Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise A, Question 1

Question:

Use your calculator to find, to 2 decimal places, the value of

a $\sinh 4$

b $\cosh(\frac{1}{2})$

c $\tanh(-2)$

d $\text{sech}\ 5.$

Solution:

a $\sinh 4 = 27.29 \quad (2\text{ d.p.})$ ← Direct from calculator.

$\left(\frac{e^4 - e^{-4}}{2} = 27.29\right)$

b $\cosh(\frac{1}{2}) = 1.13\ (2\text{ d.p.})$ ← Direct from calculator.

$\left(\frac{e^{0.5} + e^{-0.5}}{2} = 1.13\right)$

c $\tanh(-2) = -0.96\ (2\text{ d.p.})$ ← Direct from calculator.

$\left(\frac{e^{-4} - 1}{e^{-4} + 1} = -0.96\right)$

d $\text{sech}\,5 = \frac{1}{\cosh 5} = 0.01\ (2\text{ d.p.})$

$\left(\frac{2}{e^5 + e^{-5}} = 0.01\right)$

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Hyperbolic functions
Exercise A, Question 2

Question:

Write in terms of e

a sinh 1
b cosh 4
c tanh 0.5
d sech (−1) .

Solution:

a $\sinh 1 = \frac{e^1 - e^{-1}}{2} = \frac{e - e^{-1}}{2}$

b $\cosh 4 = \frac{e^4 + e^{-4}}{2}$

c $\tanh 0.5 = \frac{e^1 - 1}{e^1 + 1}$

$= \frac{e - 1}{e + 1}$

Use $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$.

d $\text{sech}(-1) = \frac{2}{e^{-1} + e^{-(-1)}}$

$= \frac{2}{e^{-1} + e}$

Use $\text{sech} x = \frac{1}{\cosh x}$.

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Hyperbolic functions
Exercise A, Question 3

Question:

Find the exact value of

a $\sinh(\ln 2)$

b $\cosh(\ln 3)$

c $\tanh(\ln 2)$

d $\text{cosech}(\ln \pi)$.

Solution:

a $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$

$= \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$e^{\ln 2} = 2$, and $e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}$

b $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2}$

$= \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$

c $\tanh(\ln 2) = \frac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1}$

$= \frac{4 - 1}{4 + 1} = \frac{3}{5}$

$e^{2\ln 2} = e^{\ln 2^2} = 4$

d $\text{cosech}(\ln \pi) = \frac{2}{e^{\ln \pi} - e^{-\ln \pi}}$

$= \frac{2}{\pi - \frac{1}{\pi}} = \frac{2\pi}{\pi^2 - 1}$

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Hyperbolic functions
Exercise A, Question 4

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the values of x for which $\cosh x = 2$.

Solution:

$$\frac{e^x + e^{-x}}{2} = 2$$

$$e^x + e^{-x} = 4$$

Multiply throughout by e^x.

$$e^{2x} + 1 = 4e^x$$

$$e^{2x} - 4e^x + 1 = 0$$

$$e^x = \frac{4 \pm \sqrt{16-4}}{2}$$

Solve as a quadratic in e^x.

$$e^x = 3.732 \text{ or } e^x = 0.268$$

$$x = \ln 3.732 = 1.32 \text{ (2 d.p.)}$$

$$x = \ln 0.268 = -1.32 \text{ (2 d.p.)}$$

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Hyperbolic functions
Exercise A, Question 5

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the value of x for which $\sinh x = 1$.

Solution:

$$\frac{e^x - e^{-x}}{2} = 1$$

$$e^x - e^{-x} = 2$$

Multiply throughout by e^x.

$$e^{2x} - 1 = 2e^x$$

$$e^{2x} - 2e^x - 1 = 0$$

$$e^x = \frac{2 \pm \sqrt{4+4}}{2}$$

Solve as a quadratic in e^x.

$$e^x = 2.414 \text{ or } e^x = -0.414$$

$$e^x = 2.414$$

e^x cannot be negative.

$$x = \ln 2.414 = 0.88 \text{ (2 d.p.)}$$

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Hyperbolic functions
Exercise A, Question 6

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the value of x for which $\tanh x = -\frac{1}{2}$.

Solution:

$$\frac{e^{2x}-1}{e^{2x}+1} = -\frac{1}{2}$$
$$2(e^{2x}-1) = -(e^{2x}+1)$$
$$2e^{2x}-2 = -e^{2x}-1$$
$$3e^{2x} = 1$$
$$e^{2x} = \frac{1}{3}$$
$$2x = \ln\left(\frac{1}{3}\right)$$
$$x = \frac{1}{2}\ln\left(\frac{1}{3}\right) = -0.55 \text{ (2d.p.)}$$

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Hyperbolic functions
Exercise A, Question 7

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.
Find, to 2 decimal places, the value of x for which $\coth x = 10$.

Solution:

$$\coth x = \frac{e^{2x}+1}{e^{2x}-1}$$

← $\coth x = \frac{1}{\tanh x}$

$$\frac{e^{2x}+1}{e^{2x}-1} = 10$$

$$e^{2x}+1 = 10e^{2x}-10$$

$$9e^{2x} = 11$$

$$e^{2x} = \frac{11}{9}$$

$$2x = \ln\left(\frac{11}{9}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{11}{9}\right) = 0.10 \text{ (2 d.p.)}$$

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Hyperbolic functions
Exercise A, Question 8

Question:

Use definitions of the hyperbolic functions (in terms of exponentials) to find your answer, then check your answer using an inverse hyperbolic function on your calculator.

Find, to 2 decimal places, the values of x for which $\text{sech}\, x = \frac{1}{8}$.

Solution:

$$\text{sech} x = \frac{2}{e^x + e^{-x}}$$ ← $\text{sech} x = \frac{1}{\cosh x}$

$$\frac{2}{e^x + e^{-x}} = \frac{1}{8}$$

$$16 = e^x + e^{-x}$$

$$16e^x = e^{2x} + 1$$ ← Multiply throughout by e^x.

$$e^{2x} - 16e^x + 1 = 0$$

$$e^x = \frac{16 \pm \sqrt{256 - 4}}{2}$$ ← Solve as a quadratic in e^x.

$$e^x = 15.937 \text{ or } e^x = 0.0627$$

$$x = \ln 15.937 = 2.77 \text{ (2 d.p.)}$$

$$x = \ln 0.0627 = -2.77 \text{ (2 d.p.)}$$

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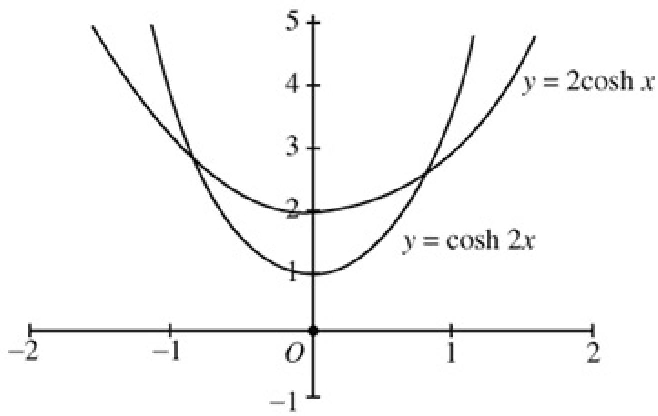
Hyperbolic functions
Exercise B, Question 1

Question:

On the same diagram, sketch the graphs of $y = \cosh 2x$ and $y = 2\cosh x$.

Solution:



For $f(x) = \cosh x$,
$f(2x) = \cosh 2x$, a horizontal sketch of scale factor $\frac{1}{2}$.

For $f(x) = \cosh x$,
$2f(x) = 2\cosh x$, a vertical stretch of scale factor 2.

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Hyperbolic functions
Exercise B, Question 2

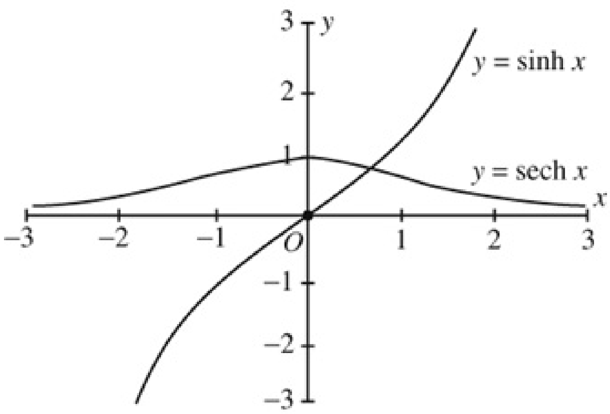
Question:

a On the same diagram, sketch the graphs of $y = \text{sech } x$ and $y = \sinh x$.

b Show that, at the point of intersection of the graphs, $x = \frac{1}{2}\ln(2+\sqrt{5})$.

Solution:

a



b At the intersection,

$$\text{sech}x = \sinh x$$

$$\frac{2}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{2}$$

$$4 = (e^x - e^{-x})(e^x + e^{-x})$$

$$4 = e^{2x} - e^{-2x}$$

Multiply throughout by e^{2x}.

$$4e^{2x} = e^{4x} - 1$$

$$e^{4x} - 4e^{2x} - 1 = 0$$

$$e^{2x} = \frac{4 \pm \sqrt{16+4}}{2}$$

Solve as a quadratic in e^{2x}.

$$e^{2x} = 2 \pm \sqrt{5}$$

$$2x = \ln(2+\sqrt{5})$$

$2-\sqrt{5}$ is negative, and e^{2x} cannot be negative.

$$x = \frac{1}{2}\ln(2+\sqrt{5})$$

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Hyperbolic functions
Exercise B, Question 3

Question:

Find the range of each hyperbolic function.

a $f(x) = \sinh x, x \in \mathbb{R}$

b $f(x) = \cosh x, x \in \mathbb{R}$

c $f(x) = \tanh x, x \in \mathbb{R}$

d $f(x) = \text{sech } x, x \in \mathbb{R}$

e $f(x) = \text{cosech} x, x \in \mathbb{R}, x \neq 0$

f $f(x) = \coth x, x \in \mathbb{R}, x \neq 0$

Solution:

a $f(x) \in \mathbb{R}$ (All real numbers)

b $f(x) \geq 1$

Check the graph of each hyperbolic function to see which y values are possible.

c $-1 < f(x) < 1$
$|f(x)| < 1$

d $0 < f(x) \leq 1$

e $f(x) \in \mathbb{R}$, $f(x) \neq 0$
(All real numbers except zero.)

f $f(x) < -1$, $f(x) > 1$
$|f(x)| > 1$

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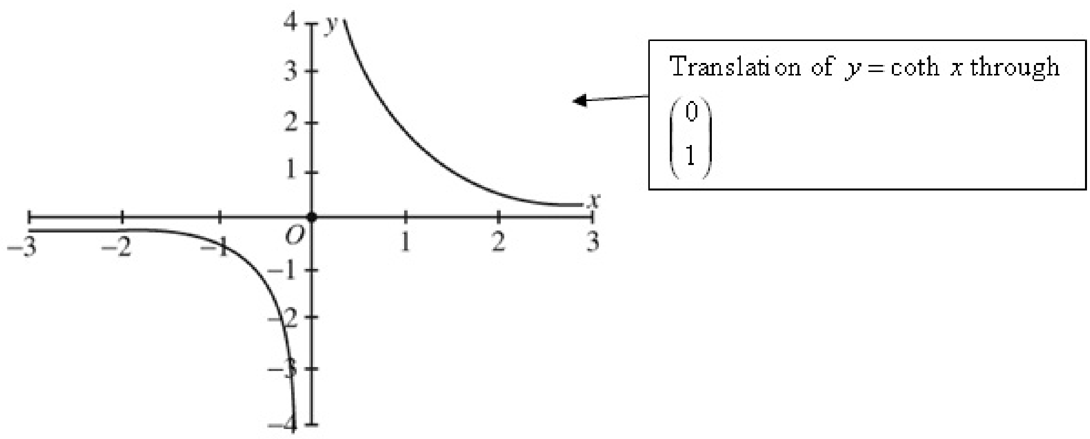
Hyperbolic functions
Exercise B, Question 4

Question:

a Sketch the graph of $y = 1 + \coth x$, $x \in \mathbb{R}$, $x \neq 0$.
b Write down the equations of the asymptotes to this curve.

Solution:

a $y = \coth x + 1$



b $x = 0$
$y = 2$
$y = 0$

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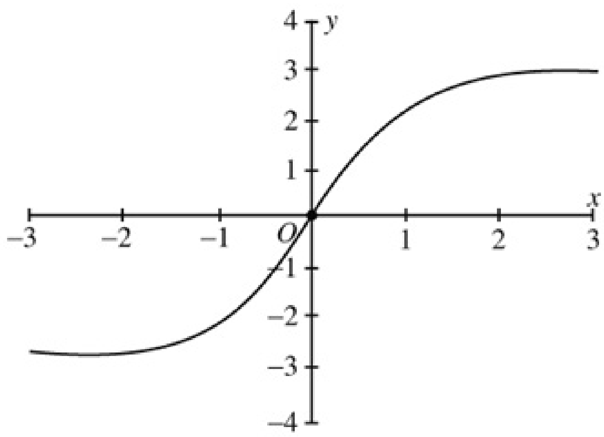
Hyperbolic functions
Exercise B, Question 5

Question:

a Sketch the graph of $y = 3\tanh x, x \in \mathbb{R}$.
b Write down the equations of the asymptotes to this curve.

Solution:

a $y = 3\tanh x$



b $y = -3$
$y = 3$

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Hyperbolic functions
Exercise C, Question 1

Question:

Prove the following identity, using the definitions of $\sinh x$ and $\cosh x$.
$\sinh 2A = 2\sinh A\cosh A$

Solution:

$$\begin{aligned}\text{R.H.S.} &= 2\sinh A\cosh A\\ &= 2\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{A}+e^{-A}}{2}\right)\\ &= \frac{1}{2}(e^{2A}-1+1-e^{-2A})\\ &= \frac{e^{2A}-e^{-2A}}{2}\\ &= \sinh 2A = \text{L.H.S.}\end{aligned}$$

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Hyperbolic functions
Exercise C, Question 2

Question:

Prove the following identity, using the definitions of sinh x and cosh x.
$\cosh(A-B) = \cosh A \cosh B - \sinh A \sinh B$

Solution:

$$\text{R.H.S.} = \cosh A \cosh B - \sinh A \sinh B$$
$$= \left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right) - \left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$
$$= \frac{e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}}{4}$$
$$- \frac{e^{A+B}-e^{-A+B}-e^{A-B}+e^{-A-B}}{4}$$
$$= \frac{2(e^{-A+B}+e^{A-B})}{4}$$
$$= \frac{e^{A-B}+e^{-(A-B)}}{2}$$
$$= \cosh(A-B) = \text{L.H.S.}$$

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Hyperbolic functions
Exercise C, Question 3

Question:

Prove the following identity, using the definitions of sinh x and cosh x.

$\cosh 3A = 4\cosh^3 A - 3\cosh A$

Solution:

$$\text{R.H.S.} = 4\cosh^3 A - 3\cosh A$$
$$= 4\left(\frac{e^A + e^{-A}}{2}\right)^3 - 3\left(\frac{e^A + e^{-A}}{2}\right)$$
$$\left(e^A + e^{-A}\right)^3 = e^{3A} + 3e^{2A}e^{-A} + 3e^A e^{-2A} + e^{-3A}$$
$$= e^{3A} + 3e^A + 3e^{-A} + e^{-3A}$$

Use the expansion $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

$$\text{R.H.S.} = \frac{e^{3A} + 3e^A + 3e^{-A} + e^{-3A}}{2} - \frac{3(e^A + e^{-A})}{2}$$
$$= \frac{e^{3A} + e^{-3A}}{2}$$
$$= \cosh 3A = \text{L.H.S.}$$

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Hyperbolic functions
Exercise C, Question 4

Question:

Prove the following identity, using the definitions of sinh x and cosh x.

$$\sinh A - \sinh B = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$$

Solution:

$$\begin{aligned}
\text{R.H.S.} &= 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right) \\
&= 2\left(\frac{e^{\frac{A-B}{2}} - e^{\frac{-A+B}{2}}}{2}\right)\left(\frac{e^{\frac{A+B}{2}} + e^{\frac{-A-B}{2}}}{2}\right) \\
&= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} - e^{\frac{-A+B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}+\frac{-A-B}{2}} - e^{\frac{-A+B}{2}+\frac{-A-B}{2}}\right) \\
&= \frac{1}{2}(e^{A} - e^{B} + e^{-B} - e^{-A}) \\
&= \frac{1}{2}(e^{A} - e^{-A}) - \frac{1}{2}(e^{B} - e^{-B}) \\
&= \sinh A - \sinh B \\
&= \text{L.H.S.}
\end{aligned}$$

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Hyperbolic functions
Exercise C, Question 5

Question:

Prove the following identity, using the definitions of sinh x and cosh x.
$\coth A - \tanh A = 2\operatorname{cosech} 2A$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \coth A - \tanh A \\
&= \frac{e^{2A}+1}{e^{2A}-1} - \frac{e^{2A}-1}{e^{2A}+1} \\
&= \frac{(e^{2A}+1)^2 - (e^{2A}-1)^2}{(e^{2A}-1)(e^{2A}+1)} \\
&= \frac{e^{4A}+2e^{2A}+1-e^{4A}+2e^{2A}-1}{e^{4A}-1} \\
&= \frac{4e^{2A}}{e^{4A}-1} \\
&= \frac{4}{e^{2A}-e^{-2A}} = 2\left(\frac{2}{e^{2A}-e^{-2A}}\right) \\
&= 2\operatorname{cosech} 2A = \text{R.H.S.}
\end{aligned}$$

Divide top and bottom by e^{2A}.

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Hyperbolic functions
Exercise C, Question 6

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.
$\sin(A - B) = \sin A \cos B - \cos A \sin B$

Solution:

$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\sinh(A - B) = \sinh A \cosh B - \cosh A \sinh B$

Replace $\sin x$ by $\sinh x$ and $\cos x$ by $\cosh x$.

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Hyperbolic functions
Exercise C, Question 7

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.
$\sin 3A = 3\sin A - 4\sin^3 A$

Solution:

$$\begin{aligned}\sin 3A &= 3\sin A - 4\sin^3 A\\ &= 3\sin A - 4\sin A\sin^2 A\\ \sinh 3A &= 3\sinh A + 4\sinh^3 A\end{aligned}$$

Replace $\sin^2 A$, the product of two sine terms, by $-\sinh^2 A$.

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Hyperbolic functions
Exercise C, Question 8

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Solution:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Replace $\cos x$ by $\cosh x$.

$$\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$$

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Hyperbolic functions
Exercise C, Question 9

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

Solution:

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\cosh 2A = \frac{1+\tanh^2 A}{1-\tanh^2 A}$$

$\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$, so there is a product of two sines.
Replace $\tan^2 A$ by $-\tanh^2 A$.

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Hyperbolic functions
Exercise C, Question 10

Question:

Use Osborn's Rule to write down the hyperbolic identity corresponding to the following trigonometric identity.
$\cos 2A = \cos^4 A - \sin^4 A$

Solution:

$$\begin{aligned}\cos 2A &= \cos^4 A - \sin^4 A \\ &= \cos^4 A - (\sin^2 A)(\sin^2 A) \\ \cosh 2A &= \cosh^4 A - (-\sinh^2 A)(-\sinh^2 A) \\ &= \cosh^4 A - \sinh^4 A\end{aligned}$$

Replace $\sin^2 A$ by $-\sinh^2 A$.

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Hyperbolic functions
Exercise C, Question 11

Question:

Given that $\cosh x = 2$, find the exact value of

a $\sinh x$

b $\tanh x$

c $\cosh 2x$.

Solution:

a Using $\cosh^2 x - \sinh^2 x = 1$

$4 - \sinh^2 x = 1$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

Both positive and negative values of $\sinh x$ are possible.

b Using $\tanh x = \dfrac{\sinh x}{\cosh x}$

$\tanh x = \pm\dfrac{\sqrt{3}}{2}$

c Using $\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = (2 \times 4) - 1$

$= 7$

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Hyperbolic functions
Exercise C, Question 12

Question:

Given that $\sinh x = -1$, find the exact value of

a $\cosh x$

b $\sinh 2x$

c $\tanh 2x$.

Solution:

a Using $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x - (-1)^2 = 1$$
$$\cosh^2 x = 2$$
$$\cosh x = \sqrt{2}$$

$\cosh x$ cannot be negative.

b Using $\sinh 2x = 2\sinh x \cosh x$

$$\sinh 2x = 2 \times (-1) \times \sqrt{2}$$
$$= -2\sqrt{2}$$

c Using $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

Alternatively use $\dfrac{\sinh 2x}{\cosh 2x} = \dfrac{\sinh 2x}{2\cosh^2 x - 1}$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-1}{\sqrt{2}}$$
$$\tanh 2x = \frac{\left(-\frac{2}{\sqrt{2}}\right)}{1+\left(\frac{1}{2}\right)}$$
$$= \frac{-2}{\sqrt{2}} \times \frac{2}{3}$$
$$= \frac{-4}{3\sqrt{2}} = -\frac{2\sqrt{2}}{3}$$

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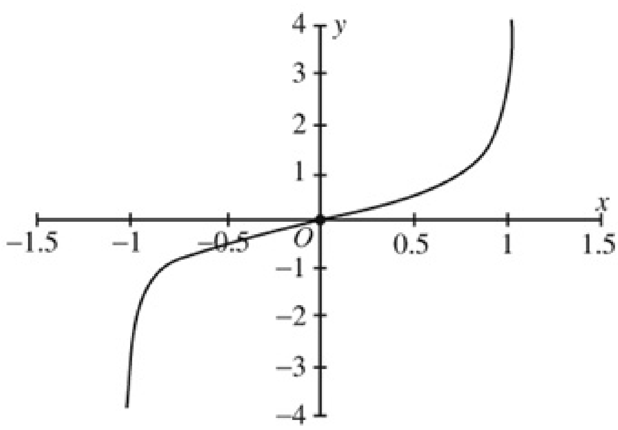
Hyperbolic functions
Exercise D, Question 1

Question:

Sketch the graph of $y = \text{artanh}x, |x| < 1$.

Solution:



$y = \text{artanh}x, |x| < 1$.

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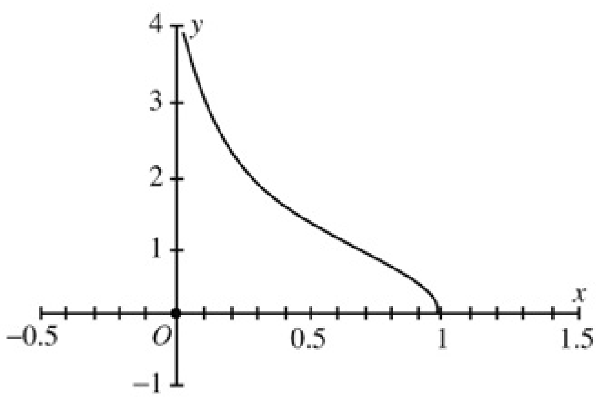
Hyperbolic functions
Exercise D, Question 2

Question:

Sketch the graph of $y = \text{arsech}x, 0 < x \leq 1$.

Solution:



$y = \text{arsech}x, 0 < x \leq 1$

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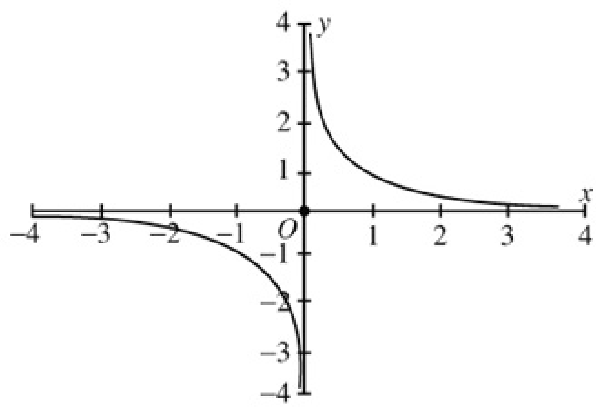
Hyperbolic functions
Exercise D, Question 3

Question:

Sketch the graph of $y = \text{arcosech}x, x \neq 0$.

Solution:



$y = \text{arcosech}x, x \neq 0$

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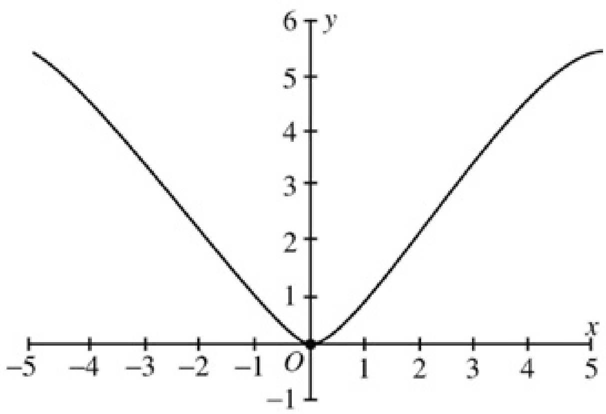
Hyperbolic functions
Exercise D, Question 4

Question:

Sketch the graph of $y = (\text{arsinh}x)^2$.

