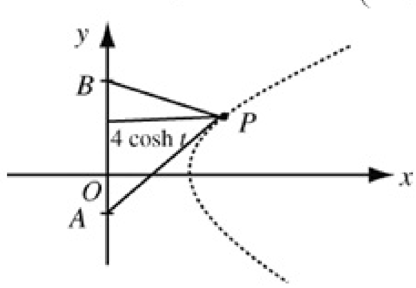
$\therefore A$ is $\left(0,-\frac{3}{\sinh t}\right)$

b Using question 4 with $a=4, b=3$

Normal has equation: $3y\cosh t+4x\sinh t=25\sinh t\cosh t$

$x=0 \Rightarrow y=\frac{25}{3}\sinh t \quad \therefore B$ is $\left(0,\frac{25}{3}\sinh t\right)$

c



Area of $\Delta=\frac{1}{2}\left(\frac{25}{3}\sinh t--\frac{3}{\sinh t}\right)4\cosh t$

$=\frac{2}{3}(25\sinh^2 t+9)\coth t$

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Further coordinate systems
Exercise D, Question 6

Question:

The tangents from the points P and Q on the hyperbola with equation $\frac{x^2}{4}-\frac{y^2}{9}=1$ meet at the point (1, 0).
Find the exact coordinates of P and Q.

Solution:

$$\frac{x^2}{4}-\frac{y^2}{9}=1 \quad x=2\sec t, a=2$$
$$y=3\tan t, b=3$$

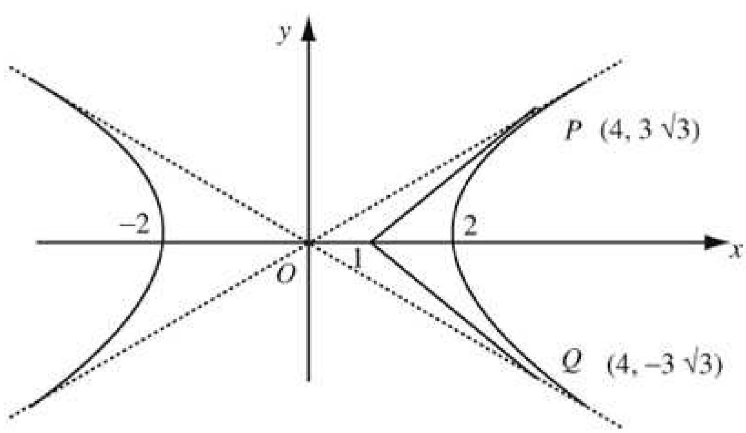
From question 3 the equation of the tangent is:
$3x\sec t-2y\tan t=6$

Tangents meet at (1, 0) so let $x=1, y=0$

$$\Rightarrow \quad 3\sec t=6$$
$$\text{i.e.} \quad \frac{1}{2}=\cos t$$
$$\therefore \quad t=\pm\frac{\pi}{3}$$

$$\sec\left(\pm\frac{\pi}{3}\right)=2, \quad \tan\left(\pm\frac{\pi}{3}\right)=\pm\sqrt{3}$$

$\therefore$ P and Q are $(4,3\sqrt{3})$ and $(4,-3\sqrt{3})$



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Further coordinate systems
Exercise D, Question 7

Question:

The line $y = 2x + c$ is a tangent to the hyperbola $\frac{x^2}{10} - \frac{y^2}{4} = 1$. Find the possible values of c.

Solution:

Using the result $y = mx + c$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for $b^2 + c^2 = a^2 m^2$

$y = 2x + c \quad \therefore m = 2$

$\frac{x^2}{10} - \frac{y^2}{4} = 1 \quad \therefore a^2 = 10, b^2 = 4$

$\therefore 4 + c^2 = 2^2 \times 10 = 40$

$c^2 = 36$

$c = \pm 6$

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Further coordinate systems
Exercise D, Question 8

Question:

The line $y = mx + 12$ is a tangent to the hyperbola $\frac{x^2}{49} - \frac{y^2}{25} = 1$ at the point P.
Find the possible values of m.

Solution:

Use $b^2 + c^2 = a^2m^2$ for $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$y = mx + 12 \Rightarrow c = 12$

$\frac{x^2}{49} - \frac{y^2}{25} = 1 \Rightarrow a^2 = 49, b^2 = 25$

$$\therefore 25 + 12^2 = 49m^2$$
$$169 = 49m^2$$
$$\therefore m^2 = \left(\frac{13}{7}\right)^2$$
$$\therefore m = \pm\frac{13}{7}$$

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Further coordinate systems
Exercise D, Question 9

Question:

The line $y = -x + c, c > 0$, touches the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ at the point P.

a Find the value of c.
b Find the exact coordinates of P.

Solution:

a $m = -1, a = 5, b = 4$

$\therefore \quad 16 + c^2 = 25(-1)^2$

i.e. $c^2 = 9$

$\therefore c = \pm 3 \qquad \because c > 0 \therefore c = 3$

Use $b^2 + c^2 = a^2m^2$ for $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

b $y = (3 - x)$, substitute into hyperbola

$$\frac{x^2}{25} - \frac{(3-x)^2}{16} = 1$$

$$16x^2 - 25(9 + x^2 - 6x) = 25 \times 16$$

$$-9x^2 - 225 + 150x = 400$$

$$0 = 9x^2 - 150x + 625$$

$$0 = (3x - 25)^2$$

$$\therefore x = \frac{25}{3}, y = -\frac{16}{3}$$

So P is $\left(\frac{25}{3}, \frac{-16}{3}\right)$

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Further coordinate systems
Exercise D, Question 10

Question:

The line with equation $y = mx + c$ is a tangent to both hyperbolae $\frac{x^2}{4} - \frac{y^2}{15} = 1$ and $\frac{x^2}{9} - \frac{y^2}{95} = 1$.

Find the possible values of m and c.

