Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise A, Question 1

Question:

Use your calculator to find, to 2 decimal places, the value of

a $\sinh 4$

b $\cosh(\frac{1}{2})$

c $\tanh(-2)$

d $\text{sech}\ 5$.

Solution:

a $\sinh 4 = 27.29 \quad (2\text{ d.p.})$ ← Direct from calculator.

$\left(\frac{e^4 - e^{-4}}{2} = 27.29\right)$

b $\cosh(\frac{1}{2}) = 1.13\ (2\text{ d.p.})$ ← Direct from calculator.

$\left(\frac{e^{0.5} + e^{-0.5}}{2} = 1.13\right)$

c $\tanh(-2) = -0.96\ (2\text{ d.p.})$ ← Direct from calculator.

$\left(\frac{e^{-4} - 1}{e^{-4} + 1} = -0.96\right)$

d $\text{sech}\,5 = \frac{1}{\cosh 5} = 0.01\ (2\text{ d.p.})$

$\left(\frac{2}{e^5 + e^{-5}} = 0.01\right)$

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Hyperbolic functions

Exercise A, Question 2

Question:

Write in terms of e

a sinh 1

b cosh 4

c tanh 0.5

d sech (−1) .

Solution:

a $\sinh 1 = \frac{e^1 - e^{-1}}{2} = \frac{e - e^{-1}}{2}$

b $\cosh 4 = \frac{e^4 + e^{-4}}{2}$

c $\tanh 0.5 = \frac{e^1 - 1}{e^1 + 1}$

$= \frac{e - 1}{e + 1}$

Use $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$.

d $\text{sech}(-1) = \frac{2}{e^{-1} + e^{-(-1)}}$

$= \frac{2}{e^{-1} + e}$

Use $\text{sech} x = \frac{1}{\cosh x}$.

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Hyperbolic functions
Exercise A, Question 3

Question:

Find the exact value of

a $\sinh(\ln 2)$

b $\cosh(\ln 3)$

c $\tanh(\ln 2)$

d $\text{cosech}(\ln \pi)$.

Solution:

a $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$

$= \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$e^{\ln 2} = 2$, and $e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}$

b $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2}$

$= \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$

c $\tanh(\ln 2) = \frac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1}$

$= \frac{4-1}{4+1} = \frac{3}{5}$

$e^{2\ln 2} = e^{\ln 2^2} = 4$

d $\text{cosech}(\ln \pi) = \frac{2}{e^{\ln \pi} - e^{-\ln \pi}}$

$= \frac{2}{\pi - \frac{1}{\pi}} = \frac{2\pi}{\pi^2 - 1}$

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Hyperbolic functions
Exercise A, Question 4

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the values of x for which $\cosh x = 2$.

Solution:

$$\frac{e^x + e^{-x}}{2} = 2$$

$$e^x + e^{-x} = 4$$

Multiply throughout by e^x.

$$e^{2x} + 1 = 4e^x$$

$$e^{2x} - 4e^x + 1 = 0$$

$$e^x = \frac{4 \pm \sqrt{16-4}}{2}$$

Solve as a quadratic in e^x.

$$e^x = 3.732 \text{ or } e^x = 0.268$$

$$x = \ln 3.732 = 1.32 \text{ (2 d.p.)}$$

$$x = \ln 0.268 = -1.32 \text{ (2 d.p.)}$$

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Hyperbolic functions
Exercise A, Question 5

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the value of x for which $\sinh x = 1$.

Solution:

$$\frac{e^x - e^{-x}}{2} = 1$$

$$e^x - e^{-x} = 2$$

Multiply throughout by e^x.

$$e^{2x} - 1 = 2e^x$$

$$e^{2x} - 2e^x - 1 = 0$$

$$e^x = \frac{2 \pm \sqrt{4+4}}{2}$$

Solve as a quadratic in e^x.

$$e^x = 2.414 \text{ or } e^x = -0.414$$

$$e^x = 2.414$$

e^x cannot be negative.

$$x = \ln 2.414 = 0.88 \text{ (2 d.p.)}$$

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Hyperbolic functions
Exercise A, Question 6

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the value of x for which $\tanh x = -\frac{1}{2}$.

Solution:

$$\frac{e^{2x}-1}{e^{2x}+1} = -\frac{1}{2}$$
$$2(e^{2x}-1) = -(e^{2x}+1)$$
$$2e^{2x}-2 = -e^{2x}-1$$
$$3e^{2x} = 1$$
$$e^{2x} = \frac{1}{3}$$
$$2x = \ln\left(\frac{1}{3}\right)$$
$$x = \frac{1}{2}\ln\left(\frac{1}{3}\right) = -0.55 \text{ (2d.p.)}$$

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Hyperbolic functions

Exercise A, Question 7

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the value of x for which $\coth x = 10$.

Solution:

$$\coth x = \frac{e^{2x}+1}{e^{2x}-1}$$

$\coth x = \dfrac{1}{\tanh x}$

$$\frac{e^{2x}+1}{e^{2x}-1} = 10$$

$$e^{2x}+1 = 10e^{2x}-10$$

$$9e^{2x} = 11$$

$$e^{2x} = \frac{11}{9}$$

$$2x = \ln\left(\frac{11}{9}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{11}{9}\right) = 0.10 \text{ (2 d.p.)}$$

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Hyperbolic functions
Exercise A, Question 8

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the values of x for which $\text{sech}\, x = \frac{1}{8}$.

Solution:

$\text{sech} x = \frac{2}{e^x + e^{-x}}$ ← $\text{sech} x = \frac{1}{\cosh x}$

$\frac{2}{e^x + e^{-x}} = \frac{1}{8}$

$16 = e^x + e^{-x}$

$16e^x = e^{2x} + 1$ ← Multiply throughout by e^x.

$e^{2x} - 16e^x + 1 = 0$

$e^x = \frac{16 \pm \sqrt{256 - 4}}{2}$ ← Solve as a quadratic in e^x.

$e^x = 15.937$ or $e^x = 0.0627$

$x = \ln 15.937 = 2.77$ (2 d.p.)

$x = \ln 0.0627 = -2.77$ (2 d.p.)

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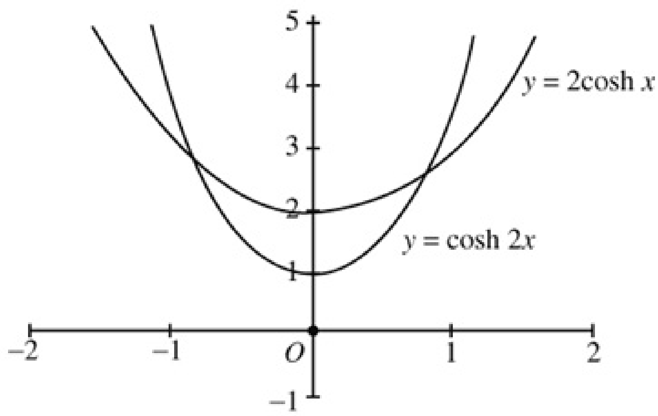
Hyperbolic functions
Exercise B, Question 1

Question:

On the same diagram, sketch the graphs of $y = \cosh 2x$ and $y = 2\cosh x$.