Solution:



$y = (\text{arsinh}x)^2$

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Hyperbolic functions
Exercise D, Question 5

Question:

Show that $\text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), |x| < 1.$

Solution:

$y = \text{artanh}x$

$x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$

$$x(e^{2y}+1) = e^{2y}-1$$
$$1+x = e^{2y}(1-x)$$
$$e^{2y} = \frac{1+x}{1-x}$$
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$
$$|x| < 1$$

For $|x| \geq 1, \ln\left(\frac{1+x}{1-x}\right)$ is not defined, since $\frac{1+x}{1-x} \leq 0$.

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Hyperbolic functions
Exercise D, Question 6

Question:

Show that $\text{arsech}\, x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), 0 < x \leq 1$.

Solution:

$y = \text{arsech}\, x$

$x = \text{sech}\, y = \frac{2}{e^y + e^{-y}}$

$x(e^y + e^{-y}) = 2$

$xe^y - 2 + xe^{-y} = 0$ ← Multiply throughout by e^y.

$xe^{2y} - 2e^y + x = 0$

$e^y = \frac{2 \pm \sqrt{4-4x^2}}{2x}$ ← Solve as a quadratic in e^y.

$e^y = \frac{1 \pm \sqrt{1-x^2}}{x}$

$y = \ln\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right)$

$\text{arsech}\, x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ ← arsech x is 'single-valued'. So only the positive value is required.

$0 < x \leq 1$ ← If $x > 1$, $\sqrt{1-x^2}$ is not real.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise D, Question 7

Question:

Express as natural logarithms.

a arsinh 2

b arcosh 3

c $\text{artanh}\,\frac{1}{2}$

Solution:

a $\text{arsinh}\,2 = \ln(2+\sqrt{2^2+1})$
$= \ln(2+\sqrt{5})$

b $\text{arcosh}\,3 = \ln(3+\sqrt{3^2-1})$
$= \ln(3+\sqrt{8})$
$= \ln(3+2\sqrt{2})$

c $\text{artanh}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$
$= \frac{1}{2}\ln 3$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise D, Question 8

Question:

Express as natural logarithms.

a $\text{arsinh}\ \sqrt{2}$

b $\text{arcosh}\ \sqrt{5}$

c $\text{artanh}\ 0.1$

Solution:

a
$$\begin{aligned}\text{arsinh}\sqrt{2} &= \ln(\sqrt{2} + \sqrt{2+1}) \\ &= \ln(\sqrt{2} + \sqrt{3})\end{aligned}$$

b
$$\begin{aligned}\text{arcosh}\sqrt{5} &= \ln(\sqrt{5} + \sqrt{5-1}) \\ &= \ln(2 + \sqrt{5})\end{aligned}$$

c
$$\begin{aligned}\text{artanh}\,0.1 &= \frac{1}{2}\ln\left(\frac{1+0.1}{1-0.1}\right) \\ &= \frac{1}{2}\ln\left(\frac{11}{9}\right)\end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise D, Question 9

Question:

Express as natural logarithms.

a $\text{arsinh}(-3)$

b $\text{arcosh}\,\frac{3}{2}$

c $\text{artanh}\,\frac{1}{\sqrt{3}}$

Solution:

a $\text{arsinh}(-3) = \ln(-3+\sqrt{(-3)^2+1})$

$= \ln(-3+\sqrt{10})$

b $\text{arcosh}\left(\frac{3}{2}\right) = \ln\left(\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2-1}\right)$

$= \ln\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)$

$= \ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$

$= \ln\left(\frac{3+\sqrt{5}}{2}\right)$

c $\text{artanh}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right)$

$= \frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

$= \frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$

$= \frac{1}{2}\ln\left(\frac{4+2\sqrt{3}}{2}\right)$

$= \frac{1}{2}\ln(2+\sqrt{3})$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise D, Question 10

Question:

Given that $\text{artanh}x + \text{artanh}y = \ln\sqrt{3}$, show that $y = \frac{2x-1}{x-2}$.

Solution:

$$\begin{aligned}
&\text{artanh}x + \text{artanh}y \\
&= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \\
&= \frac{1}{2}\ln\left(\frac{1+x}{1-x} \times \frac{1+y}{1-y}\right) \\
&= \frac{1}{2}\ln\left(\frac{1+x+y+xy}{1-x-y+xy}\right) \\
&= \ln\sqrt{\left(\frac{1+x+y+xy}{1-x-y+xy}\right)}
\end{aligned}$$

Use $\ln a + \ln b = \ln(ab)$.

Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$.

So $\frac{1+x+y+xy}{1-x-y+xy} = 3$

$$\begin{aligned}
1+x+y+xy &= 3-3x-3y+3xy \\
1+x-3+3x &= -3y+3xy-y-xy \\
2xy-4y &= 4x-2 \\
y(x-2) &= 2x-1 \\
y &= \frac{2x-1}{x-2}
\end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise E, Question 1

Question:

Solve the following equation, giving your answer as natural logarithms.

$$3\sinh x + 4\cosh x = 4$$

Solution:

$$3\sinh x + 4\cosh x = 4$$

$$\frac{3(e^x - e^{-x})}{2} + \frac{4(e^x + e^{-x})}{2} = 4$$

$$3e^x - 3e^{-x} + 4e^x + 4e^{-x} = 8$$

$$7e^x - 8 + e^{-x} = 0$$

Multiply throughout by e^x.

$$7e^{2x} - 8e^x + 1 = 0$$

$$(7e^x - 1)(e^x - 1) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{1}{7} \text{ or } e^x = 1$$

$$x = \ln\left(\frac{1}{7}\right), x = 0$$

Note that $\ln\left(\frac{1}{7}\right) = \ln(7^{-1}) = -\ln 7$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise E, Question 2

Question:

Solve the following equation, giving your answer as natural logarithms.

$7\sinh x - 5\cosh x = 1$

Solution:

$$7\sinh x - 5\cosh x = 1$$
$$\frac{7(e^x - e^{-x})}{2} - \frac{5(e^x + e^{-x})}{2} = 1$$
$$7e^x - 7e^{-x} - 5e^x - 5e^{-x} = 2$$
$$2e^x - 2 - 12e^{-x} = 0$$
$$e^x - 1 - 6e^{-x} = 0$$

Multiply throughout by e^x.

$$e^{2x} - e^x - 6 = 0$$
$$(e^x - 3)(e^x + 2) = 0$$
$$e^x = 3$$

$e^x = -2$ is not possible for real x.

$$x = \ln 3$$

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Edexcel AS and A Level Modular Mathematics

Hyperbolic functions
Exercise E, Question 3

Question:

Solve the following equation, giving your answer as natural logarithms.

$30\cosh x = 15 + 26\sinh x$

Solution:

$$30\cosh x = 15 + 26\sinh x$$

$$30\frac{(e^x + e^{-x})}{2} = 15 + 26\frac{(e^x - e^{-x})}{2}$$

$$15e^x + 15e^{-x} = 15 + 13e^x - 13e^{-x}$$

$$2e^x - 15 + 28e^{-x} = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 15e^x + 28 = 0$$

$$(2e^x - 7)(e^x - 4) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{7}{2},\ e^x = 4$$

$$x = \ln\left(\frac{7}{2}\right),\ x = \ln 4$$

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Hyperbolic functions
Exercise E, Question 4

Question:

Solve the following equation, giving your answer as natural logarithms.

$13\sinh x - 7\cosh x + 1 = 0$

Solution:

$$13\sinh x - 7\cosh x + 1 = 0$$
$$13\frac{(e^x - e^{-x})}{2} - 7\frac{(e^x + e^{-x})}{2} + 1 = 0$$
$$13e^x - 13e^{-x} - 7e^x - 7e^{-x} + 2 = 0$$
$$6e^x + 2 - 20e^{-x} = 0$$
$$3e^x + 1 - 10e^{-x} = 0$$

Multiply throughout by e^x.

$$3e^{2x} + e^x - 10 = 0$$
$$(3e^x - 5)(e^x + 2) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{5}{3}$$

$e^x = -2$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

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Hyperbolic functions
Exercise E, Question 5

Question:

Solve the following equation, giving your answer as natural logarithms.

$\cosh 2x - 5\sinh x = 13$

Solution:

$\cosh 2x - 5\sinh x = 13$

Using $\cosh 2x = 1 + 2\sinh^2 x$,

$1 + 2\sinh^2 x - 5\sinh x = 13$

$2\sinh^2 x - 5\sinh x - 12 = 0$

$(2\sinh x + 3)(\sinh x - 4) = 0$

$\sinh x = -\frac{3}{2}, \sinh x = 4$

$x = \operatorname{arsinh}\left(-\frac{3}{2}\right), x = \operatorname{arsinh} 4$ ← Use $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

$x = \ln\left(-\frac{3}{2} + \sqrt{\frac{9}{4} + 1}\right)$

$= \ln\left(\frac{-3 + \sqrt{13}}{2}\right)$

$x = \ln(4 + \sqrt{16 + 1})$

$= \ln(4 + \sqrt{17})$

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Hyperbolic functions
Exercise E, Question 6

Question:

Solve the following equation, giving your answer as natural logarithms.

$2\tanh^2 x + 5\operatorname{sech} x - 4 = 0$

Solution:

$2\tanh^2 x + 5\operatorname{sech}x - 4 = 0$

Using $\operatorname{sech}^2 x = 1 - \tanh^2 x$

$2(1 - \operatorname{sech}^2 x) + 5\operatorname{sech}x - 4 = 0$

$2\operatorname{sech}^2 x - 5\operatorname{sech}x + 2 = 0$

$(2\operatorname{sech}x - 1)(\operatorname{sech}x - 2) = 0$

$\operatorname{sech}x = \frac{1}{2}, \operatorname{sech}x = 2$

$0 < \operatorname{sech}x \le 1$, so $\operatorname{sech}x = 2$ is not possible.

$\operatorname{sech}x = \frac{1}{2}$

$\cosh x = 2$

Use $\operatorname{sech}x = \frac{1}{\cosh x}$.

$x = \operatorname{arcosh} 2, -\operatorname{arcosh} 2$

Use $\operatorname{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution. $\ln(x - \sqrt{x^2 - 1})$ is the same as $-\ln(x + \sqrt{x^2 - 1})$

$x = \ln(2 \pm \sqrt{2^2 - 1})$

$= \ln(2 \pm \sqrt{3})$

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Hyperbolic functions
Exercise E, Question 7

Question:

Solve the following equation, giving your answer as natural logarithms.

$$3\sinh^2 x - 13\cosh x + 7 = 0$$

Solution:

$3\sinh^2 x - 13\cosh x + 7 = 0$

Using $\cosh^2 x - \sinh^2 x = 1$,

$$3(\cosh^2 x - 1) - 13\cosh x + 7 = 0$$
$$3\cosh^2 x - 13\cosh x + 4 = 0$$
$$(3\cosh x - 1)(\cosh x - 4) = 0$$
$$\cosh x = \frac{1}{3}, \cosh x = 4$$

$\cosh x \geq 1$, so $\cosh x = \frac{1}{3}$ is not possible.

$$\cosh x = 4$$
$$x = \text{arcosh}4, -\text{arcosh}4$$

Use $\text{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

$$x = \ln(4 \pm \sqrt{4^2 - 1})$$
$$= \ln(4 \pm \sqrt{15})$$

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Hyperbolic functions
Exercise E, Question 8

Question:

Solve the following equation, giving your answer as natural logarithms.

$\sinh 2x - 7\sinh x = 0$

Solution:

$$\sinh 2x - 7\sinh x = 0$$
$$2\sinh x\cosh x - 7\sinh x = 0$$
$$\sinh x(2\cosh x - 7) = 0$$
$$\sinh x = 0, \cosh x = \frac{7}{2}$$
$$x = 0, x = \pm\text{arcosh}\left(\frac{7}{2}\right)$$
$$\text{arcosh}\left(\frac{7}{2}\right) = \ln\left(\frac{7}{2} + \sqrt{\frac{49}{4} - 1}\right)$$
$$= \ln\left(\frac{7 + \sqrt{45}}{2}\right)$$
$$= \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$$
$$x = 0, x = \ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$$

Use $\text{arcosh}x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

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Hyperbolic functions
Exercise E, Question 9

Question:

Solve the following equation, giving your answer as natural logarithms.

$4\cosh x + 13e^{-x} = 11$

Solution:

$$4\cosh x + 13e^{-x} = 11$$

$$4\frac{(e^x + e^{-x})}{2} + 13e^{-x} = 11$$

$$2e^x + 2e^{-x} + 13e^{-x} = 11$$

$$2e^x + 15e^{-x} - 11 = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 11e^x + 15 = 0$$

$$(2e^x - 5)(e^x - 3) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{5}{2}, e^x = 3$$

$$x = \ln\left(\frac{5}{2}\right), x = \ln 3$$

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Hyperbolic functions
Exercise E, Question 10

Question:

Solve the following equation, giving your answer as natural logarithms.

$2\tanh x = \cosh x$

Solution:

$$2\tanh x = \cosh x$$
$$\frac{2\sinh x}{\cosh x} = \cosh x$$
$$2\sinh x = \cosh^2 x$$

Using $\cosh^2 x - \sinh^2 x = 1$

$$2\sinh x = 1 + \sinh^2 x$$
$$\sinh^2 x - 2\sinh x + 1 = 0$$
$$(\sinh x - 1)^2 = 0$$
$$\sinh x = 1$$
$$x = \text{arsinh} 1$$
$$x = \ln(1 + \sqrt{1^2 + 1})$$
$$= \ln(1 + \sqrt{2})$$

Use $\text{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

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Hyperbolic functions
Exercise F, Question 1

Question:

Find the exact value of

a $\sinh(\ln 3)$

b $\cosh(\ln 5)$

c $\tanh(\ln \frac{1}{4})$.

Solution:

a $\sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2}$

$= \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$.

b $\cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2}$

$= \frac{5 + \frac{1}{5}}{2} = \frac{13}{5}$

$e^{\ln 5} = 5$, and $e^{-\ln 5} = e^{\ln 5^{-1}} = \frac{1}{5}$.

c $\tanh\left(\ln \frac{1}{4}\right) = \frac{e^{2\ln\frac{1}{4}} - 1}{e^{2\ln\frac{1}{4}} + 1}$

$= \frac{\left(\frac{1}{16} - 1\right)}{\left(\frac{1}{16} + 1\right)}$

$= -\frac{15}{17}$

$e^{2\ln\frac{1}{4}} = e^{\ln\left(\frac{1}{4}\right)^2} = \frac{1}{16}$.

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Hyperbolic functions
Exercise F, Question 2

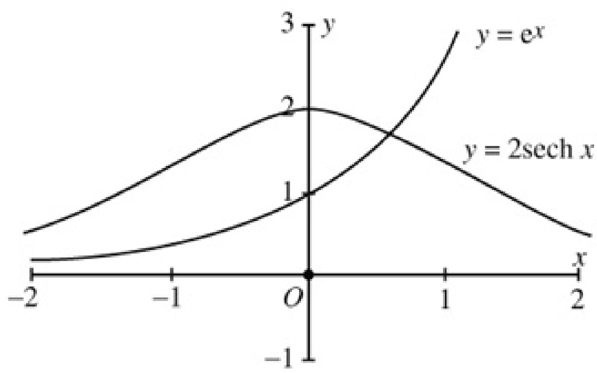
Question:

a Sketch on the same diagram the graphs of $y = 2\text{sech}\ x$ and $y = e^x$.

b Find the exact coordinates of the point of intersection of the graphs.

Solution:

a



b At the intersection,

$$2\text{sech}x = e^x$$

$$\frac{4}{e^x + e^{-x}} = e^x$$

$$4 = e^{2x} + 1$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

$$y = e^x = \sqrt{e^{2x}} = \sqrt{3}$$

coordinates are $(\frac{1}{2}\ln 3, \sqrt{3})$

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Hyperbolic functions
Exercise F, Question 3

Question:

Using the definitions of $\sinh x$ and $\cosh x$, prove that
$\sinh(A-B) = \sinh A\cosh B - \cosh A\sinh B$.

Solution:

$$\text{R.H.S.} = \sinh A\cosh B - \cosh A\sinh B$$
$$= \left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right)-\left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$
$$= \frac{e^{A+B}-e^{-A+B}+e^{A-B}-e^{-A-B}}{4}-\frac{e^{A+B}+e^{-A+B}-e^{A-B}-e^{-A-B}}{4}$$
$$= \frac{2\left(e^{A-B}-e^{-A+B}\right)}{4}$$
$$= \frac{e^{A-B}-e^{-(A-B)}}{2}$$
$$= \sinh(A-B) = \text{L.H.S.}$$

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Hyperbolic functions
Exercise F, Question 4

Question:

Using definitions in terms of exponentials, prove that $\sinh x = \dfrac{2\tanh \frac{1}{2}x}{1-\tanh^2 \frac{1}{2}x}$.

Solution:

$$\text{R.H.S.} = \frac{2\tanh \frac{1}{2}x}{1-\tanh^2 \frac{1}{2}x}$$

$$2\tanh \frac{1}{2}x \quad = \frac{2(e^x-1)}{e^x+1}$$

$$1-\tanh^2 \frac{1}{2}x = 1-\left(\frac{e^x-1}{e^x+1}\right)^2$$

$$= \frac{(e^x+1)^2-(e^x-1)^2}{(e^x+1)^2}$$

$$= \frac{4e^x}{(e^x+1)^2}$$

$$\text{So R.H.S.} = \frac{2(e^x-1)}{e^x+1} \times \frac{(e^x+1)^2}{4e^x}$$

$$= \frac{(e^x-1)(e^x+1)}{2e^x}$$

$$= \frac{e^{2x}-1}{2e^x}$$

$$= \frac{e^x-e^{-x}}{2}$$

$$= \sinh x = \text{L.H.S.}$$

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Hyperbolic functions
Exercise F, Question 5

Question:

a Given that $13\cosh x + 5\sinh x = R\cosh(x+\alpha)$, $R > 0$, use the identity $\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$ to find the values of R and α, giving the value of α to 3 decimal places.

b Write down the minimum value of $13\cosh x + 5\sinh x$.

Solution:

a $13\cosh x + 5\sinh x = R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

So $R\cosh\alpha = 13$

$R\sinh\alpha = 5$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 13^2 - 5^2$

Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 144$

$R^2 = 144$

$R = 12$

$$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{5}{13}$$

$$\tanh\alpha = \frac{5}{13}$$

$\alpha = 0.405$

Direct from calculator.

b $13\cosh x + 5\sinh x = 12\cosh(x + 0.405)$

For any value A, $\cosh A \geq 1$.

The minimum value of $13\cosh x + 5\sinh x$ is 12.

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Hyperbolic functions
Exercise F, Question 6

Question:

a Show that, for $x > 0$, $\text{arcosech}x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$.

b Use the answer to part **a** to write down the value of arcosech 3.

c Use the logarithmic form of arsinh x to verify that your answer to part **b** is the same as the value for $\text{arsinh}\left(\frac{1}{3}\right)$.

Solution:

a $y = \text{arcosech}x$

$x = \text{cosech}y = \frac{1}{\sinh y} = \frac{2}{e^y - e^{-y}}$

$x(e^y - e^{-y}) = 2$

$xe^y - 2 - xe^{-y} = 0$

$xe^{2y} - 2e^y - x = 0$ ← Multiply throughout by e^y.

$e^y = \frac{2 \pm \sqrt{4+4x^2}}{2x}$ ← Solve as a quadratic in e^y.

$e^y = \frac{1 \pm \sqrt{1+x^2}}{x}$

$e^y = \frac{1+\sqrt{1+x^2}}{x}, x > 0$ ← For $x > 0$, the positive sign gives a positive value for e^y, whereas the negative sign gives an impossible negative value for e^y.

$y = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), x > 0$

$\text{arcosech}x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), x > 0$

b $\text{arcosech } 3 = \ln\left(\frac{1+\sqrt{10}}{3}\right)$

c $\text{arsinh}\left(\frac{1}{3}\right) = \ln\left(\frac{1}{3} + \sqrt{\frac{1}{9}+1}\right)$

$= \ln\left(\frac{1}{3} + \sqrt{\frac{10}{9}}\right)$

$= \ln\left(\frac{1+\sqrt{10}}{3}\right)$

(Same as the answer to part **b**).

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Hyperbolic functions
Exercise F, Question 7

Question:

Solve, giving your answers as natural logarithms, L $9\cosh x - 5\sinh x = 15$

Solution:

$$9\cosh x - 5\sinh x = 15$$

$$9\frac{(e^x + e^{-x})}{2} - 5\frac{(e^x - e^{-x})}{2} = 15$$

$$9e^x + 9e^{-x} - 5e^x + 5e^{-x} = 30$$

$$4e^x - 30 + 14e^{-x} = 0$$

$$2e^x - 15 + 7e^{-x} = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 15e^x + 7 = 0$$

$$(2e^x - 1)(e^x - 7) = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{1}{2}, e^x = 7$$

$$x = \ln\left(\frac{1}{2}\right), x = \ln 7$$

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Hyperbolic functions
Exercise F, Question 8

Question:

Solve, giving your answers as natural logarithms, L $23\sinh x - 17\cosh x + 7 = 0$

Solution:

$$23\sinh x - 17\cosh x + 7 = 0$$

$$23\frac{(e^x - e^{-x})}{2} - 17\frac{(e^x + e^{-x})}{2} + 7 = 0$$

$$23e^x - 23e^{-x} - 17e^x - 17e^{-x} + 14 = 0$$

$$6e^x + 14 - 40e^{-x} = 0$$

$$3e^x + 7 - 20e^{-x} = 0$$

$$3e^{2x} + 7e^x - 20 = 0$$

Multiply throughout by e^x.

$$(3e^x - 5)(e^x + 4) = 0$$

$$e^x = \frac{5}{3}$$

$e^x = -4$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

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Hyperbolic functions
Exercise F, Question 9

Question:

Solve, giving your answers as natural logarithms, L $3\cosh^2 x + 11\sinh x = 17$

Solution:

$3\cosh^2 x + 11\sinh x = 17$

Using $\cosh^2 x - \sinh^2 x = 1$

$3(1+\sinh^2 x) + 11\sinh x = 17$

$3\sinh^2 x + 11\sinh x - 14 = 0$

$(3\sinh x + 14)(\sinh x - 1) = 0$

$\sinh x = -\frac{14}{3}, \sinh x = 1$

$x = \text{arsinh}\left(-\frac{14}{3}\right), x = \text{arsinh}1$

Use $\text{arsinh}x = \ln(x + \sqrt{x^2+1})$.

$x = \ln\left(-\frac{14}{3} + \sqrt{\frac{196}{9}+1}\right)$

$= \ln\left(\frac{-14+\sqrt{205}}{3}\right)$

$x = \ln(1+\sqrt{1+1})$

$= \ln(1+\sqrt{2})$

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Hyperbolic functions
Exercise F, Question 10

Question:

Solve, giving your answers as natural logarithms, L $6\tanh x - 7\operatorname{sech} x = 2$

Solution:

$$6\tanh x - 7\operatorname{sech} x = 2$$

$$\frac{6\sinh x}{\cosh x} - \frac{7}{\cosh x} = 2$$

$$6\sinh x - 7 = 2\cosh x$$

$$6\frac{(e^x - e^{-x})}{2} - 7 = 2\frac{(e^x + e^{-x})}{2}$$

$$3e^x - 3e^{-x} - 7 = e^x + e^{-x}$$

$$2e^x - 7 - 4e^{-x} = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 7e^x - 4 = 0$$

$$(2e^x + 1)(e^x - 4) = 0$$

Solve as a quadratic in e^x.

$$e^x = 4$$

$$x = \ln 4$$

$e^x = -\frac{1}{2}$ is not possible for real x.

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Hyperbolic functions
Exercise F, Question 11

Question:

Show that $\sinh[\ln(\sin x)] = -\frac{1}{2}\cos x \cot x$.

Solution:

$$\sinh(\ln(\sin x)) = \frac{e^{\ln(\sin x)} - e^{-\ln(\sin x)}}{2}$$

$$= \frac{e^{\ln(\sin x)} - e^{\ln(\sin x)^{-1}}}{2}$$

$$= \frac{\sin x - (\sin x)^{-1}}{2}$$

$$= \frac{\sin x - \operatorname{cosec} x}{2}$$

$$= \frac{\sin^2 x - 1}{2\sin x}$$

$$= -\frac{\cos^2 x}{2\sin x}$$

$$= -\frac{1}{2}\cos x\left(\frac{\cos x}{\sin x}\right)$$

$$= -\frac{1}{2}\cos x \cot x$$

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Hyperbolic functions
Exercise F, Question 12

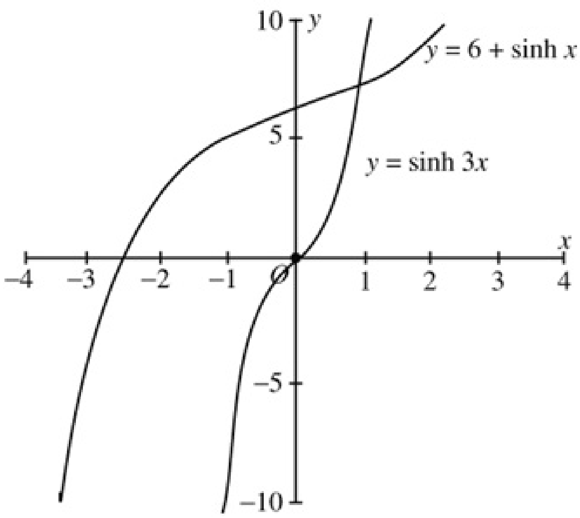
Question:

a On the same diagram, sketch the graphs of $y = 6 + \sinh x$ and $y = \sinh 3x$.

b Using the identity $\sinh 3x = 3\sinh x + 4\sinh^3 x$, show that the graphs intersect where $\sinh x = 1$ and hence find the exact coordinates of the point of intersection.