Solution:

Use $b^2 + c^2 = a^2m^2$ for $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Using $\frac{x^2}{4} - \frac{y^2}{15} = 1 \Rightarrow a^2 = 4, b^2 = 15$

$\therefore 15 + c^2 = 4m^2$ ①

Using $\frac{x^2}{9} - \frac{y^2}{95} = 1 \Rightarrow a^2 = 9, b^2 = 95$

$\therefore 95 + c^2 = 9m^2$ ②

Solving

② − ① $80 = 5m^2$

$\therefore m^2 = 16$

$m = \pm 4$

$m = \pm 4$ $c^2 = 4(16) - 15$

$= 49$ $\therefore c = \pm 7$

$\therefore m = \pm 4$ and $c = \pm 7$

i.e. lines $y = 4x \pm 7$ and $y = -4x \pm 7$

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Further coordinate systems
Exercise E, Question 1

Question:

Find the eccentricity of the following ellipses.

a $\frac{x^2}{9}+\frac{y^2}{5}=1$

b $\frac{x^2}{16}+\frac{y^2}{9}=1$

c $\frac{x^2}{4}+\frac{y^2}{8}=1$

Solution:

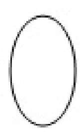
a $a^2=9 \quad b^2=5$

$b^2=a^2(1-e^2) \Rightarrow \frac{5}{9}=1-e^2 \;\therefore e^2=\frac{4}{9} \;\therefore e=\frac{2}{3}$

b $a^2=16 \quad b^2=9$

$b^2=a^2(1-e^2) \Rightarrow \frac{9}{16}=1-e^2 \;\therefore e^2=\frac{7}{16} \;\therefore e=\frac{\sqrt{7}}{4}$

c $a^2=4 \quad b^2=8$

Need to use $a^2=b^2(1-e^2)$ since ellipse is  shape.

$\frac{4}{8}=1-e^2 \Rightarrow e^2=\frac{1}{2} \therefore e=\frac{1}{\sqrt{2}}$

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Further coordinate systems
Exercise E, Question 2

Question:

Find the foci and directrices of the following ellipses.

a $\frac{x^2}{4}+\frac{y^2}{3}=1$

b $\frac{x^2}{16}+\frac{y^2}{7}=1$

c $\frac{x^2}{5}+\frac{y^2}{9}=1$

Solution:

a $a^2=4 \quad b^2=3$

$b^2=a^2(1-e^2) \Rightarrow \frac{3}{4}=1-e^2 \;\therefore e^2=\frac{1}{4} \;\therefore e=\frac{1}{2}$

Focus $(\pm ae,0)=(\pm 1,0)$; directrix $x=\pm\frac{a}{e} \Rightarrow x=\pm 4$

b $a^2=16 \quad b^2=7$

$b^2=a^2(1-e^2) \Rightarrow \frac{7}{16}=1-e^2 \;\therefore e^2=\frac{9}{16} \;\therefore e=\frac{3}{4}$

Focus $(\pm ae,0)=(\pm 3,0)$; directrix $x=\pm\frac{a}{e} \Rightarrow x=\pm\frac{16}{3}$

c $a^2=5, b^2=9$

Since $b>a$

Use $a^2=b^2(1-e^2) \Rightarrow \frac{5}{9}=1-e^2$

$\therefore e^2=\frac{4}{9} \;\therefore e=\frac{2}{3}$

Focus is $(0,\pm be)$ i.e. focus $(0,\pm 2)$

Directrix $y=\pm\frac{b}{e}$ i.e. $y=\pm\frac{9}{2}$

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Further coordinate systems
Exercise E, Question 3

Question:

An ellipse E has focus (3, 0) and the equation of the directrix is $x = 12$. Find **a** the value of the eccentricity **b** the equation of the ellipse.

Solution:

a $ae = 3 \quad \frac{a}{e} = 12$

$\Rightarrow ae \times \frac{a}{e} = a^2 = 36$

$\Rightarrow \quad a = 6, e = \frac{1}{2}$

b $b^2 = a^2(1 - e^2)$

$= 36\left(1 - \frac{1}{4}\right) = 36 \times \frac{3}{4} = 27$

$\therefore$ equation is $\frac{x^2}{36} + \frac{y^2}{27} = 1$

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Further coordinate systems
Exercise E, Question 4

Question:

An ellipse E has focus (2, 0) and the directrix has equation $x = 8$. Find **a** the value of the eccentricity **b** the equation of the ellipse.

Solution:

$ae = 2 \quad \frac{a}{e} = 8$

a $\Rightarrow \quad ae \times \frac{a}{e} = a^2 = 16$

$\Rightarrow \quad a = 4, e = \frac{1}{2}$

b $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = 16\left(1 - \frac{1}{4}\right) = 16 \times \frac{3}{4} = 12$

$\therefore$ equation is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

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Further coordinate systems
Exercise E, Question 5

Question:

Find the eccentricity of the following hyperbolae.

a $\frac{x^2}{5} - \frac{y^2}{3} = 1$

b $\frac{x^2}{9} - \frac{y^2}{7} = 1$

c $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Solution:

a $\frac{x^2}{5} - \frac{y^2}{3} = 1 \Rightarrow a^2 = 5, b^2 = 3$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{3}{5} = e^2 - 1 \therefore e^2 = \frac{8}{5} \therefore e = \frac{2\sqrt{2}}{\sqrt{5}} = \frac{2\sqrt{10}}{5}$

b $\frac{x^2}{9} - \frac{y^2}{7} = 1 \Rightarrow a^2 = 9, b^2 = 7$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{7}{9} = e^2 - 1 \therefore e^2 = \frac{16}{9} \therefore e = \frac{4}{3}$

c $\frac{x^2}{9} - \frac{y^2}{16} = 1 \Rightarrow a^2 = 9, b^2 = 16$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{16}{9} = e^2 - 1 \therefore e^2 = \frac{25}{9} \therefore e = \frac{5}{3}$

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Further coordinate systems
Exercise E, Question 6

Question:

Find the foci of the following hyperbolae and sketch them, showing clearly the equations of the asymptotes.

a $\frac{x^2}{4} - \frac{y^2}{8} = 1$

b $\frac{x^2}{16} - \frac{y^2}{9} = 1$

c $\frac{x^2}{4} - \frac{y^2}{5} = 1$

Solution:

a $\frac{x^2}{4} - \frac{y^2}{8} = 1$

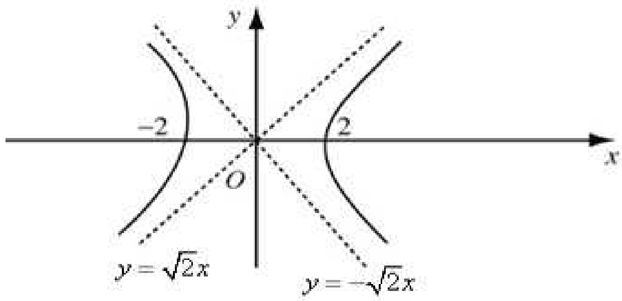
$a = 2, b = 2\sqrt{2}$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{8}{4} = e^2 - 1$