Solution:



For $\mathrm{f}(x) = \cosh x$,
$\mathrm{f}(2x) = \cosh 2x$, a horizontal sketch of scale factor $\frac{1}{2}$.

For $\mathrm{f}(x) = \cosh x$,
$2\mathrm{f}(x) = 2\cosh x$, a vertical stretch of scale factor 2.

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Hyperbolic functions
Exercise B, Question 2

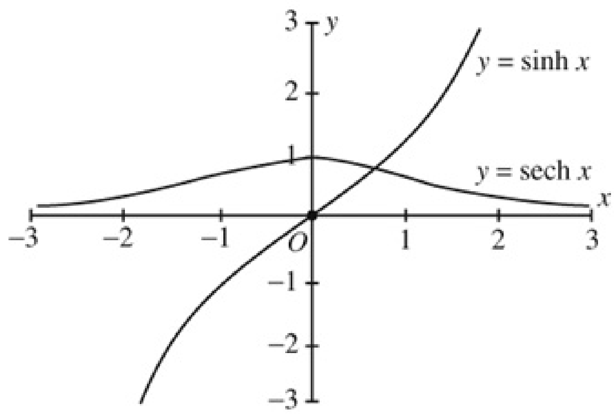
Question:

a On the same diagram, sketch the graphs of $y = \text{sech } x$ and $y = \sinh x$.

b Show that, at the point of intersection of the graphs, $x = \frac{1}{2}\ln(2+\sqrt{5})$.

Solution:

a



b At the intersection,

$\text{sech}x = \sinh x$

$\frac{2}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{2}$

$4 = (e^x - e^{-x})(e^x + e^{-x})$

$4 = e^{2x} - e^{-2x}$ ← Multiply throughout by e^{2x}.

$4e^{2x} = e^{4x} - 1$

$e^{4x} - 4e^{2x} - 1 = 0$

$e^{2x} = \frac{4 \pm \sqrt{16+4}}{2}$ ← Solve as a quadratic in e^{2x}.

$e^{2x} = 2 \pm \sqrt{5}$

$2x = \ln(2+\sqrt{5})$ ← $2-\sqrt{5}$ is negative, and e^{2x} cannot be negative.

$x = \frac{1}{2}\ln(2+\sqrt{5})$

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Hyperbolic functions
Exercise B, Question 3

Question:

Find the range of each hyperbolic function.

a $f(x) = \sinh x, x \in \mathbb{R}$

b $f(x) = \cosh x, x \in \mathbb{R}$

c $f(x) = \tanh x, x \in \mathbb{R}$

d $f(x) = \text{sech}\, x, x \in \mathbb{R}$

e $f(x) = \text{cosech} x, x \in \mathbb{R}, x \neq 0$

f $f(x) = \coth x, x \in \mathbb{R}, x \neq 0$

Solution:

a $f(x) \in \mathbb{R}$ (All real numbers)

b $f(x) \geq 1$

c $-1 < f(x) < 1$

$|f(x)| < 1$

d $0 < f(x) \leq 1$

e $f(x) \in \mathbb{R},\ f(x) \neq 0$

(All real numbers except zero.)

f $f(x) < -1,\ f(x) > 1$

$|f(x)| > 1$

Check the graph of each hyperbolic function to see which y values are possible.

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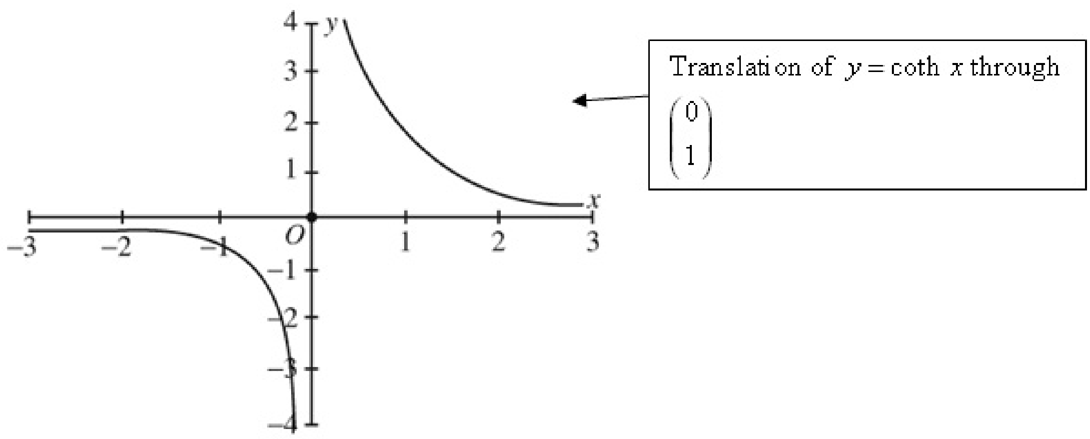
Hyperbolic functions
Exercise B, Question 4

Question:

a Sketch the graph of $y = 1 + \coth x$, $x \in \mathbb{R}, x \neq 0$.
b Write down the equations of the asymptotes to this curve.

Solution:

a $y = \coth x + 1$



b $x = 0$
$y = 2$
$y = 0$

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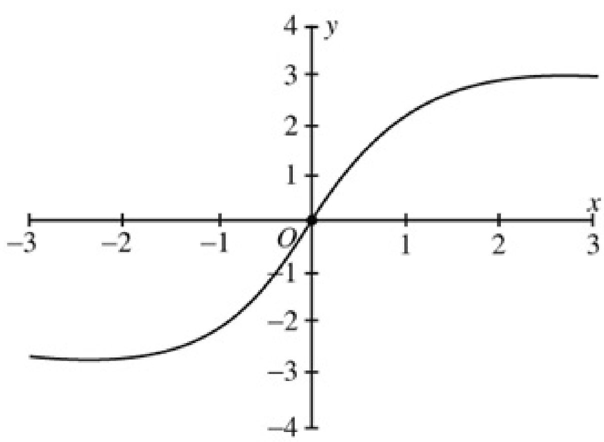
Hyperbolic functions
Exercise B, Question 5

Question:

a Sketch the graph of $y = 3\tanh x, x \in \mathbb{R}$.
b Write down the equations of the asymptotes to this curve.