Solution:

a



b At the intersection,

$$6 + \sinh x = \sinh 3x$$
$$6 + \sinh x = 3\sinh x + 4\sinh^3 x$$
$$4\sinh^3 x + 2\sinh x - 6 = 0$$
$$2\sinh^3 x + \sinh x - 3 = 0$$
$$(\sinh x - 1)(2\sinh^2 x + 2\sinh x + 3) = 0$$

You can see, by inspection, that $\sinh x = 1$ satisfies this equation.

The equation $2\sinh^2 x + 2\sinh x + 3 = 0$ has no real roots, because $b^2 - 4ac = 4 - 24 < 0$.
The only intersection is where $\sinh x = 1$

For $\sinh x = 1$,
$x = \text{arsinh} 1$
$= \ln(1 + \sqrt{1^2 + 1})$
$= \ln(1 + \sqrt{2})$
Using $y = 6 + \sinh x$
$y = 7$
Coordinates of the point of intersection are $(\ln(1 + \sqrt{2}), 7)$

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Hyperbolic functions
Exercise F, Question 13

Question:

Given that $\text{artanh}x - \text{artanh}y = \ln 5$, find y in terms of x.

Solution:

$$\text{artanh}x - \text{artanh}y$$
$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

Use $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\times\frac{1-y}{1+y}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1+x-y-xy}{1-x+y-xy}\right)$$

Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$

$$= \ln\sqrt{\left(\frac{1+x-y-xy}{1-x+y-xy}\right)}$$

So $\sqrt{\left(\frac{1+x-y-xy}{1-x+y-xy}\right)} = 5$

$$\frac{1+x-y-xy}{1-x+y-xy} = 25$$
$$1+x-y-xy = 25-25x+25y-25xy$$
$$24xy-26y = 24-26x$$
$$y(12x-13) = 12-13x$$
$$y = \frac{12-13x}{12x-13}$$

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Hyperbolic functions
Exercise F, Question 14

Question:

a Express $3\cosh x + 5\sinh x$ in the form $R\sinh(x+\alpha)$, where $R > 0$. Give α to 3 decimal places.

b Use the answer to part **a** to solve the equation $3\cosh x + 5\sinh x = 8$, giving your answer to 2 decimal places.

c Solve $3\cosh x + 5\sinh x = 8$ by using the definitions of $\cosh x$ and $\sinh x$.

Solution:

a

$$3\cosh x + 5\sinh x = R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$$

$$\text{So } R\cosh\alpha = 5$$

$$R\sinh\alpha = 3$$

$$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 5^2 - 3^2$$

Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$$R^2(\cosh^2\alpha - \sinh^2\alpha) = 16$$

$$R^2 = 16$$

$$R = 4$$

$$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{3}{5}$$

$$\tanh\alpha = \frac{3}{5}$$

$$\alpha = 0.693$$

Direct from calculator.

$$3\cosh x + 5\sinh x = 4\sinh(x + 0.693)$$

b

$$4\sinh(x + 0.693) = 8$$

$$\sinh(x + 0.693) = 2$$

$$x + 0.693 = \text{arsinh } 2$$

$$= 1.444 \text{ (3 d.p.)}$$

Direct from calculator.

$$x = 0.75 \text{ (2 d.p.)}$$

c

$$3\cosh x + 5\sinh x = 8$$

$$3\frac{(e^x + e^{-x})}{2} + 5\frac{(e^x - e^{-x})}{2} = 8$$

$$3e^x + 3e^{-x} + 5e^x - 5e^{-x} = 16$$

$$8e^x - 16 - 2e^{-x} = 0$$

$$4e^x - 8 - e^{-x} = 0$$

Multiply throughout by e^x.

$$4e^{2x} - 8e^x - 1 = 0$$

$$e^x = \frac{8 \pm \sqrt{64 + 16}}{8}$$

Solve as a quadratic in e^x.

$$e^x = 1 \pm \frac{\sqrt{80}}{8} = 1 \pm \frac{\sqrt{5}}{2}$$

$$e^x = 1 + \frac{\sqrt{5}}{2}$$

$e^x = 1 - \frac{\sqrt{5}}{2}$ is negative, so not possible for real x.

$$x = \ln\left(1 + \frac{\sqrt{5}}{2}\right)$$

$$= 0.75 \text{ (2 d.p.)}$$

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Further coordinate systems
Exercise A, Question 1

Question:

a Sketch the following ellipses showing clearly where the curve crosses the coordinate axes.

i $x^2 + 4y^2 = 16$

ii $4x^2 + y^2 = 36$

iii $x^2 + 9y^2 = 25$

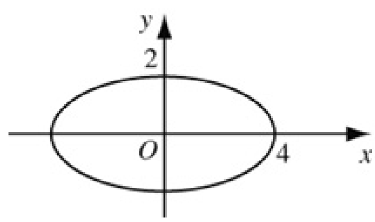
b Find parametric equations for these curves.

Solution:

i $x^2 + 4y^2 = 16$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$

a

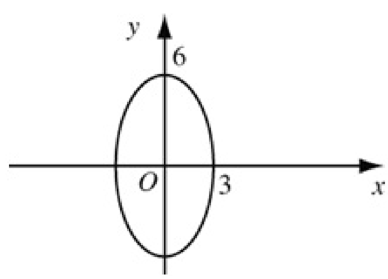


b Parametric equations $x = 4\cos\theta, y = 2\sin\theta$

ii $4x^2 + y^2 = 36$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{36} = 1$

a

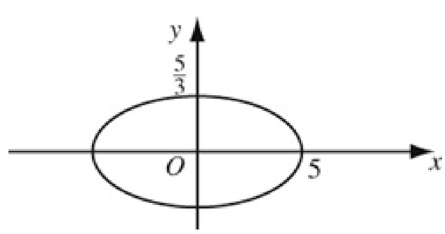


b Parametric equations $x = 3\cos\theta, y = 6\sin\theta$

iii $x^2 + 9y^2 = 25$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{\left(\frac{5}{3}\right)^2} = 1$

a



b Parametric equations $x = 5\cos\theta, y = \frac{5}{3}\sin\theta$

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Further coordinate systems
Exercise A, Question 2

Question:

a Sketch ellipses with the following parametric equations.

b Find a Cartesian equation for each ellipse.

i $x = 2\cos\theta, y = 3\sin\theta$

ii $x = 4\cos\theta, y = 5\sin\theta$

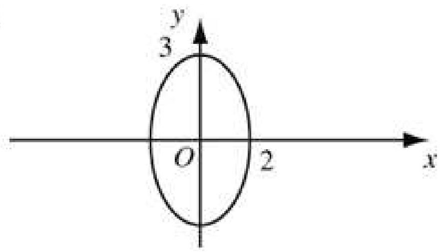
iii $x = \cos\theta, y = 5\sin\theta$

iv $x = 4\cos\theta, y = 3\sin\theta$

Solution:

i $x = 2\cos\theta, y = 3\sin\theta$
$\Rightarrow a = 2, b = 3$

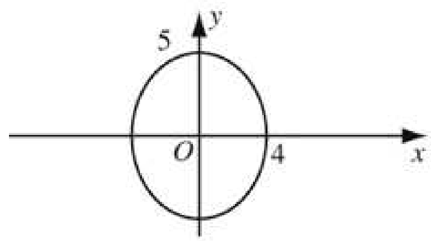
a



b Cartesian equation $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$

ii $x = 4\cos\theta, y = 5\sin\theta$
$\Rightarrow a = 4, b = 5$

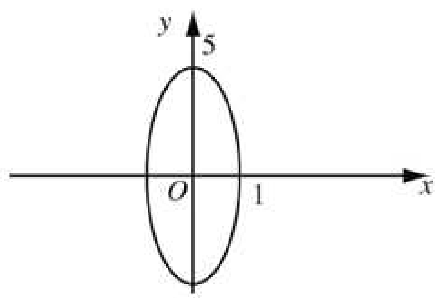
a



b Cartesian equation $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$

iii $x = \cos\theta, y = 5\sin\theta$
$\Rightarrow a = 1, b = 5$

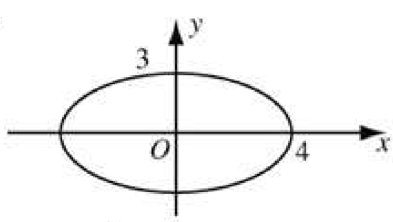
a



b Cartesian equation $x^2 + \frac{y^2}{5^2} = 1$

iv $x = 4\cos\theta, y = 3\sin\theta$
$\Rightarrow a = 4, b = 3$

a



b Cartesian equation $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$

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Further coordinate systems
Exercise B, Question 1

Question:

Find the equations of tangents and normals to the following ellipses at the points given.

a $\frac{x^2}{4}+y^2=1$ at $(2\cos\theta, \sin\theta)$

b $\frac{x^2}{25}+\frac{y^2}{9}=1$ at $(5\cos\theta, 3\sin\theta)$

Solution:

$x=a\cos\theta,\ y=b\sin\theta$

$\frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$ $\quad\therefore$ tangent is: $y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

Equation of tangent is: $ay\sin\theta+bx\cos\theta=ab$

Normal gradient is $\frac{a\sin\theta}{b\cos\theta}$ $\quad\therefore$ normal is: $y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$

Equation of normal is: $by\cos\theta-ax\sin\theta=(b^2-a^2)\sin\theta\cos\theta$

a $a=2, b=1$
So equation of tangent is: $2y\sin\theta+x\cos\theta=2$
Equation of normal is: $y\cos\theta-2x\sin\theta=-3\sin\theta\cos\theta$

b $\frac{x^2}{25}+\frac{y^2}{9}=1\Rightarrow a=5, b=3$
So equation of tangent is: $5y\sin\theta+3x\cos\theta=15$
Equation of normal is: $3y\cos\theta-5x\sin\theta=-16\sin\theta\cos\theta$

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Further coordinate systems
Exercise B, Question 2

Question:

Find equations of tangent and normals to the following ellipses at the points given.

a $\frac{x^2}{9}+\frac{y^2}{1}=1$ at $(\sqrt{5},\frac{2}{3})$

b $\frac{x^2}{16}+\frac{y^2}{4}=1$ at $(-2,\sqrt{3})$

Solution:

a $\frac{x^2}{9}+y^2=1 \Rightarrow \frac{2x}{9}+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{9y}$ so at $\left(\sqrt{5},\frac{2}{3}\right)$ $m=-\frac{\sqrt{5}}{6}$

Tangent at $\left(\sqrt{5},\frac{2}{3}\right)$ is: $y-\frac{2}{3}=-\frac{\sqrt{5}}{6}(x-\sqrt{5})$

i.e. $6y+\sqrt{5}x=9$

Normal at $\left(\sqrt{5},\frac{2}{3}\right)$ is: $y-\frac{2}{3}=\frac{6}{\sqrt{5}}(x-\sqrt{5})$

i.e. $3\sqrt{5}y-2\sqrt{5}=18x-18\sqrt{5}$ i.e. $3\sqrt{5}y=18x-16\sqrt{5}$

b $\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow \frac{x}{8}+\frac{y}{2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{4y}$ so at $(-2,\sqrt{3})$ $m=\frac{1}{2\sqrt{3}}$

Tangent at $(-2,\sqrt{3})$ is: $y-\sqrt{3}=\frac{1}{2\sqrt{3}}(x--2)$

i.e. $2\sqrt{3}y-x=8$

Normal at $(-2,\sqrt{3})$ is $y-\sqrt{3}=-2\sqrt{3}(x--2)$

i.e. $y+2\sqrt{3}x=-3\sqrt{3}$

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Further coordinate systems
Exercise B, Question 3

Question:

Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos t, b\sin t)$ is $xb\cos t + ya\sin t = ab$

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}, \text{ at } (a\cos t, b\sin t) \quad m=\frac{-b^2a\cos t}{a^2b\sin t}$$

$$\therefore m=-\frac{b\cos t}{a\sin t}$$

Equation of tangent at $(a\cos t, b\sin t)$ is:

$$y-b\sin t=-\frac{b\cos t}{a\sin t}(x-a\cos t)$$

i.e. $ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$

i.e. $bx\cos t + ay\sin t = ab$.

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Further coordinate systems
Exercise B, Question 4

Question:

a Show that the line $y = x + \sqrt{5}$ is a tangent to the ellipse with equation $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

b Find the point of contact of this tangent.

Solution:

The line $y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if

$a^2m^2 + b^2 = c^2$

a $m = 1, c = \sqrt{5} \quad (\because y = x + \sqrt{5})$

$a = 2, b = 1 \quad \left(\because \frac{x^2}{4} + \frac{y^2}{1} = 1\right)$

$$a^2m^2 + b^2 = 4 \times 1 + 1 = 5$$
$$= c^2$$

$\therefore y = x + \sqrt{5}$ is a tangent.

b Point of contact: $y = x + \sqrt{5}$

$$\frac{x^2}{4} + y^2 = 1 \Rightarrow \frac{x^2}{4} + (x + \sqrt{5})^2 = 1$$

$$\therefore x^2 + 4(x^2 + 2\sqrt{5}x + 5) = 4$$

$$5x^2 + 8\sqrt{5}x + 16 = 0$$

$$(\sqrt{5}x + 4)^2 = 0$$

$$x = -\frac{4}{\sqrt{5}} = -\frac{4}{5}\sqrt{5}$$

$$\therefore y = -\frac{4}{5}\sqrt{5} + \sqrt{5} = \frac{1}{5}\sqrt{5}$$

So point of contact is $\left(-\frac{4}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\right)$

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Further coordinate systems
Exercise B, Question 5

Question:

a Find an equation of the normal to the ellipse with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ at the point $P(3\cos\theta, 2\sin\theta)$.

This normal crosses the x-axis at the point $(-\frac{5}{6}, 0)$.

b Find the value of θ and the exact coordinates of the possible positions of P.

Solution:

a $x=3\cos\theta, y=2\sin\theta \Rightarrow \frac{dy}{dx}=\frac{2\cos\theta}{-3\sin\theta}$

$\therefore$ Gradient of normal is $\frac{3\sin\theta}{2\cos\theta}$

$\therefore$ Equation of normal is: $y-2\sin\theta=\frac{3\sin\theta}{2\cos\theta}(x-3\cos\theta)$

i.e. $2y\cos\theta-4\cos\theta\sin\theta=3\sin\theta x-9\sin\theta\cos\theta$

$2y\cos\theta-3\sin\theta x=-5\sin\theta\cos\theta$

b $y=0,\ x=-\frac{5}{6}$

$\Rightarrow\ -3\sin\theta\left(-\frac{5}{6}\right)=-5\sin\theta\cos\theta$

$\frac{5}{2}=-5\cos\theta \textit{ or } \sin\theta=0$ or $\sin\theta=0$

i.e. $\cos\theta=-\frac{1}{2}$ i.e. $\theta=0$ or 180° i.e. $\theta=0$ or 180°

$\therefore\ \theta=120^\circ, 240^\circ$

$\therefore P$ is $\left(-\frac{3}{2},\sqrt{3}\right)$ or $\left(-\frac{3}{2},-\sqrt{3}\right)$ i.e. P is $(3, 0)$ or $(-3, 0)$

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Further coordinate systems
Exercise B, Question 6

Question:

The line $y = 2x + c$ is a tangent to $x^2 + \frac{y^2}{4} = 1$. Find the possible values of c.

Solution:

$y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a^2m^2 + b^2 = c^2$

$y = 2x + c \Rightarrow m = 2, c = ?$

$x^2 + \frac{y^2}{4} = 1 \Rightarrow a = 1, b = 2$

$a^2m^2 + b^2 = c^2 \Rightarrow 1 \times 4 + 4 = c^2$

$\therefore c^2 = 8$

$\therefore c = \pm 2\sqrt{2}$

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Further coordinate systems

Exercise B, Question 7

Question:

The line with equation $y = mx + 3$ is a tangent to $x^2 + \frac{y^2}{5} = 1$.

Find the possible values of m.

Solution:

The $a^2m^2 + b^2 = c^2$ condition could be used as in question 6.

$$\left.\begin{aligned} x^2 + \frac{y^2}{5} &= 1 \\ y &= mx + 3 \end{aligned}\right\} \text{substitution} \Rightarrow x^2 + \frac{(mx+3)^2}{5} = 1$$

i.e. $5x^2 + (mx+3)^2 = 5$

$(5 + m^2 5 + m^2)x^2 + 6mx + 4 = 0$

Since the line is a tangent the discriminant of this equation must equal zero (must have equal roots).

So $36m^2 = 16(5 + m^2)$

$20m^2 = 80$

$m^2 = 4$

$\therefore m = \pm 2$

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Further coordinate systems
Exercise B, Question 8

Question:

The line $y = mx + 4$ $(m > 0)$ is a tangent to the ellipse E with equation $\frac{x^2}{3} + \frac{y^2}{4} = 1$ at the point P.

a Find the value of m.

b Find the coordinates of the point P.

The normal to E at P crosses the y-axis at the point A.

c Find the coordinates of A.

The tangent to E at P crosses the y-axis at the point B.

d Find the area of triangle APB.

Solution:

a $y = mx + 4,\quad \frac{x^2}{3} + \frac{y^2}{4} = 1 \Rightarrow c = 4, a^2 = 3, b^2 = 4$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 4 + 3m^2 = 16$

$3m^2 = 12$

$m = \pm 2$ but $m > 0$

$\therefore m = 2$

b $y = 2x + 4, \frac{x^2}{3} + \frac{y^2}{4} = 1$ substitute $\frac{x^2}{3} + \frac{(4x^2 + 16x + 16)}{4} = 1$

$\Rightarrow x^2 + 3x^2 + 12x + 12 = 3$

$4x^2 + 12x + 9 = 0$

$(2x + 3)^2 = 0$

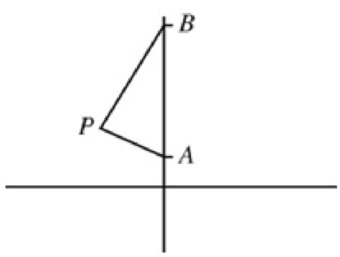
$x = -\frac{3}{2}, y = 2x + 4 = 1 \therefore P$ is $\left(-\frac{3}{2}, 1\right)$

c Gradient of normal $= -\frac{1}{2}$

Equation of normal: $y - 1 = -\frac{1}{2}\left(x - -\frac{3}{2}\right)$

$x = 0 \Rightarrow y = 1 - \frac{3}{4} = \frac{1}{4} \therefore A\left(0, \frac{1}{4}\right)$

d Tangent is $y = 2x + 4, x = 0 \Rightarrow y = 4 \therefore B(0,4)$



Area of $\Delta APB = \frac{1}{2}\left(4 - \frac{1}{4}\right) \times \frac{3}{2}$

$= \frac{1}{2} \times \frac{15}{4} \times \frac{3}{2} = \frac{45}{16}$

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Further coordinate systems
Exercise B, Question 9

Question:

The ellipse E has equation $\frac{x^2}{9}+\frac{y^2}{4}=1$.

a Show that the gradient of the tangent to E at the point $P(3\cos\theta, 2\sin\theta)$ is $-\frac{2}{3}\cot\theta$.

b Show that the point $Q(\frac{9}{5},-\frac{8}{5})$ lies on E.

c Find the gradient of the tangent to E at Q.

The tangents to E at the points P and Q are perpendicular.

d Find the value of $\tan\theta$ and hence the exact coordinates of P.

Solution:

a $\frac{dy}{d\theta}=2\cos\theta, \frac{dx}{d\theta}=-3\sin\theta \therefore \frac{dy}{dx}=-\frac{2}{3}\cot\theta$

b $\frac{\left(\frac{9}{5}\right)^2}{9}+\frac{\left(\frac{-8}{5}\right)^2}{4}=\frac{9}{25}+\frac{16}{25}=1=\text{RHS}$

$\therefore \left(\frac{9}{5},-\frac{8}{5}\right)$ lies on E

c $\frac{9}{5}=3\cos\phi \Rightarrow \cos\phi=\frac{3}{5}$

$-\frac{8}{5}=2\sin\phi \Rightarrow \sin\phi=-\frac{4}{5}$

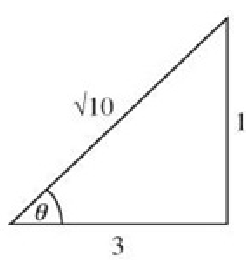
$\therefore \cot\phi=-\frac{3}{4}$ where Q is $(3\cos\phi, 2\sin\phi)$

$\therefore \frac{dy}{dx}=\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$

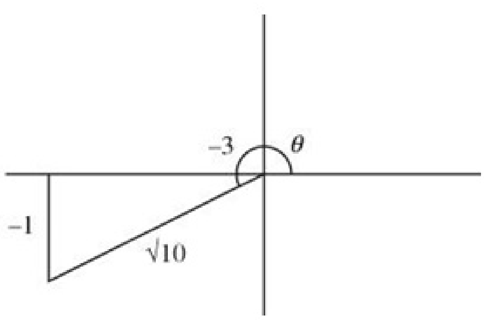
d Gradient of tangent at $P=-2$

$\therefore -2=-\frac{2}{3}\cot\theta \Rightarrow \tan\theta=\frac{1}{3}$

$\therefore P$ is $\left(3\times\frac{3}{\sqrt{10}}, 2\frac{1}{\sqrt{10}}\right)$



or



P is $\left(\frac{9}{10}\sqrt{10}, \frac{2}{10}\sqrt{10}\right)$

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Further coordinate systems

Exercise B, Question 10

Question:

The line $y = mx + c$ is a tangent to both the ellipses $\frac{x^2}{9} + \frac{y^2}{46} = 1$ and $\frac{x^2}{25} + \frac{y^2}{14} = 1$.

Find the possible values of m and c.