$\Rightarrow e = \sqrt{3}$

so foci are $(\pm 2\sqrt{3}, 0$

Asymptotes are $y = \pm\sqrt{2}x$



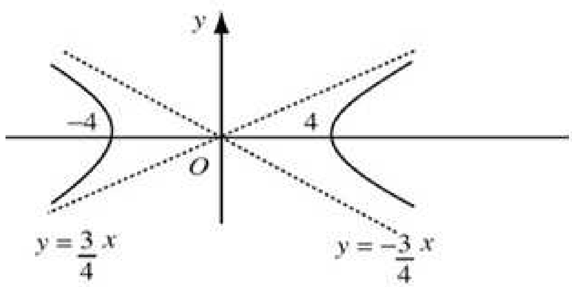
b $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4, b = 3$

$\Rightarrow 9 = 16(e^2 - 1) \Rightarrow e^2 = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

so foci are $(\pm\frac{5}{2}, 0)$

Asymptotes $y = \pm\frac{3}{4}x$



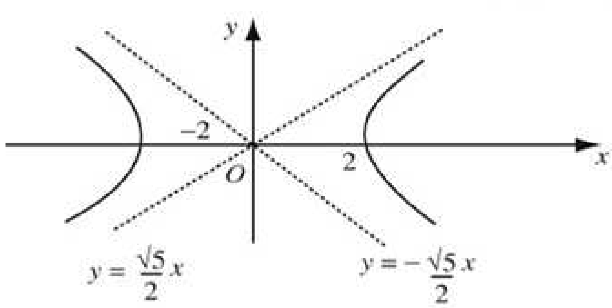
c $\frac{x^2}{4} - \frac{y^2}{5} = 1$

$a = 2, b = \sqrt{5}$

$\Rightarrow 5 = 4(e^2 - 1) \Rightarrow e^2 = \frac{9}{4} \Rightarrow e = \frac{3}{2}$

so foci are $(\pm 3, 0)$

Asymptotes $y = \pm\frac{\sqrt{5}}{2}x$



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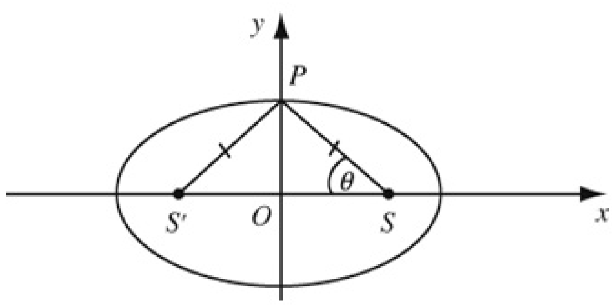
Further coordinate systems
Exercise E, Question 7

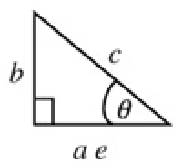
Question:

Ellipse E has equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The foci are at S and S' and the point P is $(0, b)$.
Show that $\cos(PSS')=e$, the eccentricity of E.

Solution:



Consider ΔPOS



$c^2=b^2+a^2e^2$, but $b^2=a^2(1-e^2)$

$\therefore c^2=a^2-a^2e^2+a^2e^2=a^2$

$\therefore c=a$

So $\cos\theta=\frac{ae}{a}=e$

If you use the result that $SP+S'P=2a$ then since $S'P=SP$ it is clear $SP=a$
Hence $\cos\theta=\frac{ae}{a}=e$.

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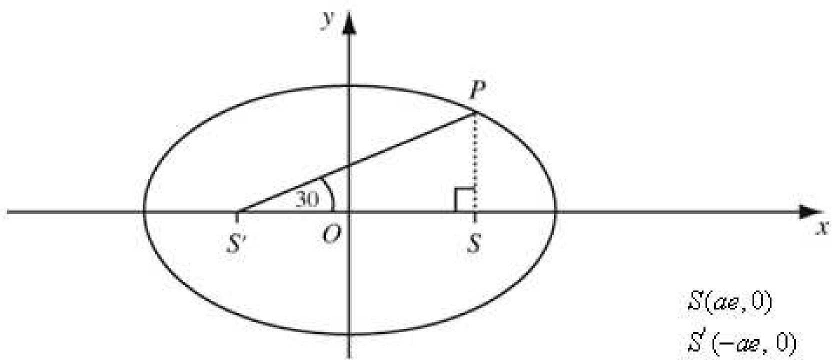
Further coordinate systems
Exercise E, Question 8

Question:

The ellipse E has foci at S and S'. The point P on E is such that angle PSS' is a right angle and angle $PS'S = 30°$.

Show that the eccentricity of the ellipse, e, is $\frac{1}{\sqrt{3}}$.

Solution:



$S(ae, 0)$
$S'(-ae, 0)$

PS is y where $\frac{a^2e^2}{a^2}+\frac{y^2}{b^2}=1$

$$y^2=b^2(1-e^2)$$
$$y=b\sqrt{1-e^2}$$
$$SS'=2ae$$

$$\tan 30=\frac{1}{\sqrt{3}}=\frac{y}{2ae}=\frac{b\sqrt{1-e^2}}{2ae}$$

But $b^2=a^2(1-e^2)$

$$\therefore \quad \frac{1}{\sqrt{3}}=\frac{a\sqrt{1-e^2}\sqrt{1-e^2}}{2ae}$$
$$\frac{2e}{\sqrt{3}}=1-e^2$$

$$e^2+\frac{2}{\sqrt{3}}e-1=0$$

$$\Rightarrow e^2+\frac{2}{\sqrt{3}}e+\frac{1}{3}=1+\frac{1}{3}$$

$$\Rightarrow \left(e+\frac{1}{\sqrt{3}}\right)^2=1+\frac{1}{3}=\frac{4}{3}$$

$$\therefore e+\frac{1}{\sqrt{3}}=\frac{2}{\sqrt{3}} \quad \therefore e=\frac{1}{\sqrt{3}}$$

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Further coordinate systems
Exercise F, Question 1

Question:

The tangent at $P(ap^2, 2ap)$ and the tangent at $Q(aq^2, 2aq)$ to the parabola with equation $y^2 = 4ax$ meet at R.

a Find the coordinates of R.

The chord PQ passes through the focus $(a, 0)$ of the parabola.

b Show that the locus of R is the line $x = -a$.

Given instead that the chord PQ has gradient 2,

c find the locus of R.