Solution:

a $y = 3\tanh x$



b $y = -3$
$y = 3$

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Hyperbolic functions
Exercise C, Question 1

Question:

Prove the following identity, using the definitions of $\sinh x$ and $\cosh x$.
$\sinh 2A = 2\sinh A\cosh A$

Solution:

$$\begin{aligned}
\text{R.H.S.} &= 2\sinh A\cosh A \\
&= 2\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{A}+e^{-A}}{2}\right) \\
&= \frac{1}{2}(e^{2A}-1+1-e^{-2A}) \\
&= \frac{e^{2A}-e^{-2A}}{2} \\
&= \sinh 2A = \text{L.H.S.}
\end{aligned}$$

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Hyperbolic functions
Exercise C, Question 2

Question:

Prove the following identity, using the definitions of $\sinh x$ and $\cosh x$.
$\cosh(A-B) = \cosh A \cosh B - \sinh A \sinh B$

Solution:

$$\text{R.H.S.} = \cosh A \cosh B - \sinh A \sinh B$$
$$= \left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right)-\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$
$$= \frac{e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}}{4}$$
$$-\frac{e^{A+B}-e^{-A+B}-e^{A-B}+e^{-A-B}}{4}$$
$$= \frac{2(e^{-A+B}+e^{A-B})}{4}$$
$$= \frac{e^{A-B}+e^{-(A-B)}}{2}$$
$$= \cosh(A-B) = \text{L.H.S.}$$

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Hyperbolic functions

Exercise C, Question 3

Question:

Prove the following identity, using the definitions of sinh x and cosh x.

$\cosh 3A = 4\cosh^3 A - 3\cosh A$

Solution:

$$\text{R.H.S.} = 4\cosh^3 A - 3\cosh A$$

$$= 4\left(\frac{e^A + e^{-A}}{2}\right)^3 - 3\left(\frac{e^A + e^{-A}}{2}\right)$$

$$\left(e^A + e^{-A}\right)^3 = e^{3A} + 3e^{2A}e^{-A} + 3e^A e^{-2A} + e^{-3A}$$

Use the expansion $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

$$= e^{3A} + 3e^A + 3e^{-A} + e^{-3A}$$

$$\text{R.H.S.} = \frac{e^{3A} + 3e^A + 3e^{-A} + e^{-3A}}{2} - \frac{3(e^A + e^{-A})}{2}$$

$$= \frac{e^{3A} + e^{-3A}}{2}$$

$$= \cosh 3A = \text{L.H.S.}$$

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Hyperbolic functions
Exercise C, Question 4

Question:

Prove the following identity, using the definitions of sinh x and cosh x.

$$\sinh A - \sinh B = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$$

Solution:

$$\begin{aligned}
\text{R.H.S.} &= 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)\\
&= 2\left(\frac{e^{\frac{A-B}{2}} - e^{\frac{-A+B}{2}}}{2}\right)\left(\frac{e^{\frac{A+B}{2}} + e^{\frac{-A-B}{2}}}{2}\right)\\
&= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} - e^{\frac{-A+B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}+\frac{-A-B}{2}} - e^{\frac{-A+B}{2}+\frac{-A-B}{2}}\right)\\
&= \frac{1}{2}(e^{A} - e^{B} + e^{-B} - e^{-A})\\
&= \frac{1}{2}(e^{A} - e^{-A}) - \frac{1}{2}(e^{B} - e^{-B})\\
&= \sinh A - \sinh B\\
&= \text{L.H.S.}
\end{aligned}$$

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Hyperbolic functions
Exercise C, Question 5

Question:

Prove the following identity, using the definitions of $\sinh x$ and $\cosh x$.

$\coth A - \tanh A = 2\operatorname{cosech} 2A$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \coth A - \tanh A \\
&= \frac{e^{2A}+1}{e^{2A}-1} - \frac{e^{2A}-1}{e^{2A}+1} \\
&= \frac{(e^{2A}+1)^2 - (e^{2A}-1)^2}{(e^{2A}-1)(e^{2A}+1)} \\
&= \frac{e^{4A}+2e^{2A}+1-e^{4A}+2e^{2A}-1}{e^{4A}-1} \\
&= \frac{4e^{2A}}{e^{4A}-1} \\
&= \frac{4}{e^{2A}-e^{-2A}} = 2\left(\frac{2}{e^{2A}-e^{-2A}}\right) \\
&= 2\operatorname{cosech} 2A = \text{R.H.S.}
\end{aligned}$$

Divide top and bottom by e^{2A}.

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Hyperbolic functions
Exercise C, Question 6

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.
$\sin(A-B) = \sin A \cos B - \cos A \sin B$

Solution:

$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\sinh(A-B) = \sinh A \cosh B - \cosh A \sinh B$

Replace $\sin x$ by $\sinh x$ and $\cos x$ by $\cosh x$.

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Hyperbolic functions
Exercise C, Question 7

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.
$\sin 3A = 3\sin A - 4\sin^3 A$

Solution:

$$\begin{aligned}\sin 3A &= 3\sin A - 4\sin^3 A \\ &= 3\sin A - 4\sin A \sin^2 A \\ \sinh 3A &= 3\sinh A + 4\sinh^3 A\end{aligned}$$

Replace $\sin^2 A$, the product of two sine terms, by $-\sinh^2 A$.

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Hyperbolic functions
Exercise C, Question 8

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Solution:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Replace $\cos x$ by $\cosh x$.

$$\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$$

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Hyperbolic functions
Exercise C, Question 9

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Solution:

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\cosh 2A = \frac{1 + \tanh^2 A}{1 - \tanh^2 A}$$

$\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$, so there is a product of two sines.
Replace $\tan^2 A$ by $-\tanh^2 A$.

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Hyperbolic functions
Exercise C, Question 10

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.
$\cos 2A = \cos^4 A - \sin^4 A$

Solution:

$$\begin{aligned}\cos 2A &= \cos^4 A - \sin^4 A \\ &= \cos^4 A - (\sin^2 A)(\sin^2 A) \\ \cosh 2A &= \cosh^4 A - (-\sinh^2 A)(-\sinh^2 A) \\ &= \cosh^4 A - \sinh^4 A\end{aligned}$$

Replace $\sin^2 A$ by $-\sinh^2 A$.

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Hyperbolic functions
Exercise C, Question 11

Question:

Given that $\cosh x = 2$, find the exact value of
a $\sinh x$
b $\tanh x$
c $\cosh 2x$.

Solution:

a Using $\cosh^2 x - \sinh^2 x = 1$

$4 - \sinh^2 x = 1$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

Both positive and negative values of $\sinh x$ are possible.

b Using $\tanh x = \frac{\sinh x}{\cosh x}$

$\tanh x = \pm\frac{\sqrt{3}}{2}$

c Using $\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = (2\times 4) - 1$

$= 7$

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Hyperbolic functions
Exercise C, Question 12

Question:

Given that $\sinh x = -1$, find the exact value of

a $\cosh x$
b $\sinh 2x$
c $\tanh 2x$.