Solution:

$y = mx + c$ and $\frac{x^2}{9} + \frac{y^2}{46} = 1 \Rightarrow a^2 = 9, b^2 = 46$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 46 + 9m^2 = c^2$ ①

$y = mx + c$ and $\frac{x^2}{25} + \frac{y^2}{14} = 1 \Rightarrow a^2 = 25, b^2 = 14$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 14 + 25m^2 = c^2$ ②

① − ② $\Rightarrow 32 - 16m^2 = 0$

$\Rightarrow m^2 = 2$

$\therefore m = \pm\sqrt{2}$

$m^2 = 2$ and $14 + 25m^2 = c^2 \Rightarrow c^2 = 64$

$\therefore c = \pm 8$

$\therefore m = \pm 2, c = \pm 8$

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Further coordinate systems
Exercise C, Question 1

Question:

Sketch the following hyperbolae showing clearly the intersections with the x-axis and the equations of the asymptotes.

a $x^2 - 4y^2 = 16$

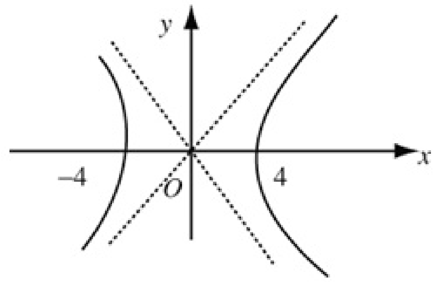
b $4x^2 - 25y^2 = 100$

c $\frac{x^2}{8} - \frac{y^2}{2} = 1$

Solution:

a $\frac{x^2}{16} - \frac{y^2}{4} = 1$

$a = 4, b = 2$

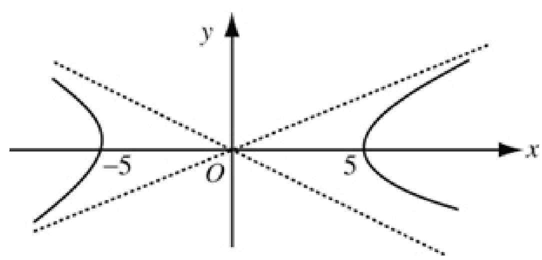


Asymptotes $y = \pm\frac{1}{2}x$

b $4x^2 - 25y^2 = 100$

$\Rightarrow \frac{x^2}{25} - \frac{y^2}{4} = 1$

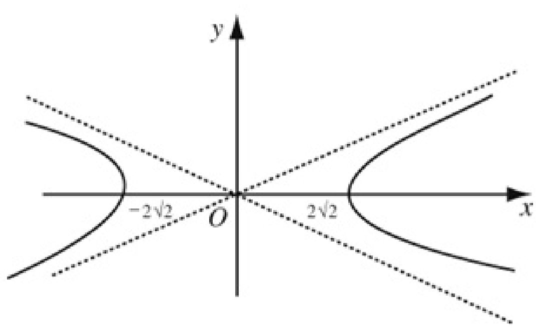
$a = 5, b = 2$



Asymptotes $y = \pm\frac{2}{5}x$

c $\frac{x^2}{8} - \frac{y^2}{2} = 1$

$a = 2\sqrt{2}, b = \sqrt{2}$



Asymptotes $y = \pm\frac{1}{2}x$

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Further coordinate systems
Exercise C, Question 2

Question:

a Sketch the hyperbolae with the following parametric equations. Give the equations of the asymptotes and show points of intersection with the x-axis.

b Find the Cartesian equation for each hyperbola.

i $x = 2\sec\theta$
$y = 3\tan\theta$

ii $x = 4\cosh t$
$y = 3\sinh t$

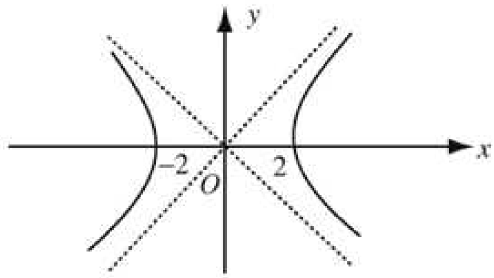
iii $x = \cosh t$
$y = 2\sinh t$

iv $x = 5\sec\theta$
$y = 7\tan\theta$

Solution:

i $x = 2\sec\theta, y = 3\tan\theta$
$a = 2, b = 3$

a

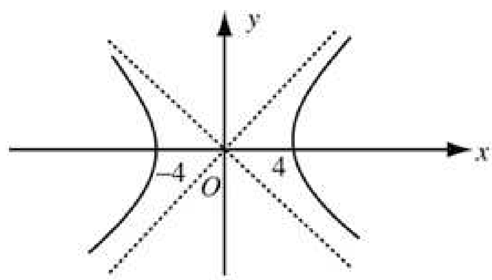


Asymptotes $y = \pm\frac{3}{2}x$

b $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$

ii $x = 4\cosh t, y = 3\sinh t$
$a = 4, b = 3$

a

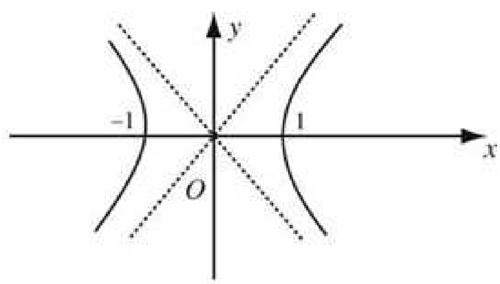


Asymptotes $y = \pm\frac{3}{4}x$

b Equation: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

iii $x = \cosh t, y = 2\sinh t$
$a = 1, b = 2$

a

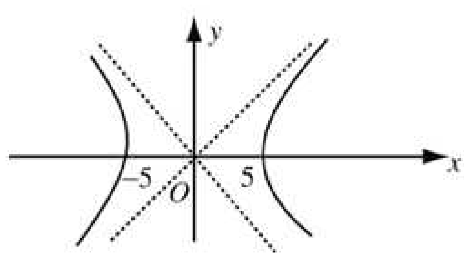


b Equation: $x^2 - \frac{y^2}{4} = 1$

Asymptotes $y = \pm 2x$

iv $x = 5\sec\theta, y = 7\tan\theta$
$a = 5, b = 7$

a



b Equation:
$\frac{x^2}{25} - \frac{y^2}{49} = 1$

Asymptotes $y = \pm\frac{7}{5}x$

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Further coordinate systems
Exercise D, Question 1

Question:

Find the equations of the tangents and normals to the hyperbolae with the following equations at the points indicated.

a $\frac{x^2}{16}-\frac{y^2}{2}=1$ at the point (12, 4)

b $\frac{x^2}{36}-\frac{y^2}{12}=1$ at the point (12, 6)

c $\frac{x^2}{25}-\frac{y^2}{3}=1$ at the point (10, 3)

Solution:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

a $a^2=16, b^2=2 \quad \therefore \frac{dy}{dx}=\frac{x}{8y}$ At (12, 4) $y'=\frac{3}{8}$

At (12, 4) equation of tangent is: $y-4=\frac{3}{8}(x-12)$

$$8y=3x-4$$

Equation of normal is: $y-4=-\frac{8}{3}(x-12)$

$$3y+8x=108$$

b $a^2=36, b^2=12 \therefore \frac{dy}{dx}=\frac{x}{3y}$ At (12, 6) $y'=\frac{2}{3}$

At (12, 6) equation of tangent is: $y-6=\frac{2}{3}(x-12)$

$$3y=2x-6$$

Equation of normal is $y-6=-\frac{3}{2}(x-12)$

$$2y+3x=48$$

c $a^2=25, b^2=3 \quad \therefore \frac{dy}{dx}=\frac{3x}{25y}$ at (10, 3) $y'=\frac{2}{5}$

At (10, 3) equation of tangent is: $y-3=\frac{2}{5}(x-10)$

$$5y=2x-5$$

Equation of normal is: $y-3=-\frac{5}{2}(x-10)$

$$2y+5x=56$$

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Further coordinate systems
Exercise D, Question 2

Question:

Find the equations of the tangents and normals to the hyperbolae with the following equations at the points indicated.

a $\frac{x^2}{25}-\frac{y^2}{4}=1$ at the point $(5\cosh t, 2\sinh t)$

b $\frac{x^2}{1}-\frac{y^2}{9}=1$ at the point $(\sec t, 3\tan t)$

Solution:

a $x=5\cosh t\ y=2\sinh t \quad \therefore \frac{dy}{dx}=\frac{2\cosh t}{5\sinh t}$

$\therefore$ Equation of tangent: $y-2\sinh t=\frac{2\cosh t}{5\sinh t}(x-5\cosh t)$

$$5y\sinh t+10=2x\cosh t$$

Equation of normal:

$$y-2\sinh t=-\frac{5\sinh t}{2\cosh t}(x-5\cosh t)$$

$$2y\cosh t+5x\sinh t=29\cosh t\sinh t$$

b $x=\sec t, y=3\tan t \quad \therefore \frac{dy}{dx}=\frac{3\sec^2 t}{\sec t\tan t}=\frac{3\sec t}{\tan t}$

$\therefore$ Equation of tangent: $y-3\tan t=\frac{3\sec t}{\tan t}(x-\sec t)$

$$y\tan t+3=3\sec tx$$

Equation of normal: $y-3\tan t=-\frac{\tan t}{3\sec t}(x-\sec t)$

$$3y\sec t+x\tan t=10\sec t\tan t$$

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Further coordinate systems
Exercise D, Question 3

Question:

Show that an equation of the tangent to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec t, b\tan t)$ is $bx\sec t - ay\tan t = ab$.

Solution:

$x = a\sec t \quad y = b\tan t$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t}$

Equation of tangent is:

$$y - b\tan t = \frac{b\sec t}{a\tan t}(x - a\sec t)$$

$$ya\tan t - ab\tan^2 t = b\sec tx - ab\sec^2 t$$

$$ab = bx\sec t - ay\tan t$$

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Further coordinate systems
Exercise D, Question 4

Question:

Show that an equation of the normal to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\cosh t, b\sinh t)$ is $b\cosh ty + a\sinh tx = (a^2 + b^2)\sinh t\cosh t$.

Solution:

$x = a\cosh t \quad y = b\sinh t$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b\cosh t}{a\sinh t}$

$\therefore$ gradient of normal $= -\frac{a\sinh t}{b\cosh t}$

$\therefore$ Equation of normal is:

$$y - b\sinh t = -\frac{a\sinh t}{b\cosh t}(x - a\cosh t)$$

$$yb\cosh t - b^2\sinh t\cosh t = -a\sinh tx + a^2\cosh t\sinh t$$

$$b\cosh ty + a\sinh tx = (a^2 + b^2)\cosh t\sinh t$$

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Further coordinate systems
Exercise D, Question 5

Question:

The point $P(4\cosh t, 3\sinh t)$ lies on the hyperbola with equation $\frac{x^2}{16}-\frac{y^2}{9}=1$.

The tangent at P crosses the y-axis at the point A.

a Find, in terms of t, the coordinates of A.

The normal to the hyperbola at P crosses the y-axis at B.

b Find, in terms of t, the coordinates of B.

c Find, in terms of t, the area of triangle APB.

Solution:

$x=4\cosh t \quad y=3\sinh t \Rightarrow \frac{dy}{dx}=\frac{3\cosh t}{4\sinh t}$

$\therefore$ Equation of tangent is: $y-3\sinh t=\frac{3\cosh t}{4\sinh t}(x-4\cosh t)$

a $x=0 \Rightarrow y=3\sinh t-\frac{3\cosh^2 t}{\sinh t}=-\frac{3}{\sinh t}$

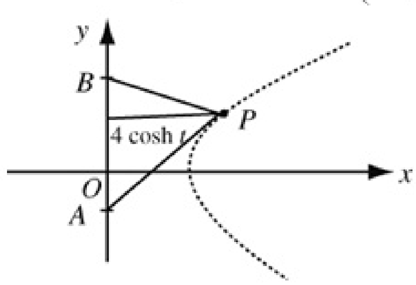
$\therefore A$ is $\left(0,-\frac{3}{\sinh t}\right)$

b Using question 4 with $a=4, b=3$

Normal has equation: $3y\cosh t+4x\sinh t=25\sinh t\cosh t$

$x=0 \Rightarrow y=\frac{25}{3}\sinh t \quad \therefore B$ is $\left(0,\frac{25}{3}\sinh t\right)$

c



Area of $\Delta=\frac{1}{2}\left(\frac{25}{3}\sinh t--\frac{3}{\sinh t}\right)4\cosh t$

$=\frac{2}{3}(25\sinh^2 t+9)\coth t$

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Further coordinate systems
Exercise D, Question 6

Question:

The tangents from the points P and Q on the hyperbola with equation $\frac{x^2}{4}-\frac{y^2}{9}=1$ meet at the point (1, 0).
Find the exact coordinates of P and Q.

Solution:

$\frac{x^2}{4}-\frac{y^2}{9}=1 \quad x=2\sec t, a=2$

$y=3\tan t, b=3$

From question 3 the equation of the tangent is:
$3x\sec t-2y\tan t=6$

Tangents meet at (1, 0) so let $x=1, y=0$

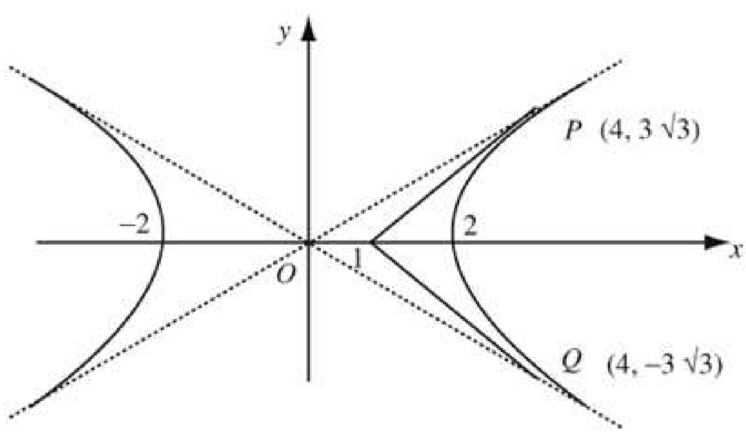
$\Rightarrow \quad 3\sec t=6$

i.e. $\frac{1}{2}=\cos t$

$\therefore \quad t=\pm\frac{\pi}{3}$

$\sec\left(\pm\frac{\pi}{3}\right)=2, \quad \tan\left(\pm\frac{\pi}{3}\right)=\pm\sqrt{3}$

$\therefore$ P and Q are $(4, 3\sqrt{3})$ and $(4, -3\sqrt{3})$



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Further coordinate systems
Exercise D, Question 7

Question:

The line $y = 2x + c$ is a tangent to the hyperbola $\frac{x^2}{10} - \frac{y^2}{4} = 1$. Find the possible values of c.

Solution:

Using the result $y = mx + c$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for $b^2 + c^2 = a^2m^2$

$y = 2x + c \quad \therefore m = 2$

$\frac{x^2}{10} - \frac{y^2}{4} = 1 \quad \therefore a^2 = 10, b^2 = 4$

$$\therefore 4 + c^2 = 2^2 \times 10 = 40$$

$$c^2 = 36$$

$$c = \pm 6$$

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Further coordinate systems
Exercise D, Question 8

Question:

The line $y = mx + 12$ is a tangent to the hyperbola $\frac{x^2}{49} - \frac{y^2}{25} = 1$ at the point P.

Find the possible values of m.

Solution:

Use $b^2 + c^2 = a^2m^2$ for $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$y = mx + 12 \Rightarrow c = 12$

$\frac{x^2}{49} - \frac{y^2}{25} = 1 \Rightarrow a^2 = 49, b^2 = 25$

$$\therefore 25 + 12^2 = 49m^2$$

$$169 = 49m^2$$

$$\therefore m^2 = \left(\frac{13}{7}\right)^2$$

$$\therefore m = \pm\frac{13}{7}$$

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Further coordinate systems
Exercise D, Question 9

Question:

The line $y = -x + c, c > 0$, touches the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ at the point P.

a Find the value of c.
b Find the exact coordinates of P.

Solution:

a $m = -1, a = 5, b = 4$

$\therefore \quad 16 + c^2 = 25(-1)^2$

i.e. $c^2 = 9$

$\therefore c = \pm 3 \qquad \because c > 0 \therefore c = 3$

Use $b^2 + c^2 = a^2m^2$ for $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

b $y = (3 - x)$, substitute into hyperbola

$$\frac{x^2}{25} - \frac{(3-x)^2}{16} = 1$$

$$16x^2 - 25(9 + x^2 - 6x) = 25 \times 16$$

$$-9x^2 - 225 + 150x = 400$$

$$0 = 9x^2 - 150x + 625$$

$$0 = (3x - 25)^2$$

$$\therefore x = \frac{25}{3}, y = -\frac{16}{3}$$

So P is $\left(\frac{25}{3}, \frac{-16}{3}\right)$

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Further coordinate systems
Exercise D, Question 10

Question:

The line with equation $y = mx + c$ is a tangent to both hyperbolae $\frac{x^2}{4} - \frac{y^2}{15} = 1$ and $\frac{x^2}{9} - \frac{y^2}{95} = 1$.

Find the possible values of m and c.

Solution:

Use $b^2 + c^2 = a^2m^2$ for $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Using $\frac{x^2}{4} - \frac{y^2}{15} = 1 \Rightarrow a^2 = 4, b^2 = 15$

$\therefore 15 + c^2 = 4m^2$ ①

Using $\frac{x^2}{9} - \frac{y^2}{95} = 1 \Rightarrow a^2 = 9, b^2 = 95$

$\therefore 95 + c^2 = 9m^2$ ②

Solving

② − ① $80 = 5m^2$

$\therefore m^2 = 16$

$m = \pm 4$

$m = \pm 4$ $c^2 = 4(16) - 15$

$= 49$ $\therefore c = \pm 7$

$\therefore m = \pm 4$ and $c = \pm 7$

i.e. lines $y = 4x \pm 7$ and $y = -4x \pm 7$

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Further coordinate systems
Exercise E, Question 1

Question:

Find the eccentricity of the following ellipses.

a $\frac{x^2}{9}+\frac{y^2}{5}=1$

b $\frac{x^2}{16}+\frac{y^2}{9}=1$

c $\frac{x^2}{4}+\frac{y^2}{8}=1$

Solution:

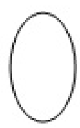
a $a^2=9 \quad b^2=5$

$b^2=a^2(1-e^2) \Rightarrow \frac{5}{9}=1-e^2 \;\therefore e^2=\frac{4}{9} \;\therefore e=\frac{2}{3}$

b $a^2=16 \quad b^2=9$

$b^2=a^2(1-e^2) \Rightarrow \frac{9}{16}=1-e^2 \;\therefore e^2=\frac{7}{16} \;\therefore e=\frac{\sqrt{7}}{4}$

c $a^2=4 \quad b^2=8$

Need to use $a^2=b^2(1-e^2)$ since ellipse is  shape.

$\frac{4}{8}=1-e^2 \Rightarrow e^2=\frac{1}{2} \therefore e=\frac{1}{\sqrt{2}}$

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Further coordinate systems
Exercise E, Question 2

Question:

Find the foci and directrices of the following ellipses.

a $\frac{x^2}{4}+\frac{y^2}{3}=1$

b $\frac{x^2}{16}+\frac{y^2}{7}=1$

c $\frac{x^2}{5}+\frac{y^2}{9}=1$

Solution:

a $a^2=4 \quad b^2=3$

$b^2=a^2(1-e^2) \Rightarrow \frac{3}{4}=1-e^2 \;\therefore e^2=\frac{1}{4} \;\therefore e=\frac{1}{2}$

Focus $(\pm ae,0)=(\pm 1,0)$; directrix $x=\pm\frac{a}{e} \Rightarrow x=\pm 4$

b $a^2=16 \quad b^2=7$

$b^2=a^2(1-e^2) \Rightarrow \frac{7}{16}=1-e^2 \;\therefore e^2=\frac{9}{16} \;\therefore e=\frac{3}{4}$

Focus $(\pm ae,0)=(\pm 3,0)$; directrix $x=\pm\frac{a}{e} \Rightarrow x=\pm\frac{16}{3}$

c $a^2=5, b^2=9$

Since $b>a$

Use $a^2=b^2(1-e^2) \Rightarrow \frac{5}{9}=1-e^2$

$\therefore e^2=\frac{4}{9} \therefore e=\frac{2}{3}$

Focus is $(0,\pm be)$ i.e. focus $(0,\pm 2)$

Directrix $y=\pm\frac{b}{e}$ i.e. $y=\pm\frac{9}{2}$

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Further coordinate systems
Exercise E, Question 3

Question:

An ellipse E has focus (3, 0) and the equation of the directrix is $x = 12$. Find **a** the value of the eccentricity **b** the equation of the ellipse.

Solution:

a $ae = 3 \quad \frac{a}{e} = 12$

$\Rightarrow ae \times \frac{a}{e} = a^2 = 36$

$\Rightarrow \quad a = 6, e = \frac{1}{2}$

b $b^2 = a^2(1 - e^2)$

$= 36\left(1 - \frac{1}{4}\right) = 36 \times \frac{3}{4} = 27$

$\therefore$ equation is $\frac{x^2}{36} + \frac{y^2}{27} = 1$

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Further coordinate systems
Exercise E, Question 4

Question:

An ellipse E has focus (2, 0) and the directrix has equation $x = 8$. Find **a** the value of the eccentricity **b** the equation of the ellipse.

Solution:

$ae = 2 \quad \frac{a}{e} = 8$

a $\Rightarrow \quad ae \times \frac{a}{e} = a^2 = 16$

$\Rightarrow \quad a = 4, e = \frac{1}{2}$

b $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = 16\left(1 - \frac{1}{4}\right) = 16 \times \frac{3}{4} = 12$

$\therefore$ equation is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

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Further coordinate systems
Exercise E, Question 5

Question:

Find the eccentricity of the following hyperbolae.

a $\frac{x^2}{5}-\frac{y^2}{3}=1$

b $\frac{x^2}{9}-\frac{y^2}{7}=1$

c $\frac{x^2}{9}-\frac{y^2}{16}=1$

Solution:

a $\frac{x^2}{5}-\frac{y^2}{3}=1 \Rightarrow a^2=5, b^2=3$

$b^2=a^2(e^2-1) \Rightarrow \frac{3}{5}=e^2-1 \therefore e^2=\frac{8}{5} \therefore e=\frac{2\sqrt{2}}{\sqrt{5}}=\frac{2\sqrt{10}}{5}$

b $\frac{x^2}{9}-\frac{y^2}{7}=1 \Rightarrow a^2=9, b^2=7$

$b^2=a^2(e^2-1) \Rightarrow \frac{7}{9}=e^2-1 \therefore e^2=\frac{16}{9} \therefore e=\frac{4}{3}$

c $\frac{x^2}{9}-\frac{y^2}{16}=1 \Rightarrow a^2=9, b^2=16$

$b^2=a^2(e^2-1) \Rightarrow \frac{16}{9}=e^2-1 \therefore e^2=\frac{25}{9} \therefore e=\frac{5}{3}$

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Further coordinate systems
Exercise E, Question 6

Question:

Find the foci of the following hyperbolae and sketch them, showing clearly the equations of the asymptotes.

a $\frac{x^2}{4} - \frac{y^2}{8} = 1$

b $\frac{x^2}{16} - \frac{y^2}{9} = 1$

c $\frac{x^2}{4} - \frac{y^2}{5} = 1$

Solution:

a $\frac{x^2}{4} - \frac{y^2}{8} = 1$

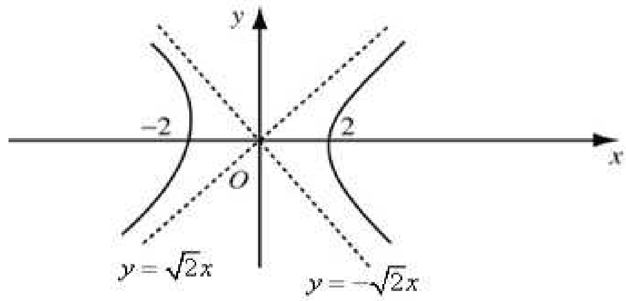
$a = 2, b = 2\sqrt{2}$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{8}{4} = e^2 - 1$

$\Rightarrow e = \sqrt{3}$

so foci are $(\pm 2\sqrt{3}, 0$

Asymptotes are $y = \pm\sqrt{2}x$



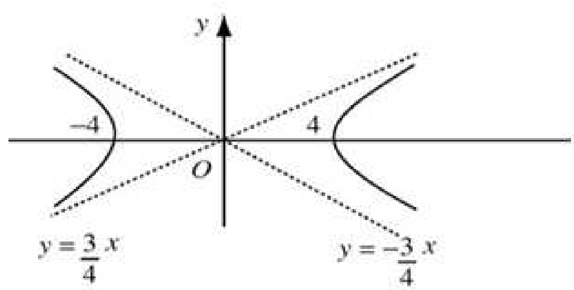
b $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4, b = 3$

$\Rightarrow 9 = 16(e^2 - 1) \Rightarrow e^2 = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

so foci are $(\pm\frac{5}{2}, 0)$

Asymptotes $y = \pm\frac{3}{4}x$



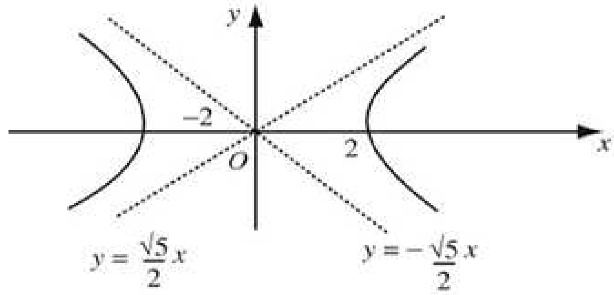
c $\frac{x^2}{4} - \frac{y^2}{5} = 1$

$a = 2, b = \sqrt{5}$

$\Rightarrow 5 = 4(e^2 - 1) \Rightarrow e^2 = \frac{9}{4} \Rightarrow e = \frac{3}{2}$

so foci are $(\pm 3, 0)$

Asymptotes $y = \pm\frac{\sqrt{5}}{2}x$



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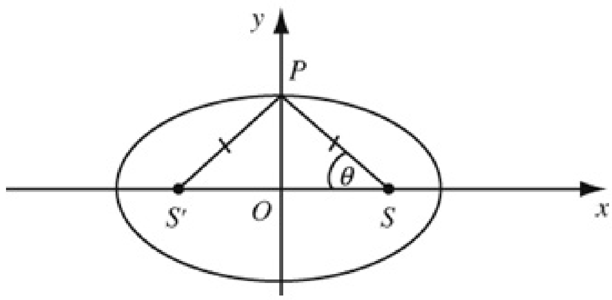
Further coordinate systems
Exercise E, Question 7

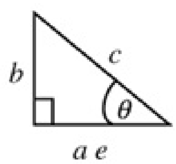
Question:

Ellipse E has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The foci are at S and S' and the point P is $(0, b)$.
Show that $\cos(PSS') = e$, the eccentricity of E.

Solution:



Consider ΔPOS



$c^2 = b^2 + a^2e^2$, but $b^2 = a^2(1 - e^2)$

$\therefore c^2 = a^2 - a^2e^2 + a^2e^2 = a^2$

$\therefore c = a$

So $\cos\theta = \frac{ae}{a} = e$

If you use the result that $SP + S'P = 2a$ then since $S'P = SP$ it is clear $SP = a$
Hence $\cos\theta = \frac{ae}{a} = e$.

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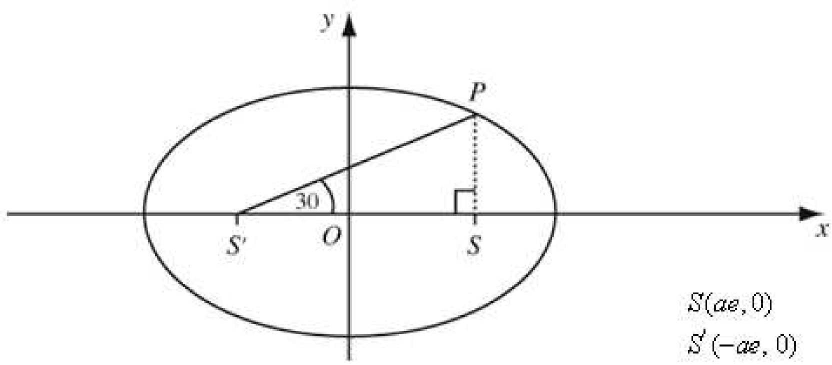
Further coordinate systems
Exercise E, Question 8

Question:

The ellipse E has foci at S and S'. The point P on E is such that angle PSS' is a right angle and angle $PS'S = 30°$.