Solution:

a Using table in Section 2.6

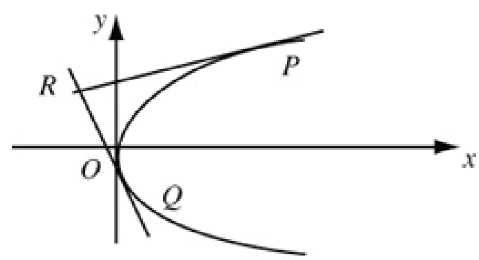
Tangent at P is $py = x + ap^2$

Tangent at Q is $qy = x + aq^2$

$(p-q)y = a(p-q)(p+q) \quad \therefore y = a(p+q)$

$\Rightarrow ap^2 + apq = x + ap^2 \quad \therefore x = apq$

So R is $(apq, a(p+q))$



b Chord PQ has gradient: $\dfrac{2ap - 2aq}{ap^2 - aq^2} = \dfrac{2a(p-q)}{a(p-q)(p+q)} = \dfrac{2}{(p+q)}$

$\therefore$ Equation of chord PQ is: $y - 2ap = \dfrac{2}{p+q}(x - ap^2)$

i.e. $y(p+q) \cancel{-2ap^2} - 2apq = 2x \cancel{-2ap^2}$

i.e. $y(p+q) = 2x + 2apq$

Chord passes through $(a, 0) \Rightarrow 0 = 2a + 2apq$ or $pq = -1$

$\therefore$ locus of R is $x = -a$

c Gradient of chord PQ is $\dfrac{2}{p+q} = 2 \Rightarrow p + q = 1$

$\therefore$ locus of R is: $y = a(p+q) = a$

i.e. $y = a$

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Further coordinate systems
Exercise F, Question 2

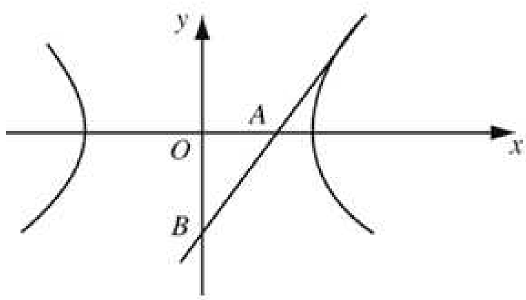
Question:

The tangent at $P(a\sec t, b\tan t)$ to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the x-axis at A and the y-axis at B.
Find the locus of the mid-point of AB.

Solution:

Equation of tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec t, b \tan t)$ is: $bx\sec t - ay\tan t = ab$

See summary



A is where $y = 0 \Rightarrow x = \frac{ab}{b\sec t} = a\cos t$
i.e. $A\ (a\cos t, 0)$
B is where $x = 0 \Rightarrow y = \frac{ab}{-a\tan t} = -b\cot t$
i.e. $B(0, -b\cot t)$

Mid-point of AB is $\left(\frac{a}{2}\cos t, -\frac{b}{2}\cot t\right)$

$x = \frac{a}{2}\cos t \Rightarrow \sec t = \frac{a}{2x}$

$y = -\frac{b}{2}\cot t \Rightarrow \tan t = -\frac{b}{2y}$

Use $\sec^2 t = 1 + \tan^2 t$

$\Rightarrow \frac{a^2}{4x^2} = 1 + \frac{b^2}{4y^2}$ which gives locus.

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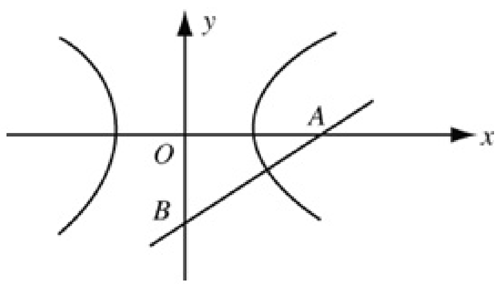
Further coordinate systems
Exercise F, Question 3

Question:

The normal at $P(a\sec t, b\tan t)$ to the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the x-axis at A and the y-axis at B.
Find the locus of the mid-point of AB.

Solution:

Normal to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec t, b\tan t)$ is $ax\sin t+by=(a^2+b^2)\tan t$



$y=0\Rightarrow x=\left(\frac{a^2+b^2}{a}\right)\sec t \quad \therefore A \text{ is } \left(\frac{\left[a^2+b^2\right]}{a}\sec t, 0\right)$

$x=0\Rightarrow y=\left(\frac{a^2+b^2}{b}\right)\tan t \quad \therefore B \text{ is } \left(0, \frac{\left[a^2+b^2\right]}{b}\tan t\right)$

Mid-point of AB is $\left(\frac{(a^2+b^2)}{2a}\sec t, \frac{(a^2+b^2)}{2b}\tan t\right)$

$x=\frac{(a^2+b^2)}{2a}\sec t\Rightarrow \sec t=\frac{2ax}{a^2+b^2}$

$y=\frac{(a^2+b^2)}{2b}\tan t\Rightarrow \tan t=\frac{2by}{a^2+b^2}$

Use $\sec^2 t=1+\tan^2 t$

$\therefore 4a^2x^2=(a^2+b^2)^2+4b^2y^2$

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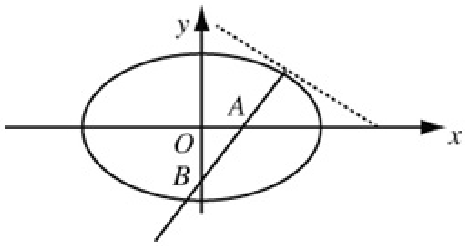
Further coordinate systems
Exercise F, Question 4

Question:

The normal at $P(a\cos t, b\sin t)$ to the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts the x-axis at A and the y-axis at B.
Find the locus of the mid-point of AB.