Solution:

a Using $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x - (-1)^2 = 1$$
$$\cosh^2 x = 2$$
$$\cosh x = \sqrt{2}$$

$\cosh x$ cannot be negative.

b Using $\sinh 2x = 2\sinh x \cosh x$

$$\sinh 2x = 2 \times (-1) \times \sqrt{2}$$
$$= -2\sqrt{2}$$

c Using $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Alternatively use $\dfrac{\sinh 2x}{\cosh 2x} = \dfrac{\sinh 2x}{2\cosh^2 x - 1}$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-1}{\sqrt{2}}$$
$$\tanh 2x = \frac{\left(-\frac{2}{\sqrt{2}}\right)}{1 + \left(\frac{1}{2}\right)}$$
$$= \frac{-2}{\sqrt{2}} \times \frac{2}{3}$$
$$= \frac{-4}{3\sqrt{2}} = -\frac{2\sqrt{2}}{3}$$

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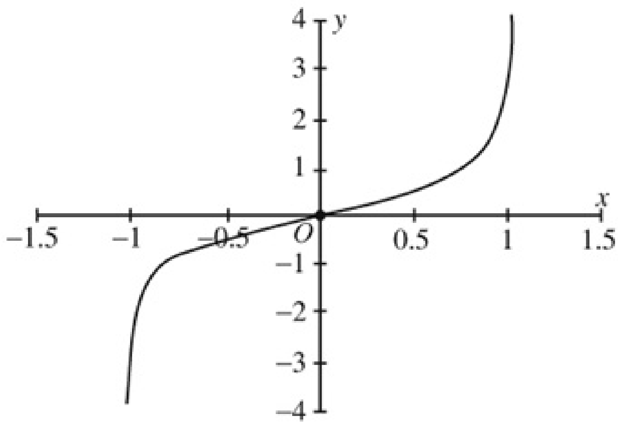
Hyperbolic functions
Exercise D, Question 1

Question:

Sketch the graph of $y = \text{artanh}x, |x| < 1$.

Solution:



$y = \text{artanh}x, |x| < 1$.

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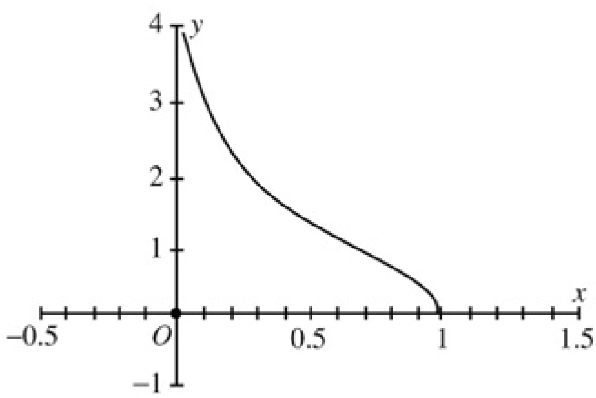
Hyperbolic functions
Exercise D, Question 2

Question:

Sketch the graph of $y = \text{arsech}x, 0 < x \leq 1$.

Solution:



$y = \text{arsech}x, 0 < x \leq 1$

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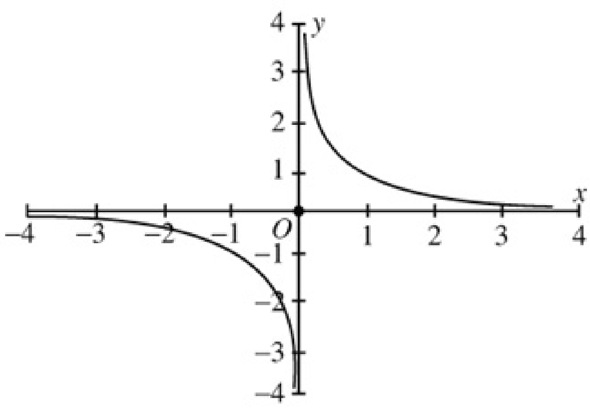
Hyperbolic functions
Exercise D, Question 3

Question:

Sketch the graph of $y = \text{arcosech} x, x \neq 0$.

Solution:



$y = \text{arcosech} x, x \neq 0$

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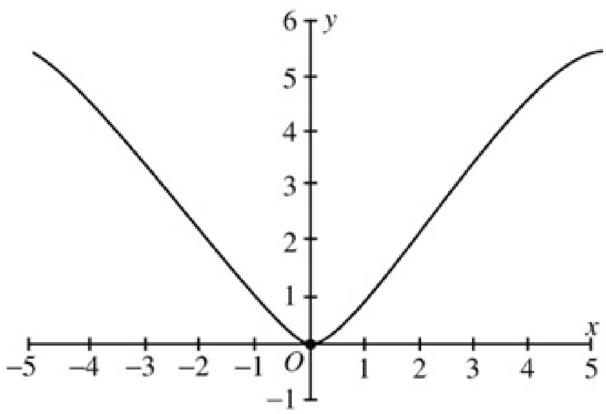
Hyperbolic functions
Exercise D, Question 4

Question:

Sketch the graph of $y = (\text{arsinh}x)^2$.

Solution:



$y = (\text{arsinh}x)^2$

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Hyperbolic functions
Exercise D, Question 5

Question:

Show that $\text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), |x| < 1$.

Solution:

$y = \text{artanh}x$

$x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$

$$\begin{aligned}
x(e^{2y}+1) &= e^{2y}-1 \\
1+x &= e^{2y}(1-x) \\
e^{2y} &= \frac{1+x}{1-x} \\
2y &= \ln\left(\frac{1+x}{1-x}\right) \\
y &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\
\text{artanh}x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \\
|x| &< 1
\end{aligned}$$

For $|x| \geq 1, \ln\left(\frac{1+x}{1-x}\right)$ is not defined, since $\frac{1+x}{1-x} \leq 0$.

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Hyperbolic functions
Exercise D, Question 6

Question:

Show that $\text{arsech}x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), 0 < x \le 1$.

Solution:

$y = \text{arsech}x$

$x = \text{sech}y = \dfrac{2}{e^y + e^{-y}}$

$x(e^y + e^{-y}) = 2$

$xe^y - 2 + xe^{-y} = 0$

$xe^{2y} - 2e^y + x = 0$ ← Multiply throughout by e^y.

$e^y = \dfrac{2 \pm \sqrt{4-4x^2}}{2x}$ ← Solve as a quadratic in e^y.

$e^y = \dfrac{1 \pm \sqrt{1-x^2}}{x}$

$y = \ln\left(\dfrac{1 \pm \sqrt{1-x^2}}{x}\right)$

$\text{arsech}x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$ ← arsech x is 'single-valued'. So only the positive value is required.

$0 < x \le 1$ ← If $x > 1$, $\sqrt{1-x^2}$ is not real.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise D, Question 7

Question:

Express as natural logarithms.

a $\text{arsinh } 2$

b $\text{arcosh } 3$

c $\text{artanh } \frac{1}{2}$

Solution:

a $\text{arsinh } 2 = \ln(2 + \sqrt{2^2 + 1})$
$= \ln(2 + \sqrt{5})$

b $\text{arcosh } 3 = \ln(3 + \sqrt{3^2 - 1})$
$= \ln(3 + \sqrt{8})$
$= \ln(3 + 2\sqrt{2})$

c $\text{artanh}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right)$
$= \frac{1}{2}\ln 3$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Hyperbolic functions

Exercise D, Question 8

Question:

Express as natural logarithms.