Show that the eccentricity of the ellipse, e, is $\frac{1}{\sqrt{3}}$.

Solution:



$S(ae, 0)$
$S'(-ae, 0)$

PS is y where $\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$

$$y^2 = b^2(1 - e^2)$$
$$y = b\sqrt{1 - e^2}$$
$$SS' = 2ae$$

$$\tan 30 = \frac{1}{\sqrt{3}} = \frac{y}{2ae} = \frac{b\sqrt{1-e^2}}{2ae}$$

But $b^2 = a^2(1 - e^2)$

$$\therefore \quad \frac{1}{\sqrt{3}} = \frac{a\sqrt{1-e^2}\sqrt{1-e^2}}{2ae}$$
$$\frac{2e}{\sqrt{3}} = 1 - e^2$$

$$e^2 + \frac{2}{\sqrt{3}}e - 1 = 0$$

$$\Rightarrow e^2 + \frac{2}{\sqrt{3}}e + \frac{1}{3} = 1 + \frac{1}{3}$$

$$\Rightarrow \left(e + \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\therefore e + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \quad \therefore e = \frac{1}{\sqrt{3}}$$

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Further coordinate systems
Exercise F, Question 1

Question:

The tangent at $P(ap^2, 2ap)$ and the tangent at $Q(aq^2, 2aq)$ to the parabola with equation $y^2 = 4ax$ meet at R.

a Find the coordinates of R.

The chord PQ passes through the focus $(a, 0)$ of the parabola.

b Show that the locus of R is the line $x = -a$.

Given instead that the chord PQ has gradient 2,

c find the locus of R.

Solution:

a Using table in Section 2.6

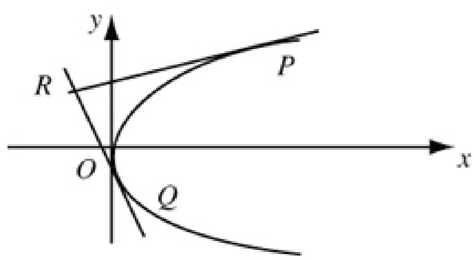
Tangent at P is $py = x + ap^2$

Tangent at Q is $qy = x + aq^2$

$(p-q)y = a(p-q)(p+q) \quad \therefore y = a(p+q)$

$\Rightarrow ap^2 + apq = x + ap^2 \quad \therefore x = apq$

So R is $(apq, a(p+q))$



b Chord PQ has gradient: $\dfrac{2ap-2aq}{ap^2-aq^2} = \dfrac{2a(p-q)}{a(p-q)(p+q)} = \dfrac{2}{(p+q)}$

$\therefore$ Equation of chord PQ is: $y - 2ap = \dfrac{2}{p+q}(x - ap^2)$

i.e. $y(p+q) - \cancel{2ap^2} - 2apq = 2x - \cancel{2ap^2}$

i.e. $y(p+q) = 2x + 2apq$

Chord passes through $(a,0) \Rightarrow 0 = 2a + 2apq$ or $pq = -1$

$\therefore$ locus of R is $x = -a$

c Gradient of chord PQ is $\dfrac{2}{p+q} = 2 \Rightarrow p+q = 1$

$\therefore$ locus of R is: $y = a(p+q) = a$

i.e. $y = a$

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Further coordinate systems
Exercise F, Question 2

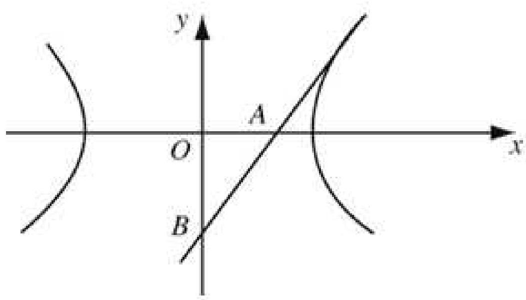
Question:

The tangent at $P(a\sec t, b\tan t)$ to the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the x-axis at A and the y-axis at B.
Find the locus of the mid-point of AB.

Solution:

Equation of tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec t, b\tan t)$ is: $bx\sec t - ay\tan t = ab$

See summary



A is where $y=0 \Rightarrow x=\frac{ab}{b\sec t}=a\cos t$

i.e. $A\,(a\cos t, 0)$

B is where $x=0 \Rightarrow y=\frac{ab}{-a\tan t}=-b\cot t$

i.e. $B(0, -b\cot t)$

Mid-point of AB is $\left(\frac{a}{2}\cos t, -\frac{b}{2}\cot t\right)$

$x = \frac{a}{2}\cos t \Rightarrow \sec t = \frac{a}{2x}$

$y = -\frac{b}{2}\cot t \Rightarrow \tan t = -\frac{b}{2y}$

Use $\sec^2 t = 1+\tan^2 t$

$\Rightarrow \frac{a^2}{4x^2}=1+\frac{b^2}{4y^2}$ which gives locus.

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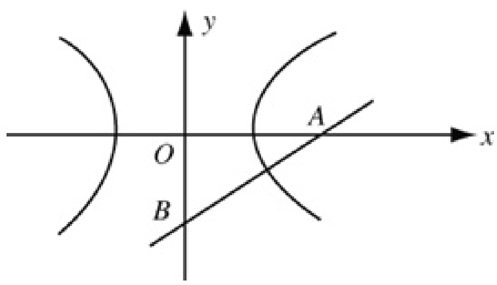
Further coordinate systems
Exercise F, Question 3

Question:

The normal at $P(a\sec t, b\tan t)$ to the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the x-axis at A and the y-axis at B.
Find the locus of the mid-point of AB.

Solution:

Normal to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec t, b\tan t)$ is $ax\sin t+by=(a^2+b^2)\tan t$



$y=0 \Rightarrow x=\left(\frac{a^2+b^2}{a}\right)\sec t \quad \therefore A \text{ is } \left(\frac{\left[a^2+b^2\right]}{a}\sec t, 0\right)$

$x=0 \Rightarrow y=\left(\frac{a^2+b^2}{b}\right)\tan t \quad \therefore B \text{ is } \left(0, \frac{\left[a^2+b^2\right]}{b}\tan t\right)$

Mid-point of AB is $\left(\frac{(a^2+b^2)}{2a}\sec t, \frac{(a^2+b^2)}{2b}\tan t\right)$

$x=\frac{(a^2+b^2)}{2a}\sec t \Rightarrow \sec t=\frac{2ax}{a^2+b^2}$

$y=\frac{(a^2+b^2)}{2b}\tan t \Rightarrow \tan t=\frac{2by}{a^2+b^2}$

Use $\sec^2 t=1+\tan^2 t$

$\therefore 4a^2x^2=(a^2+b^2)^2+4b^2y^2$

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Further coordinate systems
Exercise F, Question 4

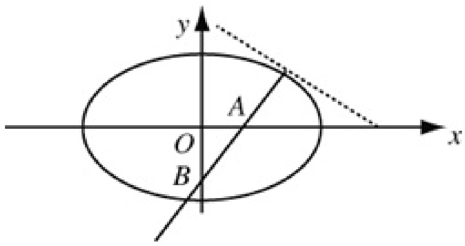
Question:

The normal at $P(a\cos t,\ b\sin t)$ to the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts the x-axis at A and the y-axis at B.
Find the locus of the mid-point of AB.

Solution:

Normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos t,\ b\sin t)$ is

$ax\sin t-by\cos t=(a^2-b^2)\cos t\sin t$



$y=\ 0\Rightarrow x=\left(\frac{a^2-b^2}{a}\right)\cos t\quad \therefore A \text{ is } \left(\frac{\left[a^2-b^2\right]}{a}\cos t, 0\right)$

$x=0\Rightarrow y=-\left(\frac{a^2-b^2}{b}\right)\sin t\quad \therefore B \text{ is } \left(0,-\frac{\left(a^2-b^2\right)}{b}\sin t\right)$

Mid-point of AB is $\left(\frac{\left[a^2-b^2\right]}{2a}\cos t,-\frac{\left[a^2-b^2\right]}{2b}\sin t\right)$

$x=\ \frac{\left(a^2-b^2\right)}{2a}\cos t\Rightarrow\cos t=\frac{2ax}{a^2-b^2}$

$y=-\frac{\left(a^2-b^2\right)}{2b}\sin t\Rightarrow\sin t=-\frac{2by}{a^2-b^2}$

Use $\sin^2 t+\cos^2 t=1$

$\therefore 4b^2y^2+4a^2x^2=\left(a^2-b^2\right)^2$

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Further coordinate systems
Exercise F, Question 5

Question:

The tangent from the point $P\left(cp, \frac{c}{p}\right)$ and the tangent from the point $Q\left(cq, \frac{c}{q}\right)$ to the rectangular hyperbola $xy = c^2$, intersect at the point R.

a Show that R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

b Show that the chord PQ has equation $ypq + x = c(p+q)$

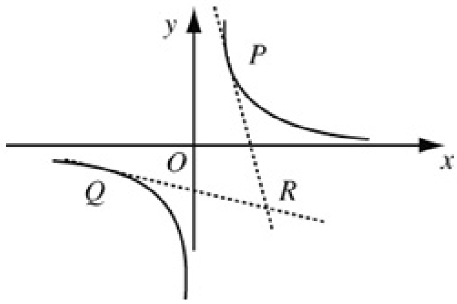
c Find the locus of R in the following cases

i when the chord PQ has gradient 2

ii when the chord PQ passes through the point (1, 0)

iii when the chord PQ passes through the point (0, 1).

Solution:



From table in Section 2.6 the equation of tangent at P is: $x + p^2 y = 2cp$

a Similarly the equation of tangent at Q is: $x + q^2 y = 2cq$

Solving: $\cancel{(p-q)}(p+q)y = 2c\,\cancel{(p-q)}$ $\quad \therefore y = \dfrac{2c}{p+q},\ x = \dfrac{2cpq}{p+q}$

$\therefore R$ is $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$

b Gradient of chord PQ is: $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

$\therefore$ Equation of chord is: $y - \dfrac{c}{p} = -\dfrac{1}{pq}(x - cp)$ i.e. $ypq + x = c(p+q)$

c **i** $-\dfrac{1}{pq} = 2 \therefore pq = -\dfrac{1}{2}$.

R is: $x = -\dfrac{c}{p+q}, y = \dfrac{2c}{p+q} \Rightarrow y = -2x$

ii Chord through $(1, 0) \Rightarrow 1 = c(p+q)$

R is $x = \dfrac{2cpq}{\frac{1}{c}}, y = \dfrac{2c}{\frac{1}{c}} \Rightarrow y = 2c^2$

iii Chord through $(0, 1) \Rightarrow pq = c(p+q)$

R is $x = \dfrac{2c^2(p+q)}{(p+q)} \Rightarrow x = 2c^2$

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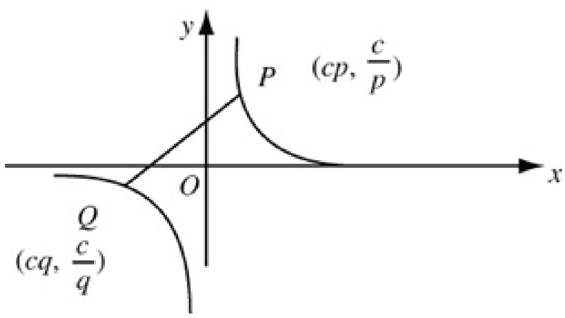
Further coordinate systems
Exercise F, Question 6

Question:

The chord PQ to the rectangular hyperbola $xy = c^2$ passes through the point (0, 1). Find the locus of the mid-point of PQ as P and Q vary.

Solution:



Gradient of chord: $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

Equation of chord: $y - \dfrac{c}{p} = -\dfrac{1}{pq}(x - cp)$

$$ypq - cq = -x + cp$$

$$\therefore \quad ypq + x = c(p+q)$$

Mid-point of chord is $\left(\dfrac{c(p+q)}{2}, \dfrac{c(p+q)}{2pq}\right)$

Chord passes through $(0, 1) \Rightarrow pq = c(p+q)$

Mid-point is: $x = \dfrac{c(p+q)}{2}$

$$y = \frac{c(p+q)}{2pq}$$

Substitute $pq = c(p+q) \Rightarrow y = \dfrac{c(p+q)}{2c(p+q)} = \dfrac{1}{2}$

$\therefore$ locus is line $y = \dfrac{1}{2}$

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Further coordinate systems
Exercise G, Question 1

Question:

A hyperbola of the form $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ has asymptotes with equations $y = \pm mx$ and passes through the point $(a, 0)$.

a Find an equation of the hyperbola in terms of x, y, a and m.

A point P on this hyperbola is equidistant from one of its asymptotes and the x-axis.

b Prove that, for all values of m, P lies on the curve with equation

$(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$ **[E]**

Solution:

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$$

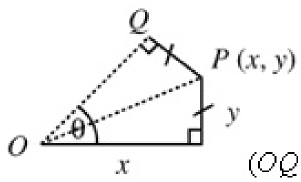
a Asymptotes are $y=\pm\frac{\beta}{\alpha}x$ $\quad\therefore m=\frac{\beta}{\alpha}$

Passes through $(a,0)\Rightarrow \frac{a^2}{\alpha^2}-0=1$ $\quad\therefore a=\alpha$

$\therefore \beta=am$

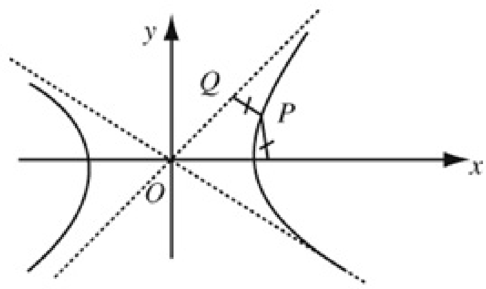
$\therefore$ Equation is $\frac{x^2}{a^2}-\frac{y^2}{a^2m^2}=1$

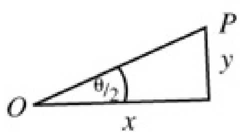
b



$(OQ=x)$

$m=\tan\theta$





$\tan\frac{\theta}{2}=\frac{y}{x}$

Using $\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\Rightarrow m=\frac{2\frac{y}{x}}{1-\frac{y^2}{x^2}}=\frac{2xy}{(x^2-y^2)}$ ①

But P lies on the hyperbola $\therefore x^2m^2-y^2=a^2m^2$

So $m^2=\frac{y^2}{x^2-a^2}$ ②

Using ①^2 and ② $\frac{4x^2y^2}{\left(x^2-y^2\right)^2}=\frac{y^2}{x^2-a^2}$

i.e. $4x^2\left(x^2-a^2\right)=\left(x^2-y^2\right)^2$

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Further coordinate systems
Exercise G, Question 2

Question:

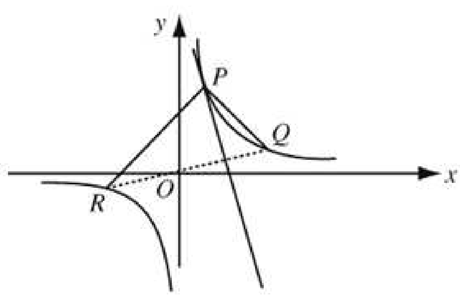
a Prove that the gradient of the chord joining the point $P\left(cp, \frac{c}{p}\right)$ and the point $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola with equation $xy = c^2$ is $-\frac{1}{pq}$.

The points P, Q and R lie on a rectangular hyperbola, the angle QPR being a right angle.

b Prove that the angle between QR and the tangent at P is also a right angle. **[E]**

Solution:

a Gradient of chord $= \dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{\not{c}(q-p)}{pq\not{c}(p-q)} = \dfrac{-1}{pq}$



$P\left(cp, \frac{c}{p}\right)$; $Q\left(cq, \frac{c}{q}\right)$; $R\left(cr, \frac{c}{r}\right)$

b Gradient of $PQ = -\frac{1}{pq}$

Gradient of $PR = -\frac{1}{pr}$

$\therefore$ If $Q\hat{P}R = 90° \Rightarrow -\frac{1}{pq} \times -\frac{1}{pr} = -1$

$\Rightarrow -1 = p^2qr$ ①

To find gradient of tangent at P let $q \to p$ for chord PQ

$\therefore$ Gradient of tangent at P is $-\frac{1}{p^2}$

Gradient of chord $RQ = -\frac{1}{qr}$.

So $\frac{-1}{qr} \times -\frac{1}{p^2} = \frac{1}{p^2qr}$

But from ① $p^2qr = -1 \therefore$ gradient of tangent at P $\times$ gradient of $QR = -1$.

Therefore tangent at P is perpendicular to chord QR.

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Further coordinate systems
Exercise G, Question 3

Question:

a Show that an equation of the tangent to the rectangular hyperbola with equation $xy = c^2$ (with $c > 0$) at the point $\left(ct, \frac{c}{t}\right)$ is $t^2y + x = 2ct$

Tangents are drawn from the point $(-3, 3)$ to the rectangular hyperbola with equation $xy = 16$.

b Find the coordinates of the points of contact of these tangents with the hyperbola. **[E]**

Solution:

a $y = ct^{-1}, x = ct \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$

$\therefore$ Equation of tangent is: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

i.e. $yt^2 - ct = -x + ct$

or $t^2y + x = 2ct$

b Let $S\left(cs, \frac{c}{s}\right)$ be another point an $xy = 16 (c = 4)$

$\therefore$ tangent at S is $s^2y + x = 2cs$

Intersection of tangents is: $(t^2 - s^2)y = 2c(t - s)$

$$y = \frac{2c}{t+s}$$

$$\therefore x = 2ct - \frac{2ct^2}{t+s} = \frac{2cts}{t+s}$$

So when $c = 4$ intersection is $\left(\frac{8ts}{t+s}, \frac{8}{t+s}\right)$

Now $x = -3, y = 3 \Rightarrow \left.\begin{array}{rl} 3(t+s) &= 8 \\ -3(t+s) &= 8ts \end{array}\right\} \Rightarrow ts = -1$

$$t = -\frac{1}{s}$$

$\therefore 3\left(s - \frac{1}{s}\right) = 8$

$\Rightarrow 3s^2 - 8s - 3 = 0$

$(3s+1)(s-3) = 0$

$\therefore s = 3$ or $-\frac{1}{3}$

$t = -\frac{1}{3}$ or 3

So points are $\left(-\frac{4}{3}, -12\right)$ and $\left(12, \frac{4}{3}\right)$

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Further coordinate systems
Exercise G, Question 4

Question:

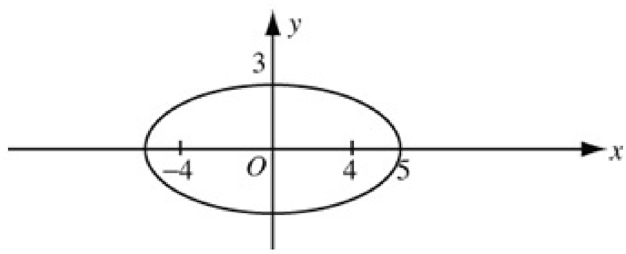
The point P lies on the ellipse with equation $9x^2 + 25y^2 = 225$, and A and B are the points $(-4, 0)$ and $(4, 0)$ respectively.

a Prove that $PA + PB = 10$

b Prove also that the normal at P bisects the angle APB. **[E]**

Solution:

a $9x^2 + 25y^2 = 225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$



$\therefore a = 5, b = 3$

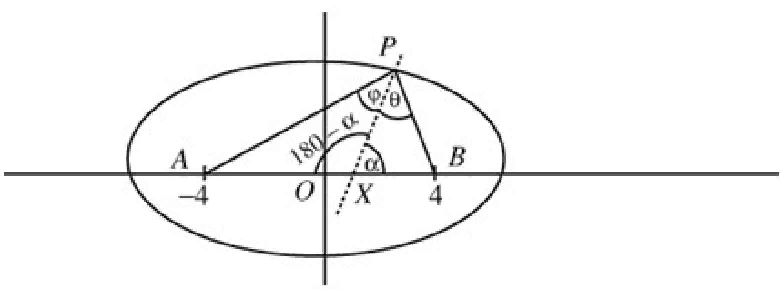
$b^2 = a^2(1 - e^2) \Rightarrow 9 = 25(1 - e^2) \quad \therefore e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

$\therefore$ Foci are $(\pm 4, 0)$ So A and B are the foci.

Since $PS + PS' = 2a$

$\therefore PA + PB = 2 \times 5 = 10$

b



Normal at P is: $5x \sin t - 3y \cos t = 16 \cos t \sin t$

$\therefore X$ is when $y = 0$ i.e. $\frac{16}{5} \cos t$

$PB^2 = (5 \cos t - 4)^2 + (3 \sin t)^2 = 25 \cos^2 t - 40 \cos t + 16 + 9 \sin^2 t$

$= 16 \cos^2 t - 40 \cos t + 25 = (4 \cos t - 5)^2$

$\therefore PB = 5 - 4 \cos t$

$\therefore PA = 10 - PB = 5 + 4 \cos t$

$AX = 4 + \frac{16}{5} \cos t, BX = 4 - \frac{16}{5} \cos t$

Consider sine rule on ΔPAX.

$$\sin\phi = \frac{\sin(180-\alpha)AX}{AP} = \frac{\sin\alpha\left(4+\frac{16}{5}\cos t\right)}{5+4\cos t}$$
$$= \frac{\sin\alpha 4(5+4\cos t)}{5(5+4\cos t)}$$
$$= \frac{4}{5}\sin\alpha$$

Consider sine rule on ΔPBX

$$\sin\theta = \frac{BX\sin\alpha}{PB} = \frac{\sin\alpha\left(4-\frac{16}{5}\cos t\right)}{5-4\cos t}$$
$$= \frac{\sin\alpha 4(5-4\cos t)}{5(5-4\cos t)}$$
$$= \frac{4}{5}\sin\alpha$$

$\therefore \sin\phi = \sin\theta$ and since both $< 90°$ $\theta = \phi$

$\therefore$ Normal bisects APB.

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Further coordinate systems
Exercise G, Question 5

Question:

A curve is given parametrically by $x = ct, y = \frac{c}{t}$.

Show that an equation of the tangent to the curve at the point $\left(ct, \frac{c}{t}\right)$ is $t^2y + x = 2ct$

The point P is the foot of the perpendicular from the origin to this tangent.

b Show that the locus of P is the curve with equation $\left(x^2 + y^2\right)^2 = 4c^2xy$ **[E]**

Solution:

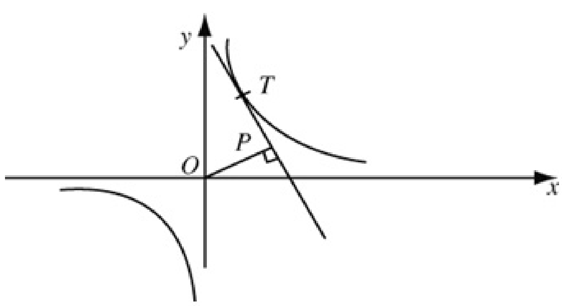
a $y = ct^{-1}, x = ct \quad \therefore \frac{dy}{dx} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$

$\therefore$ Equation of tangent is: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

i.e. $yt^2 - ct = -x + ct$

or $t^2y + x = 2ct$

b



Gradient of tangent is $-\frac{1}{t^2}$

$\therefore$ Gradient of OP is t^2

$\therefore$ Equation of OP is $y = t^2x$

Equation of tangent is $t^2y = 2ct - x$

Solving $t^4x = 2ct - x$

$$\therefore \quad x = \frac{2ct}{1+t^4}, y = \frac{2ct^3}{1+t^4}$$

$$x^2 + y^2 = \frac{4c^2t^2 + 4c^2t^6}{(1+t^4)^2} = \frac{4c^2t^2\,\cancel{(1+t^4)}}{(1+t^4)^{\cancel{2}}}$$

$$\left.\begin{aligned} \therefore (x^2+y^2)^2 &= \frac{16c^4t^4}{(1+t^4)^2} \\ xy &= \frac{4c^2t^4}{(1+t^4)^2} \end{aligned}\right\} \therefore (x^2+y^2)^2 = 4c^2xy$$

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Further coordinate systems
Exercise G, Question 6

Question:

a Find the gradient of the parabola with equation $y^2 = 4ax$ at the point $P(at^2, 2at)$.

b Hence show that the equation of the tangent at this point is $x - ty + at^2 = 0$.

The tangent meets the y-axis at T, and O is the origin.

c Show that the coordinates of the centre of the circle through O, P and T are

$$\left(\frac{at^2}{2} + a, \frac{at}{2}\right).$$

d Deduce that, as t varies, the locus of the centre of this circle is another parabola. **[E]**

Solution:

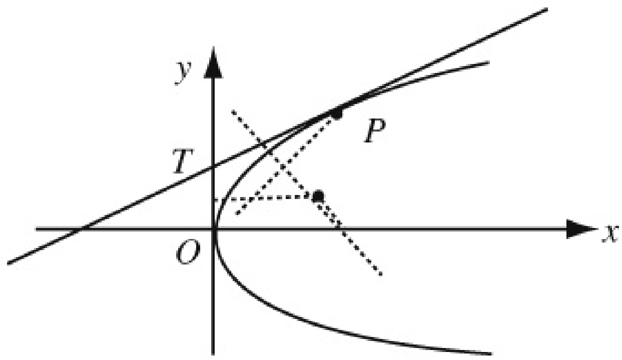
a $\left.\begin{array}{l} y = 2at \\ x = at^2 \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

b Equation of tangent is: $y - 2at = \frac{1}{t}(x - at^2)$

or $yt - 2at^2 = x - at^2$

i.e. $yt = x + at^2$

i.e. $x - yt + at^2 = 0$



T is $(0, at)$

c Centre of circle will be intersection of perpendicular bisectors of OT and OP.