Solution:

Normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos t, b\sin t)$ is
$ax\sin t - by\cos t = (a^2-b^2)\cos t\sin t$



$y = 0 \Rightarrow x = \left(\frac{a^2-b^2}{a}\right)\cos t \quad \therefore A \text{ is } \left(\frac{[a^2-b^2]}{a}\cos t, 0\right)$

$x = 0 \Rightarrow y = -\left(\frac{a^2-b^2}{b}\right)\sin t \quad \therefore B \text{ is } \left(0, -\frac{(a^2-b^2)}{b}\sin t\right)$

Mid-point of AB is $\left(\frac{[a^2-b^2]}{2a}\cos t, -\frac{[a^2-b^2]}{2b}\sin t\right)$

$x = \frac{(a^2-b^2)}{2a}\cos t \Rightarrow \cos t = \frac{2ax}{a^2-b^2}$

$y = -\frac{(a^2-b^2)}{2b}\sin t \Rightarrow \sin t = -\frac{2by}{a^2-b^2}$

Use $\sin^2 t + \cos^2 t = 1$

$\therefore 4b^2y^2 + 4a^2x^2 = (a^2-b^2)^2$

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Further coordinate systems
Exercise F, Question 5

Question:

The tangent from the point $P\left(cp, \frac{c}{p}\right)$ and the tangent from the point $Q\left(cq, \frac{c}{q}\right)$ to the rectangular hyperbola $xy = c^2$, intersect at the point R.

a Show that R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

b Show that the chord PQ has equation $ypq + x = c(p+q)$

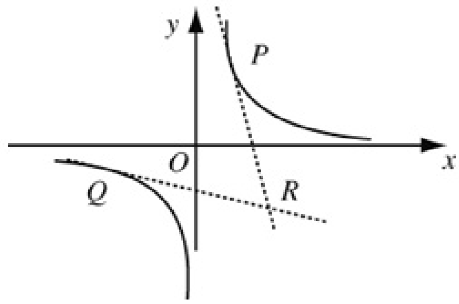
c Find the locus of R in the following cases

i when the chord PQ has gradient 2

ii when the chord PQ passes through the point (1, 0)

iii when the chord PQ passes through the point (0, 1).

Solution:



From table in Section 2.6 the equation of tangent at P is: $x + p^2 y = 2cp$

a Similarly the equation of tangent at Q is: $x + q^2 y = 2cq$

Solving: $\cancel{(p-q)}(p+q)y = 2c\,\cancel{(p-q)}$ $\quad \therefore y = \dfrac{2c}{p+q}$, $x = \dfrac{2cpq}{p+q}$

$\therefore R$ is $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$

b Gradient of chord PQ is: $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

$\therefore$ Equation of chord is: $y - \dfrac{c}{p} = -\dfrac{1}{pq}(x - cp)$ i.e. $ypq + x = c(p+q)$

c **i** $-\dfrac{1}{pq} = 2 \therefore pq = -\dfrac{1}{2}$.

R is: $x = -\dfrac{c}{p+q}, y = \dfrac{2c}{p+q} \Rightarrow y = -2x$

ii Chord through $(1, 0) \Rightarrow 1 = c(p+q)$

R is $x = \dfrac{2cpq}{\frac{1}{c}}, y = \dfrac{2c}{\frac{1}{c}} \Rightarrow y = 2c^2$

iii Chord through $(0, 1) \Rightarrow pq = c(p+q)$

R is $x = \dfrac{2c^2(p+q)}{(p+q)} \Rightarrow x = 2c^2$

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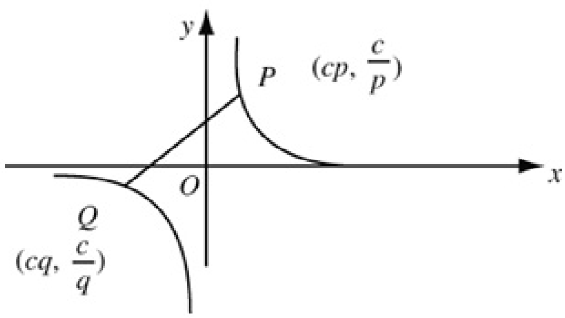
Further coordinate systems
Exercise F, Question 6

Question:

The chord PQ to the rectangular hyperbola $xy = c^2$ passes through the point (0, 1). Find the locus of the mid-point of PQ as P and Q vary.

Solution:



Gradient of chord: $\dfrac{\frac{c}{p}-\frac{c}{q}}{cp-cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

Equation of chord: $y - \dfrac{c}{p} = -\dfrac{1}{pq}(x - cp)$

$$ypq - cq = -x + cp$$

$$\therefore \quad ypq + x = c(p+q)$$

Mid-point of chord is $\left(\dfrac{c(p+q)}{2}, \dfrac{c(p+q)}{2pq}\right)$

Chord passes through $(0,1) \Rightarrow pq = c(p+q)$

Mid-point is: $x = \dfrac{c(p+q)}{2}$

$$y = \frac{c(p+q)}{2pq}$$

Substitute $pq = c(p+q) \Rightarrow y = \dfrac{c(p+q)}{2c(p+q)} = \dfrac{1}{2}$

$\therefore$ locus is line $y = \dfrac{1}{2}$

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Further coordinate systems
Exercise G, Question 1

Question:

A hyperbola of the form $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$ has asymptotes with equations $y=\pm mx$ and passes through the point $(a, 0)$.

a Find an equation of the hyperbola in terms of x, y, a and m.

A point P on this hyperbola is equidistant from one of its asymptotes and the x-axis.

b Prove that, for all values of m, P lies on the curve with equation

$(x^2-y^2)^2=4x^2(x^2-a^2)$ **[E]**

Solution:

$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$

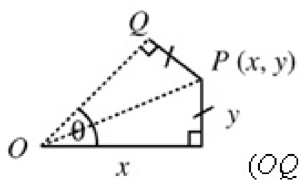
a Asymptotes are $y = \pm\frac{\beta}{\alpha}x$ $\quad\therefore m = \frac{\beta}{\alpha}$

Passes through $(a, 0) \Rightarrow \frac{a^2}{\alpha^2} - 0 = 1$ $\quad\therefore a = \alpha$

$\therefore \beta = am$

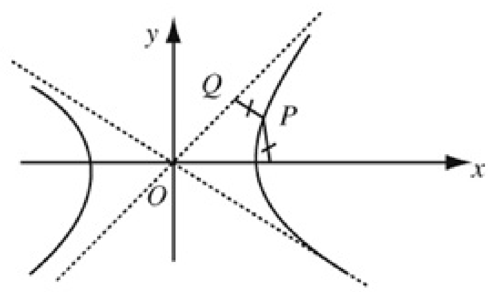
$\therefore$ Equation is $\frac{x^2}{a^2} - \frac{y^2}{a^2m^2} = 1$