a arsinh $\sqrt{2}$

b arcosh $\sqrt{5}$

c artanh 0.1

Solution:

a $\text{arsinh}\,\sqrt{2} = \ln(\sqrt{2} + \sqrt{2+1})$

$= \ln(\sqrt{2} + \sqrt{3})$

b $\text{arcosh}\,\sqrt{5} = \ln(\sqrt{5} + \sqrt{5-1})$

$= \ln(2 + \sqrt{5})$

c $\text{artanh}\,0.1 = \frac{1}{2}\ln\left(\frac{1+0.1}{1-0.1}\right)$

$= \frac{1}{2}\ln\left(\frac{11}{9}\right)$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise D, Question 9

Question:

Express as natural logarithms.

a $\text{arsinh}(-3)$

b $\text{arcosh}\,\frac{3}{2}$

c $\text{artanh}\,\frac{1}{\sqrt{3}}$

Solution:

a
$$\text{arsinh}(-3) = \ln(-3+\sqrt{(-3)^2+1})$$
$$= \ln(-3+\sqrt{10})$$

b
$$\text{arcosh}\left(\frac{3}{2}\right) = \ln\left(\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2-1}\right)$$
$$= \ln\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)$$
$$= \ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$$
$$= \ln\left(\frac{3+\sqrt{5}}{2}\right)$$

c
$$\text{artanh}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right)$$
$$= \frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$
$$= \frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$
$$= \frac{1}{2}\ln\left(\frac{4+2\sqrt{3}}{2}\right)$$
$$= \frac{1}{2}\ln(2+\sqrt{3})$$

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Hyperbolic functions
Exercise D, Question 10

Question:

Given that $\text{artanh}x + \text{artanh}y = \ln\sqrt{3}$, show that $y = \frac{2x-1}{x-2}$.

Solution:

$\text{artanh}x + \text{artanh}y$

$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$ ← Use $\ln a + \ln b = \ln(ab)$.

$= \frac{1}{2}\ln\left(\frac{1+x}{1-x} \times \frac{1+y}{1-y}\right)$

$= \frac{1}{2}\ln\left(\frac{1+x+y+xy}{1-x-y+xy}\right)$ ← Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$.

$= \ln\sqrt{\left(\frac{1+x+y+xy}{1-x-y+xy}\right)}$

So $\frac{1+x+y+xy}{1-x-y+xy} = 3$

$1+x+y+xy = 3-3x-3y+3xy$

$1+x-3+3x = -3y+3xy-y-xy$

$2xy-4y = 4x-2$

$y(x-2) = 2x-1$

$y = \frac{2x-1}{x-2}$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise E, Question 1

Question:

Solve the following equation, giving your answer as natural logarithms.

$3\sinh x + 4\cosh x = 4$

Solution:

$$3\sinh x + 4\cosh x = 4$$
$$\frac{3(e^x - e^{-x})}{2} + \frac{4(e^x + e^{-x})}{2} = 4$$
$$3e^x - 3e^{-x} + 4e^x + 4e^{-x} = 8$$
$$7e^x - 8 + e^{-x} = 0$$

Multiply throughout by e^x.

$$7e^{2x} - 8e^x + 1 = 0$$
$$(7e^x - 1)(e^x - 1) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{1}{7} \text{ or } e^x = 1$$
$$x = \ln\left(\frac{1}{7}\right), x = 0$$

Note that $\ln\left(\frac{1}{7}\right) = \ln(7^{-1}) = -\ln 7$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise E, Question 2

Question:

Solve the following equation, giving your answer as natural logarithms.

$7\sinh x - 5\cosh x = 1$

Solution:

$$7\sinh x - 5\cosh x = 1$$
$$\frac{7(e^x - e^{-x})}{2} - \frac{5(e^x + e^{-x})}{2} = 1$$
$$7e^x - 7e^{-x} - 5e^x - 5e^{-x} = 2$$
$$2e^x - 2 - 12e^{-x} = 0$$
$$e^x - 1 - 6e^{-x} = 0$$

Multiply throughout by e^x.

$$e^{2x} - e^x - 6 = 0$$
$$(e^x - 3)(e^x + 2) = 0$$
$$e^x = 3$$

$e^x = -2$ is not possible for real x.

$$x = \ln 3$$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise E, Question 3

Question:

Solve the following equation, giving your answer as natural logarithms.

$30\cosh x = 15 + 26\sinh x$

Solution:

$$30\cosh x = 15 + 26\sinh x$$

$$30\frac{(e^x + e^{-x})}{2} = 15 + 26\frac{(e^x - e^{-x})}{2}$$

$$15e^x + 15e^{-x} = 15 + 13e^x - 13e^{-x}$$

$$2e^x - 15 + 28e^{-x} = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 15e^x + 28 = 0$$

$$(2e^x - 7)(e^x - 4) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{7}{2},\ e^x = 4$$

$$x = \ln\left(\frac{7}{2}\right),\ x = \ln 4$$

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Hyperbolic functions
Exercise E, Question 4

Question:

Solve the following equation, giving your answer as natural logarithms.

$13\sinh x - 7\cosh x + 1 = 0$

Solution:

$$13\sinh x - 7\cosh x + 1 = 0$$
$$13\frac{(e^x - e^{-x})}{2} - 7\frac{(e^x + e^{-x})}{2} + 1 = 0$$
$$13e^x - 13e^{-x} - 7e^x - 7e^{-x} + 2 = 0$$
$$6e^x + 2 - 20e^{-x} = 0$$
$$3e^x + 1 - 10e^{-x} = 0$$

Multiply throughout by e^x.

$$3e^{2x} + e^x - 10 = 0$$
$$(3e^x - 5)(e^x + 2) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{5}{3}$$

$e^x = -2$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

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Hyperbolic functions
Exercise E, Question 5

Question:

Solve the following equation, giving your answer as natural logarithms.

$\cosh 2x - 5\sinh x = 13$

Solution:

$\cosh 2x - 5\sinh x = 13$

Using $\cosh 2x = 1 + 2\sinh^2 x$,

$1 + 2\sinh^2 x - 5\sinh x = 13$

$2\sinh^2 x - 5\sinh x - 12 = 0$

$(2\sinh x + 3)(\sinh x - 4) = 0$

$\sinh x = -\frac{3}{2}, \sinh x = 4$

$x = \text{arsinh}\left(-\frac{3}{2}\right), x = \text{arsinh} 4$

Use $\text{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

$x = \ln\left(-\frac{3}{2} + \sqrt{\frac{9}{4} + 1}\right)$

$= \ln\left(\frac{-3 + \sqrt{13}}{2}\right)$

$x = \ln(4 + \sqrt{16 + 1})$

$= \ln(4 + \sqrt{17})$

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Hyperbolic functions
Exercise E, Question 6

Question:

Solve the following equation, giving your answer as natural logarithms.

$2\tanh^2 x + 5\operatorname{sech} x - 4 = 0$

Solution:

$2\tanh^2 x + 5\operatorname{sech}x - 4 = 0$

Using $\operatorname{sech}^2 x = 1 - \tanh^2 x$

$2(1 - \operatorname{sech}^2 x) + 5\operatorname{sech}x - 4 = 0$

$2\operatorname{sech}^2 x - 5\operatorname{sech}x + 2 = 0$

$(2\operatorname{sech}x - 1)(\operatorname{sech}x - 2) = 0$

$\operatorname{sech}x = \frac{1}{2}, \operatorname{sech}x = 2$ ← $0 < \operatorname{sech}x \le 1$, so $\operatorname{sech}x = 2$ is not possible.