Mid-point of OP is $\left(\frac{at^2}{2}, at\right)$

Gradient of $OP = \frac{2at}{at^2} = \frac{2}{t}$ $\therefore$ Equation of perpendicular bisector of OP is:

$y - at = -\frac{t}{2}\left(x - \frac{at^2}{2}\right)$

Intersects $y = \frac{at}{2}$. When $\frac{at}{2} = +\frac{t}{2}\left(x - \frac{at^2}{2}\right)$

$\therefore$ Centre of circle is $\left(a + \frac{at^2}{2}, \frac{at}{2}\right)$

d $X = a + \frac{at^2}{2} \Rightarrow at^2 = 2(X - a)$

$Y = \frac{at}{2} \Rightarrow 2at = 4Y$

$\therefore (4Y)^2 = 4a \times 2(X - a)$ or $2Y^2 = a(X - a)$

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Further coordinate systems
Exercise G, Question 7

Question:

The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola with equation $y^2 = 4ax$.

The angle $POQ = 90°$, where O is the origin.

a Prove that $pq = -4$

Given that the normal at P to the parabola has equation

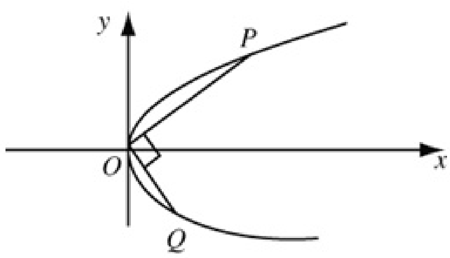
$y + xp = ap^3 + 2ap$

b write down an equation of the normal to the parabola at Q.

c Show that these two normals meet at the point R, with coordinates $(ap^2 + aq^2 - 2a, 4a[p+q])$

d Show that, as p and q vary, the locus of R has equation $y^2 = 16ax - 96a^2$. **[E]**

Solution:



a Gradient $OP = \frac{2ap}{ap^2} = \frac{2}{p}$, gradient of $OQ = \frac{2}{q}$

Since perpendicular $\frac{4}{pq} = -1 \quad \therefore pq = -4$

b Normal at Q is $y + xq = aq^3 + 2aq$

c Normal at P is $y + xp = ap^3 + 2ap$

Solving $x(q-p) = a(q^3 - p^3) + 2a(q-p)$

$x\cancel{(q-p)} = a\cancel{(q-p)}(q^2 + qp + p^2) + 2a\cancel{(q-p)}$

$x = a\left[q^2 + p^2 + qp + 2\right]$

$y = ap^3 + \cancel{2ap} - apq^2 - ap^3 - aqp^2 - \cancel{2ap}$ i.e. $y = -apq(q+p)$

But if $pq = -4$ R is $\left[aq^2 + ap^2 - 2a, 4a(p+q)\right]$

d $X = a((p+q)^2 - 2pq - 2) = a\left[(p+q)^2 + 6\right]$

$Y = 4a(p+q) \Rightarrow p + q = \frac{Y}{4a}$

$\therefore X = a\left[\frac{Y^2}{16a^2} + 6\right]$

$X - 6a = \frac{Y^2}{16a} \quad \therefore Y^2 = 16aX - 96a^2$

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Further coordinate systems
Exercise G, Question 8

Question:

Show that for all values of m, the straight lines with equations $y = mx \pm \sqrt{b^2 + a^2m^2}$ are tangents to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[E]**

Solution:

$y = mx + c$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2(mx+c)^2 = a^2b^2$

i.e. $b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$

i.e. $x^2(b^2 + a^2m^2) + 2a^2mcx + a^2(c^2 - b^2) = 0$

For a tangent the discriminant $= 0$

i.e. $4a^4m^2c^2 = 4(b^2 + a^2m^2)a^2(c^2 - b^2)$

i.e. $a^2m^2c^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$

i.e. $a^2m^2b^2 + b^4 = b^2c^2$

i.e. $c^2 = a^2m^2 + b^2$

$\therefore \quad c = \pm\sqrt{a^2m^2 + b^2}$

i.e. lines $y = mx \pm \sqrt{a^2m^2 + b^2}$ are tangents

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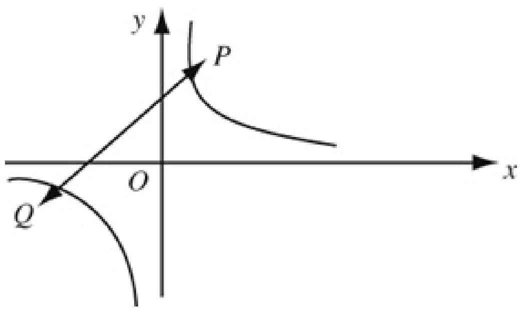
Further coordinate systems
Exercise G, Question 9

Question:

The chord PQ, where P and Q are points on $xy = c^2$, has gradient 1. Show that the locus of the point of intersection of the tangents from P and Q is the line $y = -x$.

Solution:



Chord PQ has gradient $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

If gradient $= 1$ then $pq = -1$

Tangent at P is $p^2y + x = 2cp$

Tangent at Q is $q^2y + x = 2cq$

Intersection $(p^2 - q^2)y = 2c(p-q) \Rightarrow y = \dfrac{2c}{p+q}$

$\therefore x = 2cp - \dfrac{2cp^2}{p+q} = \dfrac{2cpq}{p+q}$

So R is $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$

But $pq = -1 \therefore$ locus of R is $x = \dfrac{-2c}{p+q}$

$$y = \frac{2c}{p+q}$$

i.e. $y = -x$

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Further coordinate systems
Exercise G, Question 10

Question:

a Show that the asymptotes of the hyperbola H with equation $x^2 - y^2 = 1$ are perpendicular.

Using $(\sec t, \tan t)$ as a general point on H and the rotation matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b show that a rotation of 45° will transform H into a rectangular hyperbola with equation $xy = c^2$ and find the positive value of c.

Solution:

a Asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$

For $x^2-y^2=1, a^2=b^2=1$ $\quad\therefore$ Asymptotes are $y=\pm x$ i.e. perpendicular

b Let $\begin{pmatrix}\sec t\\ \tan t\end{pmatrix}$ be the position vector of a point on $x^2-y^2=1$

The matrix $R=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ represents rotation of $45°$ about (0, 0)

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sec t\\ \tan t\end{pmatrix}=\begin{pmatrix}\frac{\sec t}{\sqrt{2}}-\frac{\tan t}{\sqrt{2}}\\ \frac{\sec t}{\sqrt{2}}+\frac{\tan t}{\sqrt{2}}\end{pmatrix}$$

i.e. $X=\frac{1}{\sqrt{2}}(\sec t-\tan t)$

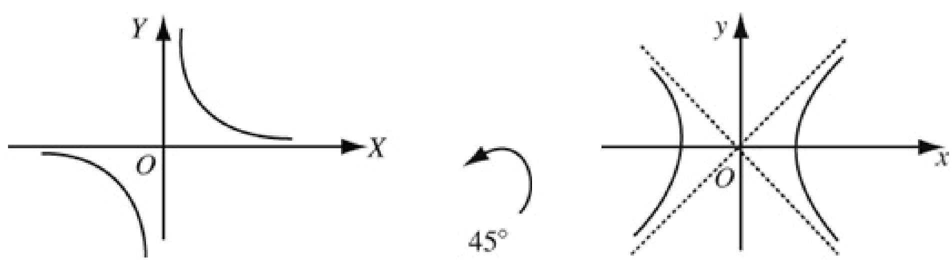
$Y=\frac{1}{\sqrt{2}}(\sec t+\tan t)$

$XY=\frac{1}{2}\left[(\sec t-\tan t)(\sec t+\tan t)\right]$

i.e. $XY=\frac{1}{2}(\sec^2 t-\tan^2 t)=\frac{1}{2}$

$\therefore$ the hyperbola $x^2-y^2=1$ when rotated by $45°$ gives the rectangular hyperbola

$XY=\frac{1}{2}$, $c=\frac{1}{\sqrt{2}}$



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Differentiation
Exercise A, Question 1

Question:

Differentiate with respect to x.
$\sinh 2x$

Solution:

$$\frac{d}{dx}(\sinh 2x) = 2\cosh 2x$$

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Differentiation
Exercise A, Question 2

Question:

Differentiate with respect to x.
$\cosh 5x$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh 5x) = \frac{-1}{(\cosh 2x)^2} \times 2\sinh 2x$$
$$= -2\frac{\sinh 2x}{\cos 2x} \times \frac{1}{\cos 2x}$$
$$= -2\tan 2x\ \mathrm{sech} 2\mathrm{x}$$

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Differentiation
Exercise A, Question 3

Question:

Differentiate with respect to x.
$\tanh 2x$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh 2x) = 2\operatorname{sech}^2 2x$$

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Differentiation
Exercise A, Question 4

Question:

Differentiate with respect to x.
$\sinh 3x$

Solution:

$$\frac{d}{dx}(\sinh 3x) = 3\cosh 3x$$

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Differentiation
Exercise A, Question 5

Question:

Differentiate with respect to x.
$\coth 4x$

Solution:

$$\frac{d}{dx}(\coth 4x) = -4\operatorname{cosech}^2 4x$$

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Differentiation
Exercise A, Question 6

Question:

Differentiate with respect to x.

$\operatorname{sech} 2x$

Hint: $\operatorname{sech} 2x = \dfrac{1}{\cosh 2x}$

Solution:

$$\begin{aligned}\frac{d}{dx}(\operatorname{sech} 2x) &= \frac{-1}{(\cosh 2x)^2} \times 2\sinh 2x \\ &= -2\frac{\sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x} \\ &= -2\tanh 2x \operatorname{sech} 2x\end{aligned}$$

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Differentiation
Exercise A, Question 7

Question:

Differentiate with respect to x.
$e^{-x}\sinh x$

Solution:

$$\frac{d}{dx}(e^{-x}\sinh x) = -e^{-x}\sinh x + e^{-x}\cosh x$$
$$= e^{-x}(\cosh x - \sinh x)$$

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Differentiation
Exercise A, Question 8

Question:

Differentiate with respect to x.
$x \cosh 3x$

Solution:

$$\frac{d}{dx}(x\cosh 3x) = \cosh 3x + 3x\sinh 3x$$

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Differentiation
Exercise A, Question 9

Question:

Differentiate with respect to x.

$$\frac{\sinh x}{3x}$$

Solution:

$$\frac{d}{dx}\left(\frac{\sinh x}{3x}\right) = \frac{\cosh x}{3x} - \frac{\sinh x}{3x^2}$$
$$= \frac{x\cosh x - \sinh x}{3x^2}$$

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Differentiation
Exercise A, Question 10

Question:

Differentiate with respect to x.
$x^2 \cosh 3x$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \cosh 3x\right) = 2x\cosh 3x + x^2 \times 3\sinh 3x$$
$$= x\left(2\cosh 3x + 3x\sinh 3x\right)$$

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Differentiation
Exercise A, Question 11

Question:

Differentiate with respect to x.
$\sinh 2x \cosh 3x$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh 2x \cosh 3x\right) = 2\cosh 2x \cosh 3x + \sinh 2x \times 3\sinh 3x$$
$$= 2\cosh 2x \cosh 3x + 3\sinh 2x \sinh 3x$$

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Differentiation

Exercise A, Question 12

Question:

Differentiate with respect to x.

$\ln(\cosh x)$

Solution:

$$\frac{d}{dx}(\ln \cosh x) = \frac{1}{\cosh x} \times \sinh x$$
$$= \tanh x$$

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Differentiation
Exercise A, Question 13

Question:

Differentiate with respect to x.
$\sinh x^3$

Solution:

$$\frac{d}{dx}\left(\sinh x^3\right) = 3x^2 \cosh x^3$$

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Differentiation
Exercise A, Question 14

Question:

Differentiate with respect to x.
$\cosh^2 2x$

Solution:

$$\frac{d}{dx}\left(\cosh^2 2x\right) = 2\cosh 2x 2\sinh 2x$$
$$= 4\cosh 2x \sinh 2x$$

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Differentiation
Exercise A, Question 15

Question:

Differentiate with respect to x.

$e^{\cosh x}$

Solution:

$$\frac{d}{dx}\left(e^{\cosh x}\right)=\sinh x e^{\cosh x}$$

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Differentiation
Exercise A, Question 16

Question:

Differentiate with respect to x.
$\operatorname{cosech} x$

Hint: $\operatorname{cosech} x = \frac{1}{\sinh x}$.

Solution:

$$\frac{d}{dx}(\operatorname{cosech} x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{0 - 1 \times \cosh x}{\sinh^2 x}$$
$$= -\coth x \operatorname{cosech} x$$

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Differentiation
Exercise A, Question 17

Question:

If $y = a\cosh nx + b\sinh nx$, where a and b are constants, prove that $\frac{d^2y}{dx^2} = n^2 y$.

Solution:

$y = a\cosh nx + b\sinh nx$

Differentiate with respect to x

$\frac{dy}{dx} = an\sinh nx + nb\cosh nx$

$\frac{d^2y}{dx^2} = an^2\cosh nx + bn^2\sinh nx$

$= n^2(a\cosh nx + b\sinh nx)$

$\frac{d^2y}{dx^2} = n^2 y$

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Differentiation
Exercise A, Question 18

Question:

Find the stationary values of the curve with equation $y = 12\cosh x - \sinh x$.

Solution:

$y = 12\cosh x - \sinh x$

$\frac{dy}{dx} = 12\sinh x - \cosh x$

At stationary values $\frac{dy}{dx} = 0$

$0 = 12\sinh x - \cosh x$

$\cosh x = 12\sinh x$

$\frac{1}{12} = \tanh x$

$x = \tanh^{-1}\frac{1}{12}$

$x = 0.0835$

The stationary value is therefore $y = 12\cosh 0.0835 - \sinh 0.0835$

$= 12.13$

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Differentiation
Exercise A, Question 19

Question:

Given that $y = \cosh 3x \sinh x$, find $\frac{d^2y}{dx^2}$.

Solution:

$$y = \cosh 3x \sinh x$$
$$\frac{dy}{dx} = 3\sinh 3x \sinh x + \cosh 3x \cosh x$$
$$\frac{d^2y}{dx^2} = 9\cosh 3x \sinh x + 3\sinh 3x \cosh x + 3\sinh 3x \cosh x + \cosh 3x \sinh x$$
$$= 10\cosh 3x \sinh x + 6\sinh 3x \cosh x$$
$$= 2(5\cosh 3x \sinh x + 3\sinh 3x \cosh x)$$

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Differentiation
Exercise A, Question 20

Question:

Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{256} - \frac{y^2}{16} = 1$ at the point $(16 \cosh q, 4 \sinh q)$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dq}}{\frac{dx}{dq}} = \frac{4\cosh q}{16 \sinh q} = \frac{\cosh q}{4 \sinh q}$$

Equation of tangent

$$y - 4\sinh q = \frac{\cosh q}{4 \sinh q}(x - 16\cosh q)$$

$$4y\sinh q - 16\sinh^2 q = x\cosh q - 16\cosh^2 q$$

$$4y\sinh q - x\cosh q = 16(\sinh^2 q - \cosh^2 q)$$

$$4y\sinh q - x\cosh q = -16$$

or $x\cosh q - 4y\sinh q = 16$

Equation of normal

$$y - 4\sinh q = \frac{-4\sinh q}{\cosh q}(x - 16\cosh q)$$

i.e. $y\cosh q - 4\sinh q\cosh q = -4x\sinh q + 64\sinh q\cosh q$

i.e. $y\cosh q + 4x\sinh q = 68\sinh q\cosh q$

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Differentiation
Exercise B, Question 1

Question:

Differentiate

a $\text{arcosh } 2x$
b $\text{arsinh}(x+1)$
c $\text{artanh } 3x$
d $\text{arsech } x$
e $\text{arcosh} x^2$
f $\text{arcosh } 3x$
g $x^2 \text{arcosh} x$
h $\text{arsinh } \frac{x}{2}$
i $e^{x^3} \text{arsinh} x$
j $\text{arsinh } x \text{ arcosh } x$
k $\text{arcosh } x \text{ sech } x$
l $x \text{ arcosh } 3x$

Solution:

a Let $y = \text{arcosh}\,2x$ then $\cosh y = 2x$

Differentiate with respect to x

$$\sinh y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sinh y}$$

$$= \frac{2}{\sqrt{\cosh^2 y - 1}} \text{ but } \cosh y = 2x$$

$$\text{so } \frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}$$

b Let $y = \text{arsinh}(x+1)$ then $\sinh y = x + 1$

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{\sinh^2 y + 1}} \text{ but } \sinh y = x + 1$$

$$\text{so } \frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2 + 1}}$$

c Let $y = \text{artanh}\,3x$

$$\tanh y = 3x$$

$$\text{sech}^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\text{sech}^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - \tanh^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - 9x^2}$$

d Let $y = \text{arsech}\, x$

$$\text{sech}\, y = x$$

$$\frac{1}{\cosh y} = x$$

$$1 = x\cosh y$$

Differentiate with respect to x

$$0 = \cosh y + x\sinh y\frac{dy}{dx}$$

$$x\sinh y\frac{dy}{dx} = -\cosh y$$

$$\frac{dy}{dx} = \frac{-\cosh y}{x\sinh y}$$

$$= \frac{1}{x\tanh y}$$

$$= \frac{1}{x\left(1-\text{sech}^2 y\right)^{\frac{1}{2}}}$$

$$= \frac{-1}{x\left(1-x^2\right)^{\frac{1}{2}}}$$

e Let $y = \text{arcosh}\, x^2$

Let $t = x^2 \quad y = \text{arcosh}\, t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2-1}}$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$$

f $y = \text{arcosh}\, 3x$

Let $t = 3x \quad y = \text{arcosh}\, t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}}$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2-1}}$$

g $y = x^2 \text{arcosh}\, x$

$$\frac{dy}{dx} = 2x\,\text{arcosh}\, x + \frac{x^2}{\sqrt{x^2-1}}$$

h $y = \text{arsinh}\,\frac{x}{2}$

Let $t = \frac{x}{2}$ $\quad y = \text{arsinh}\,t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\left(\frac{x}{2}\right)^2+1}}$$

$$= \frac{1}{\sqrt{x^2+4}}$$

i $y = e^{x^3}\text{arsinh}\,x$

$$\frac{dy}{dx} = 3x^2e^{x^3}\text{arsinh}\,x + \frac{e^{x^3}}{\sqrt{x^2+1}}$$

j $y = \text{arsinh}\,x\,\text{arcosh}\,x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\text{arcosh}\,x + \frac{1}{\sqrt{x^2-1}}\text{arsinh}\,x$$

k $y = \text{arcosh}\,x\,\text{sech}\,x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}\text{sech}\,x - \text{arcosh}\,x\tanh x\,\text{sech}\,x$$

$$= \text{sech}\,x\left(\frac{1}{\sqrt{x^2-1}} - \text{arcosh}\,x\tanh x\right)$$

l $y = x\,\text{arcosh}\,3x$

$$\frac{dy}{dx} = \text{arcosh}\,3x + x \times \frac{3}{\sqrt{9x^2-1}}$$

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Differentiation
Exercise B, Question 2

Question:

Prove that

a $\frac{d}{dx}(\text{arcosh}x) = \frac{1}{\sqrt{x^2-1}}$

b $\frac{d}{dx}(\text{artanh}x) = \frac{1}{1-x^2}$

Solution:

a
$$y = \text{arcosh}x$$
$$\cosh y = x$$
$$\sinh y \frac{dy}{dx} = 1 \Rightarrow$$
$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}}$$
but $\cosh y = x$ so
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

b
$$y = \text{artanh}x$$
$$\tanh y = x$$
$$\text{sech}^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y}$$
but $\tanh y = x$ so
$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

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Differentiation
Exercise B, Question 3

Question:

Given that $y = \text{artanh}\left(\frac{e^x}{2}\right)$, prove that $(4 - e^{2x})\frac{dy}{dx} = 2e^x$.

Solution:

$y = \text{artanh}\,\frac{e^x}{2}$

Let $t = \frac{e^x}{2}$ $\quad y = \text{artanh}\,t$

$\frac{dt}{dx} = \frac{e^x}{2} \quad \frac{dy}{dt} = \frac{1}{1-t^2}$

Then $\frac{dy}{dx} = \frac{1}{1-t^2} \times \frac{e^x}{2}$

$= \frac{1}{1-\left(\frac{e^x}{2}\right)^2} \times \frac{e^x}{2}$

$= \frac{\frac{e^x}{2}}{\frac{4-e^{2x}}{4}}$

$\frac{dy}{dx} = \frac{2e^x}{4-e^{2x}}$

$(4-e^{2x})\frac{dy}{dx} = 2e^x$

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Differentiation
Exercise B, Question 4

Question:

Given that $y = \text{arsinh}x$, show that

$$(1+x^2)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Solution:

$$y = \text{ar}\sinh x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+1)^{-\frac{3}{2}} \times 2x$$

$$= \frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{-1(x^2+1)^{\frac{3}{2}} - \frac{3}{2}(x^2+1)^{\frac{1}{2}} \times 2x \times -x}{(x^2+1)^3}$$

$$= \frac{3x^2(x^2+1)^{\frac{1}{2}} - (x^2+1)^{\frac{3}{2}}}{(x^2+1)^3}$$

$$= \frac{3x^2}{(x^2+1)^{\frac{5}{2}}} - \frac{1}{(x^2+1)^{\frac{3}{2}}}$$

$$(x^2+1)\frac{d^3y}{dx^3} = \frac{3x^2}{(x^2+1)^{\frac{3}{2}}} - \frac{1}{(x^2+1)^{\frac{1}{2}}}$$

$$= -3x\frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

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Differentiation
Exercise B, Question 5

Question:

If $y = (\text{arcosh}x)^2$, find $\frac{\text{d}^2y}{\text{d}x^2}$.

Solution:

$$y = (\text{arcosh}x)^2$$

$$\frac{\text{d}y}{\text{d}x} = 2\text{arcosh}x \times \frac{1}{\sqrt{x^2-1}}$$

$$= 2(x^2-1)^{-\frac{1}{2}}\text{arcosh}x$$

$$\frac{\text{d}^2y}{\text{d}x^2} = -(x^2-1)^{-\frac{3}{2}}2x\text{arcosh}x + 2(x^2-1)^{-\frac{1}{2}} \times \frac{1}{\sqrt{x^2-1}}$$

$$= \frac{-2x\text{arcosh}x}{(x^2-1)^{\frac{3}{2}}} + \frac{2}{x^2-1}$$

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Differentiation
Exercise B, Question 6

Question:

Find the equation of the tangent at the point where $x = \frac{12}{13}$ on the curve with equation $y = \text{artanh} x$.