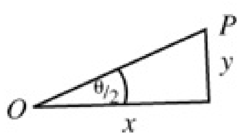
b



$(OQ = x)$

$m = \tan\theta$





$\tan\frac{\theta}{2} = \frac{y}{x}$

Using $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} \Rightarrow m = \frac{2\frac{y}{x}}{1 - \frac{y^2}{x^2}} = \frac{2xy}{(x^2 - y^2)}$ ①

But P lies on the hyperbola $\therefore x^2m^2 - y^2 = a^2m^2$

So $m^2 = \frac{y^2}{x^2 - a^2}$ ②

Using ①^2 and ② $\frac{4x^2y^2}{(x^2 - y^2)^2} = \frac{y^2}{x^2 - a^2}$

i.e. $4x^2(x^2 - a^2) = (x^2 - y^2)^2$

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Further coordinate systems
Exercise G, Question 2

Question:

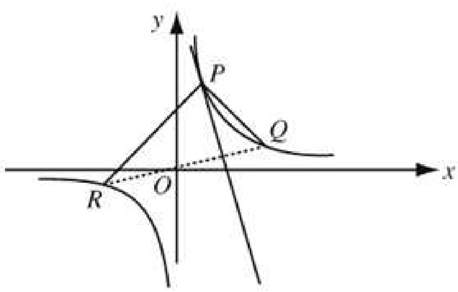
a Prove that the gradient of the chord joining the point $P\left(cp, \frac{c}{p}\right)$ and the point $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola with equation $xy = c^2$ is $-\frac{1}{pq}$.

The points P, Q and R lie on a rectangular hyperbola, the angle QPR being a right angle.

b Prove that the angle between QR and the tangent at P is also a right angle. **[E]**

Solution:

a Gradient of chord $= \dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{\not{c}(q-p)}{pq\not{c}(p-q)} = \dfrac{-1}{pq}$



$P\left(cp, \frac{c}{p}\right)$; $Q\left(cq, \frac{c}{q}\right)$; $R\left(cr, \frac{c}{r}\right)$

b Gradient of $PQ = -\frac{1}{pq}$

Gradient of $PR = -\frac{1}{pr}$

$\therefore$ If $Q\hat{P}R = 90° \Rightarrow -\frac{1}{pq} \times -\frac{1}{pr} = -1$

$\Rightarrow -1 = p^2qr$ ①

To find gradient of tangent at P let $q \to p$ for chord PQ

$\therefore$ Gradient of tangent at P is $-\frac{1}{p^2}$

Gradient of chord $RQ = -\frac{1}{qr}$.

So $\frac{-1}{qr} \times -\frac{1}{p^2} = \frac{1}{p^2qr}$

But from ① $p^2qr = -1$ $\therefore$ gradient of tangent at P $\times$ gradient of $QR = -1$.

Therefore tangent at P is perpendicular to chord QR.

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Further coordinate systems
Exercise G, Question 3

Question:

a Show that an equation of the tangent to the rectangular hyperbola with equation $xy = c^2$ (with $c > 0$) at the point $\left(ct, \frac{c}{t}\right)$ is $t^2y + x = 2ct$

Tangents are drawn from the point $(-3, 3)$ to the rectangular hyperbola with equation $xy = 16$.

b Find the coordinates of the points of contact of these tangents with the hyperbola.

[E]

Solution:

a $y = ct^{-1}, x = ct \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$

$\therefore$ Equation of tangent is: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

i.e. $yt^2 - ct = -x + ct$

or $t^2y + x = 2ct$

b Let $S\left(cs, \frac{c}{s}\right)$ be another point an $xy = 16 (c = 4)$

$\therefore$ tangent at S is $s^2y + x = 2cs$

Intersection of tangents is: $(t^2 - s^2)y = 2c(t - s)$

$$y = \frac{2c}{t + s}$$

$$\therefore x = 2ct - \frac{2ct^2}{t + s} = \frac{2cts}{t + s}$$

So when $c = 4$ intersection is $\left(\frac{8ts}{t + s}, \frac{8}{t + s}\right)$

Now $x = -3, y = 3 \Rightarrow \left.\begin{aligned} 3(t + s) &= 8 \\ -3(t + s) &= 8ts \end{aligned}\right\} \Rightarrow ts = -1$

$$t = -\frac{1}{s}$$

$\therefore 3\left(s - \frac{1}{s}\right) = 8$

$\Rightarrow 3s^2 - 8s - 3 = 0$

$(3s + 1)(s - 3) = 0$

$\therefore s = 3$ or $-\frac{1}{3}$

$t = -\frac{1}{3}$ or 3

So points are $\left(-\frac{4}{3}, -12\right)$ and $\left(12, \frac{4}{3}\right)$

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Further coordinate systems
Exercise G, Question 4

Question:

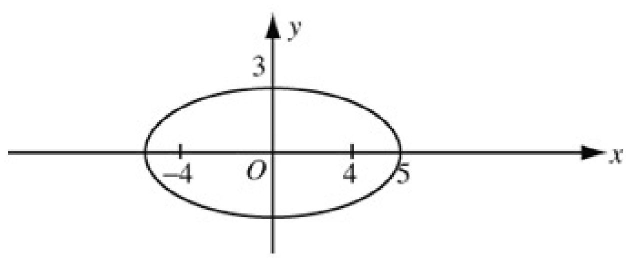
The point P lies on the ellipse with equation $9x^2 + 25y^2 = 225$, and A and B are the points $(-4, 0)$ and $(4, 0)$ respectively.

a Prove that $PA + PB = 10$

b Prove also that the normal at P bisects the angle APB. **[E]**

Solution:

a $9x^2 + 25y^2 = 225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$



$\therefore a = 5, b = 3$

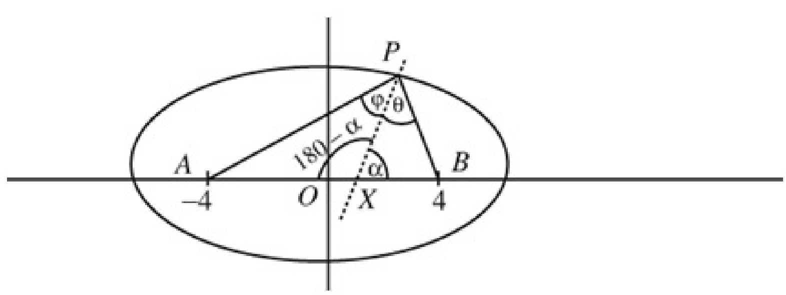
$b^2 = a^2(1 - e^2) \Rightarrow 9 = 25(1 - e^2) \quad \therefore e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

$\therefore$ Foci are $(\pm 4, 0)$ So A and B are the foci.