$\operatorname{sech}x = \frac{1}{2}$

$\cosh x = 2$ ← Use $\operatorname{sech}x = \frac{1}{\cosh x}$.

$x = \operatorname{arcosh}2, -\operatorname{arcosh}2$ ← Use $\operatorname{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution. $\ln(x - \sqrt{x^2 - 1})$ is the same as $-\ln(x + \sqrt{x^2 - 1})$

$x = \ln(2 \pm \sqrt{2^2 - 1})$

$= \ln(2 \pm \sqrt{3})$

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Hyperbolic functions
Exercise E, Question 7

Question:

Solve the following equation, giving your answer as natural logarithms.

$3\sinh^2 x - 13\cosh x + 7 = 0$

Solution:

$3\sinh^2 x - 13\cosh x + 7 = 0$

Using $\cosh^2 x - \sinh^2 x = 1$,

$3(\cosh^2 x - 1) - 13\cosh x + 7 = 0$

$3\cosh^2 x - 13\cosh x + 4 = 0$

$(3\cosh x - 1)(\cosh x - 4) = 0$

$\cosh x = \frac{1}{3}, \cosh x = 4$

$\cosh x \geq 1$, so $\cosh x = \frac{1}{3}$ is not possible.

$\cosh x = 4$

$x = \text{arcosh}4, -\text{arcosh}4$

Use $\text{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

$x = \ln(4 \pm \sqrt{4^2 - 1})$

$= \ln(4 \pm \sqrt{15})$

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Hyperbolic functions
Exercise E, Question 8

Question:

Solve the following equation, giving your answer as natural logarithms.

$\sinh 2x - 7\sinh x = 0$

Solution:

$$\sinh 2x - 7\sinh x = 0$$
$$2\sinh x\cosh x - 7\sinh x = 0$$
$$\sinh x(2\cosh x - 7) = 0$$
$$\sinh x = 0, \cosh x = \frac{7}{2}$$
$$x = 0, x = \pm\text{arcosh}\left(\frac{7}{2}\right)$$

Use $\text{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

$$\text{arcosh}\left(\frac{7}{2}\right) = \ln\left(\frac{7}{2} + \sqrt{\frac{49}{4} - 1}\right)$$
$$= \ln\left(\frac{7 + \sqrt{45}}{2}\right)$$
$$= \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$$
$$x = 0, x = \ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$$

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Hyperbolic functions
Exercise E, Question 9

Question:

Solve the following equation, giving your answer as natural logarithms.

$4\cosh x + 13e^{-x} = 11$

Solution:

$$4\cosh x + 13e^{-x} = 11$$

$$4\frac{(e^{x} + e^{-x})}{2} + 13e^{-x} = 11$$

$$2e^{x} + 2e^{-x} + 13e^{-x} = 11$$

$$2e^{x} + 15e^{-x} - 11 = 0$$

Multiply throughout by e^{x}.

$$2e^{2x} - 11e^{x} + 15 = 0$$

$$(2e^{x} - 5)(e^{x} - 3) = 0$$

Solve as a quadratic in e^{x}.

$$e^{x} = \frac{5}{2}, e^{x} = 3$$

$$x = \ln\left(\frac{5}{2}\right), x = \ln 3$$

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Hyperbolic functions
Exercise E, Question 10

Question:

Solve the following equation, giving your answer as natural logarithms.

$2\tanh x = \cosh x$

Solution:

$$2\tanh x = \cosh x$$
$$\frac{2\sinh x}{\cosh x} = \cosh x$$
$$2\sinh x = \cosh^2 x$$
Using $\cosh^2 x - \sinh^2 x = 1$
$$2\sinh x = 1 + \sinh^2 x$$
$$\sinh^2 x - 2\sinh x + 1 = 0$$
$$(\sinh x - 1)^2 = 0$$
$$\sinh x = 1$$
$$x = \text{arsinh} 1$$
$$x = \ln(1 + \sqrt{1^2 + 1})$$
$$= \ln(1 + \sqrt{2})$$

Use $\text{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

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Hyperbolic functions
Exercise F, Question 1

Question:

Find the exact value of

a $\sinh(\ln 3)$

b $\cosh(\ln 5)$

c $\tanh(\ln \frac{1}{4})$.

Solution:

a $\sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2}$

$= \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$.

b $\cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2}$

$= \frac{5 + \frac{1}{5}}{2} = \frac{13}{5}$

$e^{\ln 5} = 5$, and $e^{-\ln 5} = e^{\ln 5^{-1}} = \frac{1}{5}$.

c $\tanh\left(\ln \frac{1}{4}\right) = \frac{e^{2\ln\frac{1}{4}} - 1}{e^{2\ln\frac{1}{4}} + 1}$

$= \frac{\left(\frac{1}{16} - 1\right)}{\left(\frac{1}{16} + 1\right)}$

$= -\frac{15}{17}$

$e^{2\ln\frac{1}{4}} = e^{\ln\left(\frac{1}{4}\right)^2} = \frac{1}{16}$.

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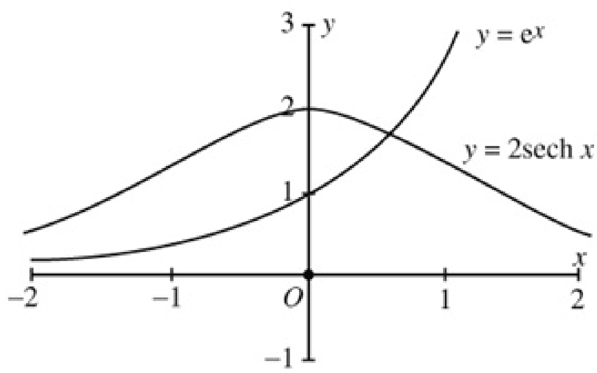
Hyperbolic functions
Exercise F, Question 2

Question:

a Sketch on the same diagram the graphs of $y = 2\text{sech}\ x$ and $y = e^x$.
b Find the exact coordinates of the point of intersection of the graphs.

Solution:

a



b At the intersection,

$$2\text{sech}x = e^x$$

$$\frac{4}{e^x + e^{-x}} = e^x$$

$$4 = e^{2x} + 1$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

$$y = e^x = \sqrt{e^{2x}} = \sqrt{3}$$

coordinates are $(\frac{1}{2}\ln 3, \sqrt{3})$

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Hyperbolic functions
Exercise F, Question 3

Question:

Using the definitions of $\sinh x$ and $\cosh x$, prove that
$\sinh(A-B)=\sinh A\cosh B-\cosh A\sinh B$.