Solution:

$y = \text{artanh} x \quad x = \frac{12}{13} \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln 25 = \ln 5$

$\frac{dy}{dx} = \frac{1}{1-x^2} = \frac{1}{1-\left(\frac{12}{13}\right)^2} = \frac{169}{25}$

Tangent is

$(y - \ln 5) = \frac{169}{25}\left(x - \frac{12}{13}\right)$

$25y - 25\ln 5 = 169x - 156$

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Differentiation
Exercise C, Question 1

Question:

Given that $y = \arccos x$ prove that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Solution:

$$y = \arccos x$$
$$\cos y = x$$
$$-\sin y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-1}{\sin y}$$
$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$

since $\cos y = x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

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Differentiation
Exercise C, Question 2

Question:

Differentiate with respect to x

a $\arccos 2x$

b $\arctan \frac{x}{2}$

c $\arcsin 3x$

d $\text{arccot}\, x$

e $\text{arcsec}\, x$

f $\text{arccosec}\, x$

g $\arcsin\left(\frac{x}{x-1}\right)$

h $\arccos x^2$

i $e^x \arccos x$

j $\arcsin x \cos x$

k $x^2 \arccos x$

l $e^{\arctan x}$

Solution:

a Let $y = \arccos 2x$

Let $t = 2x \quad y = \arccos t$

then $\frac{dt}{dx} = 2 \quad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-t^2}} \times 2$$

$$= \frac{-2}{\sqrt{1-4x^2}}$$

b Let $y = \arctan \frac{x}{2}$

Let $t = \frac{x}{2} \quad y = \arctan t$

$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{1+t^2}$

$$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{1}{2} = \frac{1}{2\left(1+\frac{x^2}{4}\right)} = \frac{2}{4+x^2} \text{ or } \frac{2}{x^2+4}$$

c Let $y = \arcsin 3x$

$\sin y = 3x$

$$\cos y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\cos y} = \frac{3}{\sqrt{1-\sin^2 y}}$$

$$= \frac{3}{\sqrt{1-9x^2}}$$

$$= \frac{3}{\sqrt{1-9x^2}}$$

d Let $y = \text{arccot}\, x$

$$\cot y = x$$

$$-\text{cosec}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y}$$

$$= \frac{-1}{1 + \cot^2 y}$$

$$= \frac{-1}{1 + x^2}$$

e Let $y = \text{arcsec}\, x$

$$\sec y = x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{x\sqrt{x^2 - 1}}$$

f Let $y = \text{arccosec}\, x$

$$\text{cosec}\, y = x$$

$$-\text{cosec}\, y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\text{cosec}\, y \cot y}$$

$$= \frac{-1}{\text{cosec}\, y \sqrt{(\text{cosec}^2 y - 1)}}$$

$$= \frac{-1}{x\sqrt{x^2 - 1}}$$

g Let $y = \arcsin\left(\frac{x}{x-1}\right)$

$$\sin y = \frac{x}{x-1}$$

$$\cos y \frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \times \frac{-1}{(x-1)^2}$$

$$= \frac{1}{\sqrt{1 - \frac{x^2}{(x-1)^2}}} \times \frac{-1}{(x-1)^2}$$

$$= \frac{1}{\sqrt{\frac{(x-1)^2 - x^2}{(x-1)^2}}} \times \frac{-1}{(x-1)^2}$$

$$= \frac{1}{\frac{\sqrt{1-2x}}{x-1}} \times \frac{-1}{(x-1)^2}$$

$$= \frac{-1}{(x-1)\sqrt{1-2x}}$$

h Let $y = \arccos x^2$

Let

$t = x^2 \quad y = \arccos t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-t^2}} \times 2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

i Let $y = e^x \arccos x$

$$\frac{dy}{dx} = e^x \arccos x - e^x \frac{1}{\sqrt{1-x^2}}$$

$$= e^x \left(\arccos x - \frac{1}{\sqrt{1-x^2}} \right)$$

j Let $y = \arcsin x \cos x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cos x + \arcsin x \times -\sin x$$

$$= \frac{\cos x}{\sqrt{1-x^2}} - \sin x \arcsin x$$

k Let $y = x^2 \arccos x$

$$\frac{dy}{dx} = 2x \arccos x - x^2 \times \frac{1}{\sqrt{1-x^2}}$$

$$= 2x \arccos x - \frac{x^2}{\sqrt{1-x^2}}$$

$$= x\left(2\arccos x - \frac{x}{\sqrt{1-x^2}} \right)$$

l Let $y = e^{\arctan x}$

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

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Differentiation
Exercise C, Question 3

Question:

If $\tan y = x \arctan x$, find $\frac{dy}{dx}$.

Solution:

$$\tan y = x \arctan x$$

$$\sec^2 y \frac{dy}{dx} = \arctan x + \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(\arctan x + \frac{x}{1+x^2}\right)$$

$$= \frac{1}{1+x^2(\arctan x)^2}\left(\arctan x + \frac{x}{1+x^2}\right)$$

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Differentiation
Exercise C, Question 4

Question:

Given that $y = \arcsin x$ prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$ **[E]**

Solution:

$$y = \arcsin x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{0 - \frac{1}{2}\left(1 - x^2\right)^{-\frac{1}{2}} \times -2x}{\left(\sqrt{1 - x^2}\right)^2}$$

$$= \frac{x\left(1 - x^2\right)^{-\frac{1}{2}}}{\left(1 - x^2\right)}$$

$$= \frac{x}{\sqrt{1 - x^2}\left(1 - x^2\right)}$$

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx}$$

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

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Differentiation
Exercise C, Question 5

Question:

Find an equation of the tangent to the curve with equation $y = \arcsin 2x$ at the point where $x = \frac{1}{4}$.

Solution:

$$y = \arcsin 2x \quad x = \frac{1}{4} \quad y = \arcsin\left(\frac{2}{4}\right) = \frac{\pi}{6}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{4}{\sqrt{3}}$$

Tangent is

$$\left(y - \frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$$

$$\sqrt{3}y - \frac{\pi\sqrt{3}}{6} = 4x - 1$$

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Differentiation
Exercise D, Question 1

Question:

Given $y = \cosh 2x$, find $\frac{dy}{dx}$.

Solution:

$$y = \cosh 2x$$
$$\frac{dy}{dx} = 2\sinh 2x$$

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Differentiation
Exercise D, Question 2

Question:

Differentiate with respect to x.

a $\text{arsinh}\, 3x$

b $\text{arsinh}\, x^2$

c $\text{arcosh}\, \frac{x}{2}$

d $x^2 \text{arcosh}\, 2x$

Solution:

a $y = \operatorname{arsinh} 3x$

Let $t = 3x \quad y = \operatorname{arsinh} t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 3$$

$$= \frac{3}{\sqrt{9x^2+1}}$$

b $y = \operatorname{arsinh} x^2$

Let $t = x^2 \quad y = \operatorname{arsinh} t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}} \times 2x$$

$$= \frac{2x}{\sqrt{x^4+1}}$$

c $y = \operatorname{arcosh} \frac{x}{2}$

Let $t = \frac{x}{2} \quad y = \operatorname{arcosh} t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2-1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}} \times \frac{1}{2}$$

$$= \frac{1}{2\sqrt{\frac{x^2}{4}-1}} = \frac{1}{\sqrt{x^2-4}}$$

d $y = x^2 \operatorname{arcosh} 2x$

$$\frac{dy}{dx} = 2x \operatorname{arcosh} 2x + x^2 \times \frac{2}{\sqrt{4x^2-1}}$$

$$= 2x\left(\operatorname{arcosh} 2x + \frac{x}{\sqrt{4x^2-1}}\right)$$

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Differentiation
Exercise D, Question 3

Question:

Given that $y = \arctan x$, prove that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

Solution:

$y = \arctan x$

then $\tan y = x$

$\tan y = x$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

but $\sec^2 y = 1 + \tan^2 y = 1 + x^2$

$$\text{so } \frac{dy}{dx} = \frac{1}{1+x^2}$$

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Differentiation
Exercise D, Question 4

Question:

Given that $y = (\text{arsinh}x)^2$ prove that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$$

Solution:

$$y = \left(\text{arsinh}x\right)^2$$

$$\frac{dy}{dx} = \frac{2\left(\text{arsinh}x\right)^1}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2}{\sqrt{x^2+1}} \times \sqrt{x^2+1} - \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x \times 2\text{arsinh}x}{\left(\sqrt{x^2+1}\right)^2}$$

$$\left(x^2+1\right)\frac{d^2y}{dx^2} = 2 - 2x\left(x^2+1\right)^{-\frac{1}{2}}\text{arsinh}x$$

$$= 2 - x\frac{dy}{dx}$$

$$\left(x^2+1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$$

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Differentiation
Exercise D, Question 5

Question:

Given $y = 5\cosh x - 3\sinh x$

a find $\frac{dy}{dx}$

b find the minimum turning points.

Solution:

$y = 5\cosh x - 3\sinh x$

$\frac{dy}{dx} = 5\sinh x - 3\cosh x$

At maximum and minimum $\frac{dy}{dx} = 0$

$0 = 5\sinh x - 3\cosh x$

$3\cosh x = 5\sinh x$

$\frac{3}{5} = \tanh x$

$x = \text{artanh}\frac{3}{5}$

Use $\text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$x = \frac{1}{2}\ln\left(\frac{\frac{8}{5}}{\frac{2}{5}}\right)$

$x = \frac{1}{2}\ln 4$

$= \ln 2$

$y = 6\frac{1}{4} - 2\frac{1}{4}$

$= 4$

$\Rightarrow$ turning point is $(\ln 2, 4)$

$\frac{d^2y}{dx^2} = 5\cosh x - 3\sinh x = 4$ at $x = \ln 2$

$\therefore \frac{d^2y}{dx^2} > 0$ at $(\ln 2, 4)$ so this point is a minimum

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Differentiation
Exercise D, Question 6

Question:

Given that $y = (\arcsin x)^2$ show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Solution:

$$y = (\arcsin x)^2$$

$$\frac{dy}{dx} = 2(\arcsin x)\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{2\times\frac{1}{\sqrt{1-x^2}}\times\sqrt{1-x^2} - 2\arcsin x \times \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 2 + \frac{x \times 2\arcsin x}{(1-x^2)^{\frac{1}{2}}}$$

$$= 2 + x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

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Differentiation
Exercise D, Question 7

Question:

Differentiate arcosh (sinh $2x$).

Solution:

$$y = \text{arcosh}\left(\sinh 2x\right)$$

Let $t = \sinh 2x \quad y = \text{arcosh}\, t$

$$\frac{dt}{dx} = 2\cosh 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2 - 1}} \times 2\cosh 2x$$

$$= \frac{2\cosh 2x}{\sqrt{\sinh^2 2x - 1}}$$

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Differentiation
Exercise D, Question 8

Question:

Given that $y = x - \arctan x$, prove that $\frac{d^2y}{dx^2} = 2x\left(1 - \frac{dy}{dx}\right)^2$

Solution:

$$y = x - \arctan x$$

$$\frac{dy}{dx} = 1 - \frac{1}{1+x^2}$$

$$\frac{d^2y}{dx^2} = 0 - \frac{(0-2x)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$$= 2x\left(1 - \left(1 - \frac{1}{1+x^2}\right)\right)^2$$

$$\frac{d^2y}{dx^2} = 2x\left(1 - \frac{dy}{dx}\right)^2$$

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Differentiation
Exercise D, Question 9

Question:

Differentiate $\arcsin \frac{x}{\sqrt{1+x^2}}$.

Solution:

$$y = \arcsin \frac{x}{\sqrt{1+x^2}}$$

Let $t = \frac{x}{\sqrt{1+x^2}}$ $\qquad y = \arcsin t$

$$\frac{dt}{dx} = \frac{1(1+x^2)^{\frac{1}{2}} - \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \times 2x \times x}{(1+x^2)} \qquad \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-t^2}}\left(\frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2}\right)$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}\left((1+x^2)^{-\frac{1}{2}}\frac{[1+x^2-x^2]}{(1+x^2)}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{1+x^2}}}(1+x^2)^{-\frac{1}{2}}\frac{[1]}{(1+x^2)} = \frac{1}{x^2+1}$$

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Differentiation
Exercise D, Question 10

Question:

Show that the curve with equation $y = \text{sech}\, x$ has $\frac{d^2y}{dx^2} = 0$ at the point where $x = \pm \ln p$ and state a value of p.

Solution:

$$y = \operatorname{sech} x$$

$$\frac{dy}{dx} = -\tanh x \operatorname{sech} x$$

$$\frac{d^2y}{dx^2} = \operatorname{sech}^2 x \operatorname{sech} x + \tanh x(-\tanh x \sec x)$$

$$= \operatorname{sech}^3 x - \operatorname{sech} x \tanh^2 x$$

$$= \operatorname{sech} x\left(\operatorname{sech}^2 x - \tanh^2 x\right)$$

$$= \operatorname{sech} x\left(1 - \tanh^2 x - \tanh^2 x\right)$$

$$= \operatorname{sech} x(1 - 2\tanh^2 x)$$

When $\frac{d^2y}{dx^2} = 0$

$$0 = \operatorname{sech} x\left(1 - 2\tanh^2 x\right)$$

so $\tanh^2 x = \frac{1}{2} \Rightarrow \tanh x = \pm\frac{1}{\sqrt{2}}$

$$x = \operatorname{artanh} \pm\frac{1}{\sqrt{2}} = \pm\operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)$$

$$x = \pm\frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)$$

$$= \pm\frac{1}{2}\ln\left(\frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}}\right)$$

$$= \pm\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \pm\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right)$$

$$= \pm\frac{1}{2}\ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right)$$

$$= \pm\frac{1}{2}\ln\left(\sqrt{2}+1\right)^2$$

$= \pm\ln\left(\sqrt{2}+1\right)$ $\quad p = \sqrt{2}+1$ (Note $p = \sqrt{2}-1$ is also acceptable.)

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Differentiation
Exercise D, Question 11

Question:

Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \cosh q, b \sinh q)$.

Solution:

$$x = a \cosh q \quad y = b \sinh q$$

$$\frac{dy}{dx} = \frac{b \cosh q}{a \sinh q}$$

Equation of tangent $y - b \sinh q = \frac{b \cosh q}{a \sinh q}(x - a \cosh q)$

$$ay \sinh q - ab \sinh^2 q = xb \cosh q - ab \cosh^2 q$$

$$ay \sinh q - xb \cosh q + ab(\cosh^2 q - \sinh^2 q) = 0$$

$$ay \sinh q - xb \cosh q + ab = 0$$

Equation of normal $y - b \sinh q = -\frac{a \sinh q}{b \cosh q}(x - a \cosh q)$

$$by \cosh q - b^2 \sinh q \cosh q = -ax \sinh q + a^2 \sinh q \cosh q$$

$$ax \sinh q + by \cosh q - \sinh q \cosh q(a^2 + b^2) = 0$$

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Integration
Exercise A, Question 1

Question:

Integrate the following with respect to x.

a $\sinh x + 3\cosh x$

b $5\text{sech}^2 x$

c $\dfrac{1}{\sinh^2 x}$

d $\cosh x - \dfrac{1}{\cosh^2 x}$

e $\dfrac{\sinh x}{\cosh^2 x}$

f $\dfrac{3}{\sinh x \tanh x}$

g $\text{sech } x(\text{sech } x + \tanh x)$

h $(\text{sech} x + \text{cosech} x)(\text{sech } x - \text{cosech} x)$

Solution:

a $\int (\sinh x + 3\cosh x)\, dx = \cosh x + 3\sinh x + C$

b $\int 5\text{sech}^2 x\, dx = 5\tanh x + C$

c $\int \dfrac{1}{\sinh^2 x}\, dx = \int \text{cosech}^2 x\, dx = -\coth x + C$

d $\int \left(\cosh x - \dfrac{1}{\cosh^2 x}\right) dx = \int (\cosh x - \text{sech}^2 x)\, dx = \sinh x - \tanh x + C$

e $\int \dfrac{\sinh x}{\cosh^2 x}\, dx = \int \dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x}\, dx = \int \text{sech} x \tanh x\, dx = -\text{sech} x + C$

f $\int \dfrac{3}{\sinh x \tanh x}\, dx = 3\int \text{cosech} x \coth x\, dx = -3\text{cosech} x + C$

g $\int \text{sech} x(\text{sech} x + \tanh x)\, dx = \int (\text{sech}^2 x + \text{sech} x \tanh x)\, dx = \tanh x - \text{sech} x + C$

h $\int (\text{sech}^2 x - \text{cosech}^2 x)\, dx = \tanh x + \coth x + C$

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Integration
Exercise A, Question 2

Question:

Find

a $\int \sinh 2x \, dx$

b $\int \cosh\left(\frac{x}{3}\right) dx$

c $\int \text{sech}^2 (2x-1) dx$

d $\int \text{cosech}^2 5x \, dx$

e $\int \text{cosech}\, 2x \coth 2x \, dx$

f $\int \text{sech}\left(\frac{x}{\sqrt{2}}\right) \tanh\left(\frac{x}{\sqrt{2}}\right) dx$

g $\int \left(5\sinh 5x - 4\cosh 4x + 3\text{sech}^2\left(\frac{x}{2}\right)\right) dx$

Solution:

a $\int \sinh 2x \, dx = \frac{1}{2}\cosh 2x + C$

b $\int \cosh\left(\frac{x}{3}\right) dx = \frac{1}{\left(\frac{1}{3}\right)} \sinh\left(\frac{x}{3}\right) + C = 3\sinh\left(\frac{x}{3}\right) + C$

c $\int \text{sech}^2 (2x-1) dx = \frac{1}{2}\tanh(2x-1) + C$

d $\int \text{cosech}^2 5x dx = -\frac{1}{5}\coth 5x + C$

e $\int \text{cosech}\, 2x \coth 2x \, dx = -\frac{1}{2}\text{cosech}\, 2x + C$

f $\int \text{sech}\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right) dx = -\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\text{sech}\left(\frac{x}{\sqrt{2}}\right) + C = \sqrt{2}\text{sech}\left(\frac{x}{\sqrt{2}}\right) + C$

g $\int 5\sinh 5x - 4\cosh 4x + 3\text{sech}^2\left(\frac{x}{2}\right) dx = 5\times\frac{1}{5}\cosh 5x - 4\times\frac{1}{4}\sinh 4x + 3\times\frac{1}{\left(\frac{1}{2}\right)}\tanh\left(\frac{x}{2}\right) + C$

$= \cosh 5x - \sinh 4x + 6\tanh\left(\frac{x}{2}\right) + C$

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Integration
Exercise A, Question 3

Question:

Write down the results of the following. (This is a recognition exercise and also involve some integrals from C4.)

a $\int \frac{1}{1+x^2}\,dx$

b $\int \frac{1}{\sqrt{1+x^2}}\,dx$

c $\int \frac{1}{1+x}\,dx$

d $\int \frac{2x}{1+x^2}\,dx$

e $\int \frac{1}{\sqrt{1-x^2}}\,dx, |x|<1$

f $\int \frac{1}{\sqrt{x^2-1}}\,dx$

g $\int \frac{3x}{\sqrt{x^2-1}}\,dx$

h $\int \frac{3}{(1+x)^2}\,dx$

Solution:

a $\arctan x + C$

b $\text{arsinh}x + C$

c $\ln|1+x| + C$

d $\ln(1+x^2) + C$

e $\arcsin x + C$

f $\text{arcosh}x + C$

g $3\sqrt{x^2-1} + C$

h $-\frac{3}{(1+x)} + C$

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Integration
Exercise A, Question 4

Question:

Find

a $\int \frac{2x+1}{\sqrt{1-x^2}}\,dx$

b $\int \frac{1+x}{\sqrt{x^2-1}}\,dx$

c $\int \frac{x-3}{1+x^2}\,dx$

Solution:

a $$\int \frac{2x+1}{\sqrt{(1-x^2)}}\,dx = \int \frac{2x}{\sqrt{(1-x^2)}}\,dx + \int \frac{1}{\sqrt{(1-x^2)}}\,dx$$
$$= 2\int x(1-x^2)^{-\frac{1}{2}}\,dx + \int \frac{1}{\sqrt{(1-x^2)}}\,dx$$
$$= -2\sqrt{(1-x^2)} + \text{ar}\sin x + C$$

b $$\int \frac{1+x}{\sqrt{(x^2-1)}}\,dx = \int \frac{1}{\sqrt{(x^2-1)}}\,dx + \int \frac{x}{\sqrt{(x^2-1)}}\,dx$$
$$= \int \frac{1}{\sqrt{(x^2-1)}}\,dx + \int x(x^2-1)^{-\frac{1}{2}}\,dx$$
$$= \text{arcosh}x + \sqrt{(x^2-1)} + C$$

c $$\int \frac{x-3}{\sqrt{(1+x^2)}}\,dx = \int \frac{x}{\sqrt{(1+x^2)}}\,dx - \int \frac{3}{\sqrt{(1+x^2)}}\,dx$$
$$= \int x(1+x^2)^{-\frac{1}{2}}\,dx - \int \frac{3}{(1+x^2)}\,dx$$
$$= \sqrt{(1+x^2)} - 3\text{arsinh}x + C$$

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Integration
Exercise A, Question 5

Question:

a Show that $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$

b Hence find $\int \frac{x^2}{1+x^2}\,dx$

Solution:

a $\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}$

b $\int \frac{x^2}{1+x^2}\,dx = \int \left\{1 - \frac{1}{1+x^2}\right\} dx$ ← Using a.

$= x - \arctan x + C$

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Integration
Exercise B, Question 1

Question:

Find

a $\int \sinh^3 x \cosh x \, dx$

b $\int \tanh 4x \, dx$

c $\int \tanh^5 x \operatorname{sech}^2 x \, dx$

d $\int \operatorname{cosech}^7 x \coth x \, dx$

e $\int \sqrt{\cosh 2x} \sinh 2x \, dx$

f $\int \operatorname{sech}^{10} 3x \tanh 3x \, dx$

Solution:

a $\int \sinh^3 x \cosh x \, dx = \int (\sinh x)^3 \cosh x \, dx = \frac{1}{4}\sinh^4 x + C$

b $\int \tanh 4x \, dx = \int \frac{\sinh 4x}{\cosh 4x} \, dx = \frac{1}{4}\ln \cosh 4x + C$

c $\int \tanh^5 x \operatorname{sech}^2 x \, dx = \int (\tanh x)^5 \operatorname{sech}^2 x \, dx = \frac{1}{6}\tanh^6 x + C$

d $\int \operatorname{cosech}^7 x \coth x dx = \int \operatorname{cosech}^6 x (\operatorname{cosech} x \coth x) dx$

$$= -\int (\operatorname{cosech})^6 (-\operatorname{cosech} x \coth x) dx$$
$$= -\frac{1}{7}\operatorname{cosech}^7 x + C$$

e $\int \sqrt{\cosh 2x} \sinh 2x \, dx = \frac{1}{2}\int (\cosh 2x)^{\frac{1}{2}} (2\sinh 2x) dx$

$$= \frac{1}{2}\left\{\frac{(\cosh 2x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right\} + C$$
$$= \frac{1}{3}(\cosh 2x)^{\frac{3}{2}} + C$$

f $\int \operatorname{sech}^{10} 3x \tanh 3x \, dx = -\frac{1}{3}\int \operatorname{sech}^9 3x (-3\operatorname{sech} 3x \tanh 3x) dx$

$$= -\frac{1}{3}\left\{\frac{\operatorname{sech}^{10} 3x}{10}\right\} + C$$
$$= -\frac{1}{30}\operatorname{sech}^{10} 3x + C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise B, Question 2

Question:

Find

a $\int \frac{\sinh x}{2+3\cosh x}\,dx$

b $\int \frac{1+\tanh x}{\cosh^2 x}\,dx$

c $\int \frac{5\cosh x+2\sinh x}{\cosh x}\,dx$.

Solution:

a $\int \frac{\sinh x}{2+3\cosh x}\,dx = \frac{1}{3}\int \frac{3\sinh x}{2+3\cosh x}\,dx$

$= \frac{1}{3}\ln(2+3\cosh x)+C$

b $\int \frac{1+\tanh x}{\cosh^2 x}\,dx = \int (1+\tanh x)\,\text{sech}^2 x\,dx$

$= \int (\text{sech}^2 x+\tanh x\,\text{sech}^2 x)\,dx$

$= \tanh x+\frac{1}{2}\tanh^2 x+C$ or $\tanh x-\frac{1}{2}\text{sech}^2 x+C$

c $\int \frac{5\cosh x+2\sinh x}{\cosh x}\,dx = \int (5+2\tanh x)\,dx$

$= 5x+2\ln\cosh x+C$

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Integration
Exercise B, Question 3

Question:

a Show that $\int \coth x \, dx = \ln \sinh x + C$.

b Show that $\int_1^2 \coth 2x \, dx = \ln \sqrt{\left(e^2 + \frac{1}{e^2}\right)}$.

Solution:

a $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx = \ln \sinh x + C$

b $\int \coth 2x \, dx = \frac{1}{2} \ln \sinh 2x + C$

So $\int_1^2 \coth 2x = \left[\frac{1}{2} \ln \sinh 2x\right]_1^2 = \frac{1}{2}(\ln \sinh 4 - \ln \sinh 2)$

$$= \frac{1}{2} \ln\left(\frac{\sinh 4}{\sinh 2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{e^4 - e^{-4}}{e^2 - e^{-2}}\right)$$

$$= \frac{1}{2} \ln(e^2 + e^{-2})$$

Using $a^2 - b^2 = (a+b)(a-b)$ with $a = e^2, b = e^{-2}$

$$= \ln \sqrt{e^2 + \frac{1}{e^2}}$$

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Integration
Exercise B, Question 4

Question:

Use integration by parts to find

a $\int x \sinh 3x \, dx$

b $\int x \text{sech}^2 x \, dx$.

Solution:

a $\int x \sinh 3x \, dx = \frac{1}{3} x \cosh 3x - \int \frac{1}{3} \cosh 3x dx$

$= \frac{1}{3} x \cosh 3x - \frac{1}{9} \sinh 3x + C$

Using $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ with $u = x$ and $\frac{dv}{dx} = \sinh 3x$

b $\int x \text{sech}^2 x \, dx = x \tanh x - \int \tanh x \, dx$

$= x \tanh x - \ln \cosh x + C$

Using integration by parts with $u = x$ and $\frac{dv}{dx} = \text{sech}^2 x$

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Integration
Exercise B, Question 5

Question:

Find

a $\int e^x \cosh x \, dx$

b $\int e^{-2x} \sinh 3x \, dx$

c $\int \cosh x \cosh 3x \, dx$.

Solution:

a $\int e^x \cosh x \, dx = \int e^x \left(\frac{e^x + e^{-x}}{2} \right) dx$

You cannot use integration by parts.

$= \frac{1}{2} \int \left(e^{2x} + 1 \right) dx$

$= \frac{1}{4} e^{2x} + \frac{1}{2} x + C$

b $\int e^{-2x} \sinh 3x \, dx = \int e^{-2x} \left(\frac{e^{3x} - e^{-3x}}{2} \right) dx$

You could use integration by parts twice.