Since $PS + PS' = 2a$

$\therefore PA + PB = 2 \times 5 = 10$

b



Normal at P is: $5x \sin t - 3y \cos t = 16 \cos t \sin t$

$\therefore X$ is when $y = 0$ i.e. $\frac{16}{5} \cos t$

$PB^2 = (5\cos t - 4)^2 + (3\sin t)^2 = 25\cos^2 t - 40\cos t + 16 + 9\sin^2 t$

$= 16\cos^2 t - 40\cos t + 25 = (4\cos t - 5)^2$

$\therefore PB = 5 - 4\cos t$

$\therefore PA = 10 - PB = 5 + 4\cos t$

$AX = 4 + \frac{16}{5}\cos t, BX = 4 - \frac{16}{5}\cos t$

Consider sine rule on ΔPAX.

$$\sin\phi = \frac{\sin(180-\alpha)AX}{AP} = \frac{\sin\alpha\left(4+\frac{16}{5}\cos t\right)}{5+4\cos t}$$

$$= \frac{\sin\alpha 4(5+4\cos t)}{5(5+4\cos t)}$$

$$= \frac{4}{5}\sin\alpha$$

Consider sine rule on ΔPBX

$$\sin\theta = \frac{BX\sin\alpha}{PB} = \frac{\sin\alpha\left(4-\frac{16}{5}\cos t\right)}{5-4\cos t}$$

$$= \frac{\sin\alpha 4(5-4\cos t)}{5(5-4\cos t)}$$

$$= \frac{4}{5}\sin\alpha$$

$\therefore \sin\phi = \sin\theta$ and since both $< 90° \theta = \phi$

$\therefore$ Normal bisects APB.

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Further coordinate systems
Exercise G, Question 5

Question:

A curve is given parametrically by $x = ct, y = \frac{c}{t}$.

Show that an equation of the tangent to the curve at the point $\left(ct, \frac{c}{t}\right)$ is $t^2y + x = 2ct$

The point P is the foot of the perpendicular from the origin to this tangent.

b Show that the locus of P is the curve with equation $\left(x^2 + y^2\right)^2 = 4c^2xy$ **[E]**

Solution:

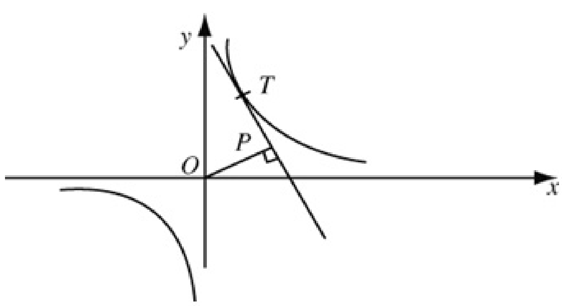
a $y = ct^{-1}, x = ct \quad \therefore \frac{dy}{dx} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$

$\therefore$ Equation of tangent is: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

i.e. $yt^2 - ct = -x + ct$

or $t^2y + x = 2ct$

b



Gradient of tangent is $-\frac{1}{t^2}$

$\therefore$ Gradient of OP is t^2

$\therefore$ Equation of OP is $y = t^2x$

Equation of tangent is $t^2y = 2ct - x$

Solving $t^4x = 2ct - x$

$$\therefore \quad x = \frac{2ct}{1+t^4}, y = \frac{2ct^3}{1+t^4}$$

$$x^2 + y^2 = \frac{4c^2t^2 + 4c^2t^6}{(1+t^4)^2} = \frac{4c^2t^2\,(1+t^4)}{(1+t^4)^2}$$

$$\therefore (x^2+y^2)^2 = \frac{16c^4t^4}{(1+t^4)^2}$$

$$xy = \frac{4c^2t^4}{(1+t^4)^2}$$

$$\therefore (x^2+y^2)^2 = 4c^2xy$$

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Further coordinate systems
Exercise G, Question 6

Question:

a Find the gradient of the parabola with equation $y^2 = 4ax$ at the point $P(at^2, 2at)$.

b Hence show that the equation of the tangent at this point is $x - ty + at^2 = 0$.

The tangent meets the y-axis at T, and O is the origin.

c Show that the coordinates of the centre of the circle through O, P and T are

$$\left(\frac{at^2}{2} + a, \frac{at}{2}\right).$$

d Deduce that, as t varies, the locus of the centre of this circle is another parabola. **[E]**

Solution:

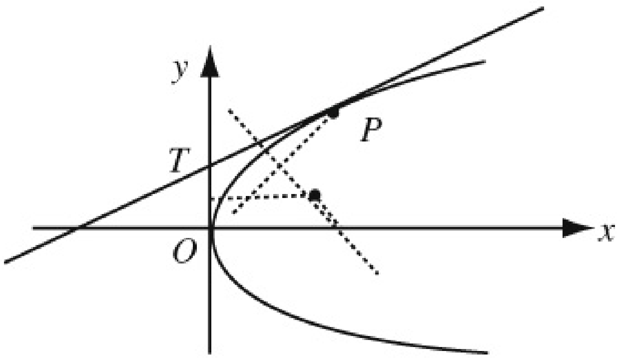
a $\left.\begin{array}{l} y = 2at \\ x = at^2 \end{array}\right\} \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

b Equation of tangent is: $y - 2at = \dfrac{1}{t}(x - at^2)$

or $yt - 2at^2 = x - at^2$

i.e. $yt = x + at^2$

i.e. $x - yt + at^2 = 0$



T is $(0, at)$

c Centre of circle will be intersection of perpendicular bisectors of OT and OP.