Solution:

$$\text{R.H.S.}=\sinh A\cosh B-\cosh A\sinh B$$
$$=\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right)-\left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$
$$=\frac{e^{A+B}-e^{-A+B}+e^{A-B}-e^{-A-B}}{4}-\frac{e^{A+B}+e^{-A+B}-e^{A-B}-e^{-A-B}}{4}$$
$$=\frac{2\left(e^{A-B}-e^{-A+B}\right)}{4}$$
$$=\frac{e^{A-B}-e^{-(A-B)}}{2}$$
$$=\sinh(A-B)=\text{L.H.S.}$$

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Hyperbolic functions
Exercise F, Question 4

Question:

Using definitions in terms of exponentials, prove that $\sinh x = \dfrac{2\tanh\frac{1}{2}x}{1-\tanh^2\frac{1}{2}x}$.

Solution:

$$\text{R.H.S.} = \frac{2\tanh\frac{1}{2}x}{1-\tanh^2\frac{1}{2}x}$$

$$2\tanh\frac{1}{2}x = \frac{2(e^x-1)}{e^x+1}$$

$$1-\tanh^2\frac{1}{2}x = 1-\left(\frac{e^x-1}{e^x+1}\right)^2$$

$$= \frac{(e^x+1)^2-(e^x-1)^2}{(e^x+1)^2}$$

$$= \frac{4e^x}{(e^x+1)^2}$$

$$\text{So R.H.S.} = \frac{2(e^x-1)}{e^x+1} \times \frac{(e^x+1)^2}{4e^x}$$

$$= \frac{(e^x-1)(e^x+1)}{2e^x}$$

$$= \frac{e^{2x}-1}{2e^x}$$

$$= \frac{e^x-e^{-x}}{2}$$

$$= \sinh x = \text{L.H.S.}$$

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Hyperbolic functions
Exercise F, Question 5

Question:

a Given that $13\cosh x + 5\sinh x = R\cosh(x+\alpha), R > 0$, use the identity $\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$ to find the values of R and α, giving the value of α to 3 decimal places.

b Write down the minimum value of $13\cosh x + 5\sinh x$.

Solution:

a $13\cosh x + 5\sinh x = R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

So $R\cosh\alpha = 13$

$R\sinh\alpha = 5$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 13^2 - 5^2$

Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 144$

$R^2 = 144$

$R = 12$

$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{5}{13}$

$\tanh\alpha = \frac{5}{13}$

$\alpha = 0.405$

Direct from calculator.

b $13\cosh x + 5\sinh x = 12\cosh(x + 0.405)$

The minimum value of $13\cosh x + 5\sinh x$ is 12.

For any value A, $\cosh A \geq 1$.

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Hyperbolic functions
Exercise F, Question 6

Question:

a Show that, for $x > 0$, $\text{arcosech}\,x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$.

b Use the answer to part **a** to write down the value of arcosech 3.

c Use the logarithmic form of arsinh x to verify that your answer to part **b** is the same as the value for $\text{arsinh}\left(\frac{1}{3}\right)$.

Solution:

a $y = \text{arcosech}\,x$

$x = \text{cosech}\,y = \frac{1}{\sinh y} = \frac{2}{e^y - e^{-y}}$

$x(e^y - e^{-y}) = 2$

$xe^y - 2 - xe^{-y} = 0$ ← Multiply throughout by e^y.

$xe^{2y} - 2e^y - x = 0$

$e^y = \frac{2 \pm \sqrt{4+4x^2}}{2x}$ ← Solve as a quadratic in e^y.

$e^y = \frac{1 \pm \sqrt{1+x^2}}{x}$

$e^y = \frac{1+\sqrt{1+x^2}}{x}, x > 0$ ← For $x > 0$, the positive sign gives a positive value for e^y, whereas the negative sign gives an impossible negative value for e^y.

$y = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), x > 0$

$\text{arcosech}\,x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), x > 0$

b $\text{arcosech}\,3 = \ln\left(\frac{1+\sqrt{10}}{3}\right)$

c $\text{arsinh}\left(\frac{1}{3}\right) = \ln\left(\frac{1}{3} + \sqrt{\frac{1}{9}+1}\right)$

$= \ln\left(\frac{1}{3} + \sqrt{\frac{10}{9}}\right)$

$= \ln\left(\frac{1+\sqrt{10}}{3}\right)$

(Same as the answer to part **b**).

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Hyperbolic functions
Exercise F, Question 7

Question:

Solve, giving your answers as natural logarithms, L $9\cosh x - 5\sinh x = 15$

Solution:

$$9\cosh x - 5\sinh x = 15$$

$$9\frac{(e^x + e^{-x})}{2} - 5\frac{(e^x - e^{-x})}{2} = 15$$

$$9e^x + 9e^{-x} - 5e^x + 5e^{-x} = 30$$

$$4e^x - 30 + 14e^{-x} = 0$$

$$2e^x - 15 + 7e^{-x} = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 15e^x + 7 = 0$$

$$(2e^x - 1)(e^x - 7) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{1}{2}, e^x = 7$$

$$x = \ln\left(\frac{1}{2}\right), x = \ln 7$$

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Hyperbolic functions
Exercise F, Question 8

Question:

Solve, giving your answers as natural logarithms, L $23\sinh x - 17\cosh x + 7 = 0$

Solution:

$$23\sinh x - 17\cosh x + 7 = 0$$

$$23\frac{(e^x - e^{-x})}{2} - 17\frac{(e^x + e^{-x})}{2} + 7 = 0$$

$$23e^x - 23e^{-x} - 17e^x - 17e^{-x} + 14 = 0$$

$$6e^x + 14 - 40e^{-x} = 0$$

$$3e^x + 7 - 20e^{-x} = 0$$

$$3e^{2x} + 7e^x - 20 = 0$$

Multiply throughout by e^x.

$$(3e^x - 5)(e^x + 4) = 0$$

$$e^x = \frac{5}{3}$$

$e^x = -4$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

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Hyperbolic functions
Exercise F, Question 9

Question:

Solve, giving your answers as natural logarithms, L $3\cosh^2 x + 11\sinh x = 17$

Solution:

$3\cosh^2 x + 11\sinh x = 17$

Using $\cosh^2 x - \sinh^2 x = 1$

$3(1 + \sinh^2 x) + 11\sinh x = 17$

$3\sinh^2 x + 11\sinh x - 14 = 0$

$(3\sinh x + 14)(\sinh x - 1) = 0$

$\sinh x = -\frac{14}{3}, \sinh x = 1$

$x = \text{arsinh}\left(-\frac{14}{3}\right), x = \text{arsinh}1$

Use $\text{arsinh}x = \ln(x + \sqrt{x^2 + 1})$.