Mid-point of OP is $\left(\dfrac{at^2}{2}, at\right)$

Gradient of $OP = \dfrac{2at}{at^2} = \dfrac{2}{t}$ $\therefore$ Equation of perpendicular bisector of OP is:

$y - at = -\dfrac{t}{2}\left(x - \dfrac{at^2}{2}\right)$

Intersects $y = \dfrac{at}{2}$. When $\dfrac{at}{2} = +\dfrac{t}{2}\left(x - \dfrac{at^2}{2}\right)$

$\therefore$ Centre of circle is $\left(a + \dfrac{at^2}{2}, \dfrac{at}{2}\right)$

d $X = a + \dfrac{at^2}{2} \Rightarrow at^2 = 2(X - a)$

$Y = \dfrac{at}{2} \Rightarrow 2at = 4Y$

$\therefore (4Y)^2 = 4a \times 2(X - a)$ or $2Y^2 = a(X - a)$

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Further coordinate systems
Exercise G, Question 7

Question:

The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola with equation $y^2 = 4ax$.
The angle $POQ = 90°$, where O is the origin.

a Prove that $pq = -4$

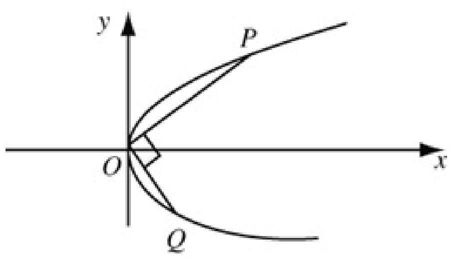
Given that the normal at P to the parabola has equation
$y + xp = ap^3 + 2ap$

b write down an equation of the normal to the parabola at Q.

c Show that these two normals meet at the point R, with coordinates
$(ap^2 + aq^2 - 2a, 4a[p+q])$

d Show that, as p and q vary, the locus of R has equation $y^2 = 16ax - 96a^2$. **[E]**

Solution:



a Gradient $OP = \frac{2ap}{ap^2} = \frac{2}{p}$, gradient of $OQ = \frac{2}{q}$

Since perpendicular $\frac{4}{pq} = -1 \quad \therefore pq = -4$

b Normal at Q is $y + xq = aq^3 + 2aq$

c Normal at P is $y + xp = ap^3 + 2ap$

Solving $x(q-p) = a(q^3 - p^3) + 2a(q-p)$

$x\cancel{(q-p)} = a\cancel{(q-p)}(q^2 + qp + p^2) + 2a\cancel{(q-p)}$

$x = a\left[q^2 + p^2 + qp + 2\right]$

$y = ap^3 + \cancel{2ap} - apq^2 - ap^3 - aqp^2 - \cancel{2ap}$ i.e. $y = -apq(q+p)$

But if $pq = -4$ R is $\left[aq^2 + ap^2 - 2a, 4a(p+q)\right]$

d $X = a((p+q)^2 - 2pq - 2) = a\left[(p+q)^2 + 6\right]$

$Y = 4a(p+q) \Rightarrow p + q = \frac{Y}{4a}$

$\therefore X = a\left[\frac{Y^2}{16a^2} + 6\right]$

$X - 6a = \frac{Y^2}{16a} \quad \therefore Y^2 = 16aX - 96a^2$

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Further coordinate systems
Exercise G, Question 8

Question:

Show that for all values of m, the straight lines with equations $y = mx \pm \sqrt{b^2 + a^2m^2}$ are tangents to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[E]**

Solution:

$y = mx + c$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2(mx + c)^2 = a^2b^2$

i.e. $b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$

i.e. $x^2(b^2 + a^2m^2) + 2a^2mcx + a^2(c^2 - b^2) = 0$

For a tangent the discriminant $= 0$

i.e. $4a^4m^2c^2 = 4(b^2 + a^2m^2)a^2(c^2 - b^2)$

i.e. $a^2m^2c^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$

i.e. $a^2m^2b^2 + b^4 = b^2c^2$

i.e. $c^2 = a^2m^2 + b^2$

$\therefore \quad c = \pm\sqrt{a^2m^2 + b^2}$

i.e. lines $y = mx \pm \sqrt{a^2m^2 + b^2}$ are tangents

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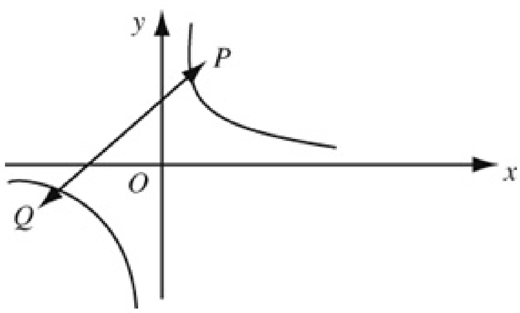
Further coordinate systems
Exercise G, Question 9

Question:

The chord PQ, where P and Q are points on $xy = c^2$, has gradient 1. Show that the locus of the point of intersection of the tangents from P and Q is the line $y = -x$.

Solution:



Chord PQ has gradient $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

If gradient $= 1$ then $pq = -1$

Tangent at P is $p^2y + x = 2cp$

Tangent at Q is $q^2y + x = 2cq$

Intersection $(p^2 - q^2)y = 2c(p-q) \Rightarrow y = \dfrac{2c}{p+q}$

$\therefore x = 2cp - \dfrac{2cp^2}{p+q} = \dfrac{2cpq}{p+q}$

So R is $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$

But $pq = -1 \therefore$ locus of R is $x = \dfrac{-2c}{p+q}$

$$y = \frac{2c}{p+q}$$

i.e. $y = -x$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further coordinate systems
Exercise G, Question 10

Question:

a Show that the asymptotes of the hyperbola H with equation $x^2 - y^2 = 1$ are perpendicular.

Using $(\sec t, \tan t)$ as a general point on H and the rotation matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b show that a rotation of $45°$ will transform H into a rectangular hyperbola with equation $xy = c^2$ and find the positive value of c.

Solution:

a Asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$

For $x^2-y^2=1, a^2=b^2=1$ $\therefore$ Asymptotes are $y=\pm x$ i.e. perpendicular

b Let $\begin{pmatrix}\sec t\\ \tan t\end{pmatrix}$ be the position vector of a point on $x^2-y^2=1$

The matrix $R=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ represents rotation of $45°$ about (0, 0)

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sec t\\ \tan t\end{pmatrix}=\begin{pmatrix}\frac{\sec t}{\sqrt{2}}-\frac{\tan t}{\sqrt{2}}\\ \frac{\sec t}{\sqrt{2}}+\frac{\tan t}{\sqrt{2}}\end{pmatrix}$$

i.e. $X=\frac{1}{\sqrt{2}}(\sec t-\tan t)$

$Y=\frac{1}{\sqrt{2}}(\sec t+\tan t)$

$XY=\frac{1}{2}\left[(\sec t-\tan t)(\sec t+\tan t)\right]$

i.e. $XY=\frac{1}{2}(\sec^2 t-\tan^2 t)=\frac{1}{2}$

$\therefore$ the hyperbola $x^2-y^2=1$ when rotated by $45°$ gives the rectangular hyperbola

$XY=\frac{1}{2}$, $c=\frac{1}{\sqrt{2}}$