$x = \ln\left(-\frac{14}{3} + \sqrt{\frac{196}{9} + 1}\right)$

$= \ln\left(\frac{-14 + \sqrt{205}}{3}\right)$

$x = \ln(1 + \sqrt{1 + 1})$

$= \ln(1 + \sqrt{2})$

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Hyperbolic functions
Exercise F, Question 10

Question:

Solve, giving your answers as natural logarithms, L $6\tanh x - 7\text{sech } x = 2$

Solution:

$$6\tanh x - 7\text{sech } x = 2$$

$$\frac{6\sinh x}{\cosh x} - \frac{7}{\cosh x} = 2$$

$$6\sinh x - 7 = 2\cosh x$$

$$6\frac{(e^x - e^{-x})}{2} - 7 = 2\frac{(e^x + e^{-x})}{2}$$

$$3e^x - 3e^{-x} - 7 = e^x + e^{-x}$$

Multiply throughout by e^x.

$$2e^x - 7 - 4e^{-x} = 0$$

$$2e^{2x} - 7e^x - 4 = 0$$

Solve as a quadratic in e^x.

$$(2e^x + 1)(e^x - 4) = 0$$

$$e^x = 4$$

$e^x = -\frac{1}{2}$ is not possible for real x.

$$x = \ln 4$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Hyperbolic functions

Exercise F, Question 11

Question:

Show that $\sinh[\ln(\sin x)] = -\frac{1}{2}\cos x \cot x$.

Solution:

$$\sinh(\ln(\sin x)) = \frac{e^{\ln(\sin x)} - e^{-\ln(\sin x)}}{2}$$

$$= \frac{e^{\ln(\sin x)} - e^{\ln(\sin x)^{-1}}}{2}$$

$$= \frac{\sin x - (\sin x)^{-1}}{2}$$

$$= \frac{\sin x - \operatorname{cosec} x}{2}$$

$$= \frac{\sin^2 x - 1}{2\sin x}$$

$$= -\frac{\cos^2 x}{2\sin x}$$

$$= -\frac{1}{2}\cos x\left(\frac{\cos x}{\sin x}\right)$$

$$= -\frac{1}{2}\cos x \cot x$$

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Exercise F, Question 12

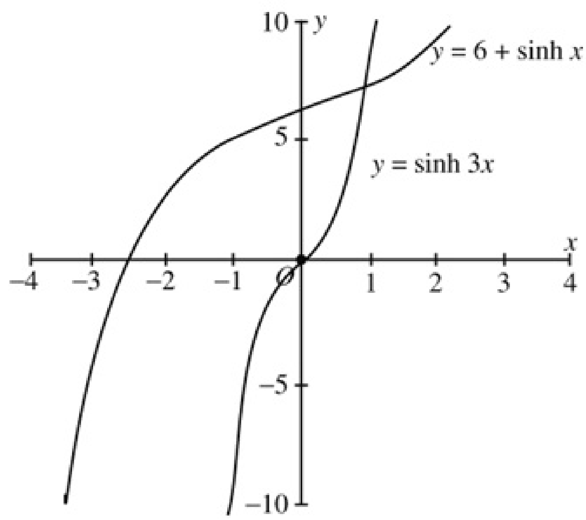
Question:

a On the same diagram, sketch the graphs of $y = 6 + \sinh x$ and $y = \sinh 3x$.

b Using the identity $\sinh 3x = 3\sinh x + 4\sinh^3 x$, show that the graphs intersect where $\sinh x = 1$ and hence find the exact coordinates of the point of intersection.

Solution:

a



b At the intersection,

$$6 + \sinh x = \sinh 3x$$
$$6 + \sinh x = 3\sinh x + 4\sinh^3 x$$
$$4\sinh^3 x + 2\sinh x - 6 = 0$$
$$2\sinh^3 x + \sinh x - 3 = 0$$
$$(\sinh x - 1)(2\sinh^2 x + 2\sinh x + 3) = 0$$

You can see, by inspection, that $\sinh x = 1$ satisfies this equation.

The equation $2\sinh^2 x + 2\sinh x + 3 = 0$ has no real roots, because $b^2 - 4ac = 4 - 24 < 0$.

The only intersection is where $\sinh x = 1$

For $\sinh x = 1$,

$x = \text{arsinh}1$

$= \ln(1 + \sqrt{1^2 + 1})$

$= \ln(1 + \sqrt{2})$

Using $y = 6 + \sinh x$

$y = 7$

Coordinates of the point of intersection are $(\ln(1 + \sqrt{2}), 7)$

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Exercise F, Question 13

Question:

Given that $\text{artanh}x - \text{artanh}y = \ln 5$, find y in terms of x.

Solution:

$$\text{artanh}x - \text{artanh}y$$
$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

Use $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x} \times \frac{1-y}{1+y}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1+x-y-xy}{1-x+y-xy}\right)$$

Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$

$$= \ln\sqrt{\left(\frac{1+x-y-xy}{1-x+y-xy}\right)}$$

So $\sqrt{\left(\frac{1+x-y-xy}{1-x+y-xy}\right)} = 5$

$$\frac{1+x-y-xy}{1-x+y-xy} = 25$$
$$1+x-y-xy = 25-25x+25y-25xy$$
$$24xy-26y = 24-26x$$
$$y(12x-13) = 12-13x$$
$$y = \frac{12-13x}{12x-13}$$

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Hyperbolic functions
Exercise F, Question 14

Question:

a Express $3\cosh x + 5\sinh x$ in the form $R\sinh(x+\alpha)$, where $R > 0$. Give α to 3 decimal places.
b Use the answer to part **a** to solve the equation $3\cosh x + 5\sinh x = 8$, giving your answer to 2 decimal places.
c Solve $3\cosh x + 5\sinh x = 8$ by using the definitions of $\cosh x$ and $\sinh x$.

Solution:

a $3\cosh x + 5\sinh x = R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$

So $R\cosh\alpha = 5$

$R\sinh\alpha = 3$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 5^2 - 3^2$

Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 16$

$R^2 = 16$

$R = 4$

$\dfrac{R\sinh\alpha}{R\cosh\alpha} = \dfrac{3}{5}$

$\tanh\alpha = \dfrac{3}{5}$

$\alpha = 0.693$

Direct from calculator.

$3\cosh x + 5\sinh x = 4\sinh(x + 0.693)$

b $4\sinh(x + 0.693) = 8$

$\sinh(x + 0.693) = 2$

$x + 0.693 = \text{arsinh } 2$

$= 1.444$ (3 d.p.)

Direct from calculator.

$x = 0.75$ (2 d.p.)

c $3\cosh x + 5\sinh x = 8$

$3\dfrac{(e^x + e^{-x})}{2} + 5\dfrac{(e^x - e^{-x})}{2} = 8$

$3e^x + 3e^{-x} + 5e^x - 5e^{-x} = 16$

$8e^x - 16 - 2e^{-x} = 0$

$4e^x - 8 - e^{-x} = 0$

Multiply throughout by e^x.

$4e^{2x} - 8e^x - 1 = 0$

$e^x = \dfrac{8 \pm \sqrt{64 + 16}}{8}$

Solve as a quadratic in e^x.

$e^x = 1 \pm \dfrac{\sqrt{80}}{8} = 1 \pm \dfrac{\sqrt{5}}{2}$

$e^x = 1 + \dfrac{\sqrt{5}}{2}$

$e^x = 1 - \dfrac{\sqrt{5}}{2}$ is negative, so not possible for real x.

$x = \ln\left(1 + \dfrac{\sqrt{5}}{2}\right)$

$= 0.75$ (2 d.p.)