$= \frac{1}{2} \int \left(e^x - e^{-5x} \right) dx$

$= \frac{1}{2} e^x + \frac{1}{10} e^{-5x} + C$

c $\int \cosh x \cosh 3x \, dx = \int \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^{3x} + e^{-3x}}{2} \right) dx$

or write as $\frac{1}{2} (\cosh 4x + \cosh 2x)$

$= \frac{1}{4} \int \left(e^{4x} + e^{-4x} + e^{2x} + e^{-2x} \right) dx$

$= \frac{1}{16} e^{4x} - \frac{1}{16} e^{-4x} + \frac{1}{8} e^{2x} - \frac{1}{8} e^{-2x} + C$ or $\frac{1}{8} \sinh 4x + \frac{1}{4} \sinh 2x + C$

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Integration
Exercise B, Question 6

Question:

By writing $\cosh 3x$ in exponential form, find $\int \cosh^2 3x \, dx$ and show that it is equivalent to the result found in Example 5**b**.

Solution:

$$\int \cosh^2 3x \, dx = \frac{1}{4}\int (e^{3x} + e^{-3x})^2 \, dx$$
$$= \frac{1}{4}\int (e^{6x} + 2 + e^{-6x}) \, dx$$
$$= \frac{1}{24}e^{6x} - \frac{1}{24}e^{-6x} + \frac{1}{2}x + C$$
$$= \frac{1}{12}\sinh 6x + \frac{1}{2}x + C \quad \text{which was result in Example 5b}$$

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Integration
Exercise B, Question 7

Question:

Evaluate $\int_0^1 \frac{1}{\sinh x + \cosh x}\, dx$, giving your answer in terms of e.

Solution:

$$\sinh x + \cosh x = \frac{1}{2}\left(e^x - e^{-x}\right) + \frac{1}{2}\left(e^x + e^{-x}\right) = e^x$$

So $\int_0^1 \left(\frac{1}{\sinh x + \cosh x}\right) dx = \int_0^1 e^{-x}\, dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}$

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Integration
Exercise B, Question 8

Question:

Use appropriate identities to find

a $\int \sinh^2 x \, dx$

b $\int (\operatorname{sech} x - \tanh x)^2 \, dx$

c $\int \frac{\cosh^2 3x}{\sinh^2 3x} \, dx$

d $\int \sinh^2 x \cosh^2 x \, dx$

e $\int \cosh^5 x \, dx$

f $\int \tanh^3 2x \, dx$.

Solution:

a $\int \sinh^2 x \, dx = \frac{1}{2}\int (\cosh 2x - 1)dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + C$

b $\int (\text{sech}\, x - \tanh x)^2 \, dx = \int (\text{sech}^2 x - 2\text{sech}\, x \tanh x + \tanh^2 x)dx$

$= \int (\text{sech}^2 x - 2\text{sech}\, x \tanh x + 1 - \text{sech}^2 x)dx$

$= \int (1 - 2\text{sech}\, x \tanh x)dx$

$= x + 2\text{sech}\, x + C$

c $\int \frac{\cosh^2 3x}{\sinh^2 3x} dx = \int \coth^2 3x \, dx$

$= \int (1 + \text{cosech}^2 3x)dx$

$= x - \frac{1}{3}\coth 3x + C$

d $\int \sinh^2 x \cosh^2 x \, dx = \int \left(\frac{1}{2}\sinh 2x\right)^2 dx$

$= \frac{1}{4}\int \sinh^2 2x \, dx$

$= \frac{1}{4}\int \left(\frac{\cosh 4x - 1)}{2}\right)dx$ ← Using $\cosh 2u = 1 + 2\sinh^2 u$

$= -\frac{1}{8}x + \frac{1}{32}\sinh 4x + C$

e $\int \cosh^5 x dx = \int \cosh^4 x \cosh x \, dx$

$= \int (1 + \sinh^2 x)^2 \cosh x \, dx$

$= \int (1 + 2\sinh^2 x + \sinh^4 x) \cosh x dx$

$= \int (\cosh x + 2\sinh^2 x \cosh x + \sinh^4 x \cosh x)dx$

$= \sinh x + \frac{2}{3}\sinh^3 x + \frac{1}{5}\sinh^5 x + C$

f $\int \tanh^3 2x \, dx = \int \tanh^2 2x \tanh 2x \, dx$

$= \int (1 - \text{sech}^2 2x) \tanh 2x \, dx$

$= \int (\tanh 2x - \tanh 2x \text{sech}^2 2x)dx$

$= \frac{1}{2}\ln\cosh 2x - \frac{1}{4}\tanh^2 2x + C$

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Integration
Exercise B, Question 9

Question:

Show that $\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right)dx = \frac{1}{8}(3+\ln 16)$.

Solution:

$$\begin{aligned}\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right)dx &= \int_0^{\ln 2}\left(\frac{1+\cosh x}{2}\right)dx \\ &= \frac{1}{2}\left[x+\sinh x\right]_0^{\ln 2} \\ &= \frac{1}{2}\left[\ln 2+\left(\frac{e^{\ln 2}-e^{-\ln 2}}{2}\right)\right] \\ &= \frac{1}{2}\left[\ln 2+\frac{3}{4}\right] \\ &= \frac{1}{8}\left[3+4\ln 2\right] \\ &= \frac{1}{8}(3+\ln 16)\end{aligned}$$

$e^{\ln 2} = 2, e^{-\ln 2} = e^{\ln\frac{1}{2}} = \frac{1}{2}$

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Integration
Exercise B, Question 10

Question:

The region bounded by the curve $y = \sinh x$, the line $x = 1$ and the positive x-axis is rotated through $360°$ about the x-axis. Show that the volume of the solid of revolution formed is $\frac{\pi}{8e^2}\left(e^4 - 4e^2 - 1\right)$.

Solution:

$$\text{Volume} = \pi\int_0^1 \sinh^2 x \, dx = \frac{\pi}{2}\int_0^1 (\cosh 2x - 1)dx$$
$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^1$$
$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2 - 1\right]$$
$$= \frac{\pi}{2}\left[\frac{1}{4}\left(e^2 - e^{-2}\right) - 1\right]$$
$$= \frac{\pi}{8}\left[e^2 - 4 - e^{-2}\right]$$
$$= \frac{\pi}{8e^2}\left(e^4 - 4e^2 - 1\right).$$

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Integration
Exercise B, Question 11

Question:

Using the result for $\int \operatorname{sech} x \, dx$ given in Example 7, find

a $\int \frac{2}{\cosh x} \, dx$

b $\int \operatorname{sech} 2x \, dx$

c $\int \sqrt{1 - \tanh^2\left(\frac{x}{2}\right)} \, dx$.

Solution:

Using $\int \operatorname{sech} x \, dx = 2\arctan(e^x) + C$

a $\int \frac{2}{\cosh x} \, dx = \int 2 \operatorname{sech} x \, dx = 4\arctan(e^x) + C$

b Using the substitution $u = 2x$,

$\left(\text{or using} \int f'(ax+b)dx = \frac{1}{a} f(ax+b) + C6x \right)$

$\int \operatorname{sech} 2x dx = \frac{1}{2} \int \operatorname{sech} u \, du = \arctan\left(e^u\right) + C = \arctan\left(e^{2x}\right) + C$

c $\int \sqrt{1 - \tanh^2\left(\frac{x}{2}\right)} \, dx = \int \operatorname{sech}\left(\frac{x}{2}\right) dx = \frac{1}{\left(\frac{1}{2}\right)} 2\arctan\left(e^{\frac{x}{2}}\right) + C$

$= 4\arctan\left(e^{\frac{x}{2}}\right) + C$

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Integration
Exercise B, Question 12

Question:

Using the substitution $u = x^2$, or otherwise, find

a $\int x\cosh^2(x^2)\,dx$

b $\int \frac{x}{\cosh^2(x^2)}\,dx$.

Solution:

Using the substitution $u = x^2$, $du = 2x\,dx$,

a So $\int x\cosh^2(x^2)\,dx = \frac{1}{2}\int \cosh^2 u\,du$

$$= \frac{1}{4}\int (\cosh 2u + 1)\,du$$

$$= \frac{1}{8}\sinh 2u + \frac{u}{4} + C$$

$$= \frac{1}{8}\sinh(2x^2) + \frac{x^2}{4} + C$$

b So $\int \frac{x}{\cosh^2(x^2)}\,dx = \int x\,\mathrm{sech}^2(x^2)\,dx$

$$= \frac{1}{2}\int \mathrm{sech}^2 u\,du$$

$$= \frac{1}{2}\tanh u + C$$

$$= \frac{1}{2}\tanh(x^2) + C$$

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Integration
Exercise C, Question 1

Question:

Use the substitution $x = a\tan\theta$ to show that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$.

Solution:

Using $x = a\tan\theta, dx = a\sec^2\theta\, d\theta$

so $\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2+a^2\tan^2\theta}a\sec^2\theta\, d\theta$

$= \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\,d\theta$

$= \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C$

$= \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$ ← $x = a\tan\theta \Rightarrow \theta = \arctan\left(\frac{x}{a}\right)$

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Integration
Exercise C, Question 2

Question:

Use the substitution $x = \cos\theta$ to show that $\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + C$.

Solution:

Using $x = \cos\theta, dx = -\sin\theta\, d\theta$

$$\text{so} \int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\,d\theta$$
$$= -\int d\theta$$
$$= -\theta + C$$
$$= -\arccos x + C$$

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Integration
Exercise C, Question 3

Question:

Use suitable substitutions to find

a $\int \frac{3}{\sqrt{4-x^2}}\,dx$

b $\int \frac{1}{\sqrt{x^2-9}}\,dx$

c $\int \frac{4}{5+x^2}\,dx$

d $\int \frac{1}{\sqrt{4x^2+25}}\,dx$.

Solution:

a Let $x = 2\sin\theta$, so $dx = 2\cos\theta \, d\theta$

$$\int \frac{3}{\sqrt{4-x^2}}\,dx = \int \frac{3}{\sqrt{4-4\sin^2\theta}} 2\cos\theta \, d\theta$$

$$= \int \frac{6\cos\theta}{2\cos\theta}\,d\theta$$

$$= 3\int d\theta$$

$$= 3\theta + C$$

$$= 3\arcsin\left(\frac{x}{2}\right) + C$$

b Let $x = 3\cosh u$, so $dx = 3\sinh u\, du$

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{1}{\sqrt{9\cosh^2 u - 9}} 3\sinh u \, du$$

$$= \int \frac{1}{3\sqrt{\cosh^2 u - 1}} 3\sinh u \, du$$

$$= \int \frac{3\sinh u}{3\sinh u}\,du$$

$$= \int 1 \, du$$

$$= u + C$$

$$= \text{arcosh}\left(\frac{x}{3}\right) + C$$

c Let $x = \sqrt{5}\tan\theta$, so $dx = \sqrt{5}\sec^2\theta \, d\theta$

$$\int \frac{4}{5+x^2}\,dx = \int \frac{4}{5+5\tan^2\theta}\sqrt{5}\sec^2\theta \, d\theta$$

$$= \int \frac{4\sqrt{5}\sec^2\theta}{5\sec^2\theta}\,d\theta$$

$5 + 5\tan^2\theta = 5(1+\tan^2\theta) = 5\sec^2\theta$

$$= \frac{4\sqrt{5}}{5}\int d\theta$$

$$= \frac{4\sqrt{5}}{5}\theta + C$$

$$= \frac{4\sqrt{5}}{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

d You need $4x^2 = 25\sinh^2 u$, or $2x = 5\sinh u$, then $\mathrm{d}x = \frac{5}{2}\cosh u\,\mathrm{d}u$

$$\int \frac{1}{\sqrt{4x^2+25}}\,\mathrm{d}x = \int \frac{1}{\sqrt{25\sinh^2 u + 25}}\left(\frac{5}{2}\cosh u\right)\mathrm{d}u$$

$$= \frac{5}{2}\int \frac{\cosh u}{5\sqrt{\sinh^2 u + 1}}\,\mathrm{d}u$$

$$= \frac{1}{2}\int \frac{\cosh u}{\cosh u}\,\mathrm{d}u$$

$$= \frac{1}{2}\int 1\,\mathrm{d}u$$

$$= \frac{1}{2}u + C$$

$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{5}\right) + C$$

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Integration
Exercise C, Question 4

Question:

Write down the results for the following:

a $\int \frac{1}{\sqrt{25-x^2}}\,dx$

b $\int \frac{3}{\sqrt{x^2+9}}\,dx$

c $\int \frac{1}{\sqrt{x^2-2}}\,dx$

d $\int \frac{2}{16+x^2}\,dx$.

Solution:

a $\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right)+C$ ← Using $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C$

b $\int \frac{3}{\sqrt{x^2+9}}\,dx = 3\text{arsinh}\left(\frac{x}{3}\right)+C$ ← Using $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \text{arsinh}\left(\frac{x}{a}\right)+C$.

c $\int \frac{1}{\sqrt{x^2-2}}\,dx = \text{arcosh}\left(\frac{x}{\sqrt{2}}\right)+C$ ← Using $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \text{arcosh}\left(\frac{x}{a}\right)+C$

d $\int \frac{2}{16+x^2}\,dx = 2\int \frac{1}{16+x^2}\,dx$

$= 2\left\{\frac{1}{4}\arctan\left(\frac{x}{4}\right)\right\}+C$ ← Using $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

$= \frac{1}{2}\arctan\left(\frac{x}{4}\right)+C$

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Integration

Exercise C, Question 5

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Find

a $\int \frac{1}{\sqrt{4x^2-12}}\,dx$

b $\int \frac{1}{4+3x^2}\,dx$

c $\int \frac{1}{\sqrt{9x^2+16}}\,dx$

d $\int \frac{1}{\sqrt{3-4x^2}}\,dx$.

Solution:

a $\int \frac{1}{\sqrt{4x^2-12}}\,dx = \int \frac{1}{\sqrt{4(x^2-3)}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{(x^2-3)}}\,dx$

$= \frac{1}{2}\operatorname{arcosh}\left(\frac{x}{\sqrt{3}}\right) + C$

b $\int \frac{1}{4+3x^2}\,dx = \int \frac{1}{3\left\{\frac{4}{3}+x^2\right\}}\,dx$

$= \frac{1}{3}\left\{\frac{1}{\left(\frac{2}{\sqrt{3}}\right)}\arctan\left(\frac{x}{\left(\frac{2}{\sqrt{3}}\right)}\right)\right\} + C$

$= \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}x}{2}\right) + C$

c $\int \frac{1}{\sqrt{9x^2+16}}\,dx = \int \frac{1}{\sqrt{9\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,dx$

$= \frac{1}{3}\operatorname{arsinh}\left(\frac{x}{\left(\frac{4}{3}\right)}\right) + C$

$= \frac{1}{3}\operatorname{arsinh}\left(\frac{3x}{4}\right) + C$

d $\int \frac{1}{\sqrt{3-4x^2}}\,dx = \int \frac{1}{\sqrt{4\left\{\frac{3}{4}-x^2\right\}}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{\left\{\frac{3}{4}-x^2\right\}}}\,dx$

$= \frac{1}{2}\arcsin\left(\frac{x}{\frac{\sqrt{3}}{2}}\right) + C$ ← $a^2 = \frac{3}{4}$ so $a = \frac{\sqrt{3}}{2}$

$= \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{3}}\right) + C$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 6

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Evaluate

a $\int_1^3 \frac{2}{1+x^2}\,dx$

b $\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,dx$

c $\int_{-1}^2 \frac{1}{\sqrt{21-3x^2}}\,dx$.

Solution:

a $\int_1^3 \frac{2}{1+x^2}\,dx = 2\left[\arctan x\right]_1^3$

$= 2(\arctan 3 - \arctan 1)$

$= 0.927 \quad (3\text{ s.f.})$ ← Remember that you need to be in radian mode.

b $\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,dx = 3\int_1^2 \frac{1}{2\sqrt{\frac{1}{4}+x^2}}\,dx$

$= \frac{3}{2}\left[\operatorname{arsinh}\frac{x}{\left(\frac{1}{2}\right)}\right]_1^2$

$= \frac{3}{2}\left[\operatorname{arsinh}(2x)\right]_1^2$

$= \frac{3}{2}\left[\operatorname{arsinh}4 - \operatorname{arsinh}2\right]$

$= 0.977 \quad (3\text{ s.f.})$

c $\int_{-1}^2 \frac{1}{\sqrt{21-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int_{-1}^2 \frac{1}{\sqrt{7-x^2}}\,dx$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{x}{\sqrt{7}}\right)\right]_{-1}^2$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{2}{\sqrt{7}}\right) - \arcsin\left(-\frac{1}{\sqrt{7}}\right)\right]$

$= \frac{1}{\sqrt{3}}\left[0.85707\ldots - (-0.38759\ldots)\right]$ ← You need to be in radian mode

$= 0.719 \quad (3\text{ s.f.})$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 7

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Evaluate, giving your answers in terms of π or as a single natural logarithm, whichever is appropriate.

a $\int_0^4 \frac{1}{\sqrt{x^2+16}}\,dx$

b $\int_{13}^{15} \frac{1}{\sqrt{x^2-144}}\,dx$

c $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Solution:

Reminder: The logarithmic form of an inverse hyperbolic function is in the Edexcel formulae booklet.

a $\int_0^4 \frac{1}{\sqrt{x^2+16}}\,dx = \left[\text{arsinh}\left(\frac{x}{4}\right)\right]_0^4$

$= \text{arsinh}\,1 - \text{arsinh}\,0$

$= \ln\left\{1+\sqrt{2}\right\}$ ← Using $\text{arsinh}\,x = \ln\left\{x+\sqrt{x^2+1}\right\}$

b $\int_{13}^{15} \frac{1}{\sqrt{x^2-144}}\,dx = \left[\text{arcosh}\left(\frac{x}{12}\right)\right]_{13}^{15}$

$= \text{arcosh}\left(\frac{5}{4}\right) - \text{arcosh}\left(\frac{13}{12}\right)$ ← Using $\text{arcosh}\,x = \ln\left\{x+\sqrt{x^2-1}\right\}$

$= \ln\left\{\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right\} - \ln\left\{\frac{13}{12}+\sqrt{\frac{169}{144}-1}\right\}$

$= \ln\left\{\frac{5}{4}+\sqrt{\frac{9}{16}}\right\} - \ln\left\{\frac{13}{12}+\sqrt{\frac{25}{144}}\right\}$

$= \ln 2 - \ln\left(\frac{3}{2}\right)$

$= \ln\left(\frac{4}{3}\right)$ ← Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

c $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{\sqrt{2}}{2}\right)$

$= \left(\frac{\pi}{3}\right) - \left(\frac{\pi}{4}\right)$

$= \left(\frac{\pi}{12}\right)$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 8

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

The curve C has equation $y = \frac{2}{\sqrt{2x^2 + 9}}$. The region R is bounded by C, the coordinate axes and the lines $x = -1$ and $x = 3$.

a Find the area of R.

The region R is rotated through $360°$ about the x-axis.

b Find the volume of the solid generated.

Solution:

a Area of $R = \int_{-1}^{3} y \, dx = \int_{-1}^{3} \frac{2}{\sqrt{2x^2+9}} \, dx$

$$= \int_{-1}^{3} \frac{2}{\sqrt{2\left(x^2+\frac{9}{2}\right)}} \, dx$$

Curve is always above x-axis

$$= \sqrt{2}\left[\operatorname{arsinh} \frac{x}{\left(\frac{3}{\sqrt{2}}\right)}\right]_{-1}^{3}$$

$$= \sqrt{2}\left[\operatorname{arsinh}\left(\frac{\sqrt{2}x}{3}\right)\right]_{-1}^{3}$$

$$= \sqrt{2}\left[\operatorname{arsinh}\sqrt{2} - \operatorname{arsinh}\left(-\frac{\sqrt{2}}{3}\right)\right]$$

$$= 2.27 \text{ (3 s.f.)}$$

b Volume $= \pi \int_{-1}^{3} y^2 \, dx = \pi \int_{-1}^{3} \frac{4}{2x^2+9} \, dx$

$$= 2\pi \int_{-1}^{3} \frac{1}{x^2 + \left(\frac{9}{2}\right)} \, dx$$

$$= 2\pi \left[\frac{1}{\left(\frac{3}{\sqrt{2}}\right)} \arctan \frac{x}{\left(\frac{3}{\sqrt{2}}\right)}\right]_{-1}^{3}$$

$$= \left(\frac{2\sqrt{2}\pi}{3}\right)\left[\arctan\left(\frac{\sqrt{2}x}{3}\right)\right]_{-1}^{3}$$

$$= \left(\frac{2\sqrt{2}\pi}{3}\right)\left[\arctan\left(\sqrt{2}\right) - \arctan\left(-\frac{\sqrt{2}}{3}\right)\right]$$

$$= 1.32\pi \text{ (3 s.f.)} = 4.13 \text{ (3 s.f.)}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 9

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

A circle C has centre the origin and radius r.

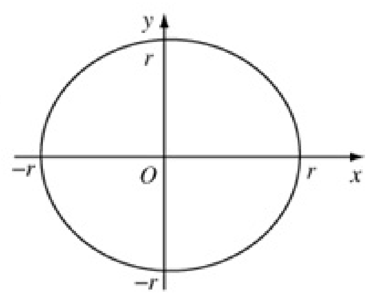
a Show that the area of C can be written as $4\int_0^r \sqrt{r^2 - x^2}\,dx$.

b Hence show that the area of C is πr^2.

Solution:

a Cartesian equation of circle is $x^2 + y^2 = r^2$.

Area of C can be written as $4\int_0^r y\,dx = 4\int_0^r \sqrt{r^2 - x^2}\,dx$



b Use substitution $x = r\sin\theta$, so $dx = r\cos\theta\,d\theta$,

$$4\int_0^r \sqrt{r^2 - x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\,d\theta$$

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 2r^2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta$$

$$= 2r^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2r^2\left(\frac{\pi}{2}\right)$$

$$= \pi r^2$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Integration

Exercise C, Question 10

Question:

a Use the substitution $x = \frac{2}{3}\tan\theta$ to find $\int \frac{x^2}{9x^2+4}\,dx$.

b Use the substitution $x = \sinh^2 u$ to find $\int \sqrt{\frac{x}{x+1}}\,dx,\ x > 0$.

Solution:

a With $x = \frac{2}{3}\tan\theta$ and $dx = \frac{2}{3}\sec^2\theta\,d\theta$,

$$9x^2+4 = 9\left(\frac{4}{9}\tan^2\theta\right)+4 = 4\tan^2\theta+4 = 4\left(\tan^2\theta+1\right) = 4\sec^2\theta$$

and $\frac{x^2}{9x^2+4} = \frac{\frac{4}{9}\tan^2\theta}{4\sec^2\theta} = \frac{\tan^2\theta}{9\sec^2\theta}$

so $\int \frac{x^2}{9x^2+4}\,dx = \int \frac{\tan^2\theta}{9\sec^2\theta}\times\frac{2}{3}\sec^2\theta\,d\theta$

$$= \frac{2}{27}\int \tan^2\theta\,d\theta$$

$$= \frac{2}{27}\int\left(\sec^2\theta-1\right)d\theta$$

$$= \frac{2}{27}\left(\tan\theta-\theta\right)+C$$

$$= \frac{2}{27}\left(\frac{3x}{2}-\arctan\frac{3x}{2}\right)+C$$

$$= \frac{x}{9}-\frac{2}{27}\arctan\frac{3x}{2}+C$$

b With $x = \sinh^2 u$ and $dx = 2\sinh u\cosh u\,du$,

and $\frac{x}{x+1} = \frac{\sinh^2 u}{\sinh^2 u+1} = \frac{\sinh^2 u}{\cosh^2 u}$

$$\int\sqrt{\frac{x}{x+1}}\,dx = \int\frac{\sinh u}{\cosh u}2\sinh u\cosh u\,du$$

$$= \int 2\sinh^2 u\,du$$

$$= \int\left(\cosh 2u-1\right)du$$

$$= \frac{\sinh 2u}{2}-u+C$$

$$= \sinh u\cosh u-\operatorname{arsinh}\left(\sqrt{x}\right)+C$$

$$= \sqrt{x}\sqrt{1+x}-\operatorname{arsinh}\left(\sqrt{x}\right)+C$$

$\sinh u = \sqrt{x}$ and
$\cosh u = \sqrt{1+\sinh^2 u}$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 11

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

By splitting up each integral into two separate integrals, or otherwise, find

a $\int \frac{x-2}{\sqrt{x^2-4}}\,dx$

b $\int \frac{2x-1}{\sqrt{2-x^2}}\,dx$

c $\int \frac{2+3x}{1+3x^2}\,dx$.

Solution:

a $\int \frac{x-2}{\sqrt{x^2-4}}\,dx = \int \frac{x}{\sqrt{x^2-4}}\,dx - \int \frac{2}{\sqrt{x^2-4}}\,dx$

$= \sqrt{x^2-4} - 2\,\text{arcosh}\left(\frac{x}{2}\right) + C$

b $\int \frac{2x-1}{\sqrt{2-x^2}}\,dx = \int \frac{2x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{2-x^2}}\,dx$

$= -2\sqrt{2-x^2} - \arcsin\left(\frac{x}{\sqrt{2}}\right) + C$

c $\int \frac{2+3x}{1+3x^2}\,dx = \int \frac{2}{1+3x^2}\,dx + \int \frac{3x}{1+3x^2}\,dx$

$= \frac{2}{3}\int \frac{1}{\left(\frac{1}{3}+x^2\right)}\,dx + \frac{1}{2}\int \frac{6x}{1+3x^2}\,dx$

$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right) + \frac{1}{2}\ln\left(1+3x^2\right) + C$

$a = \frac{1}{\sqrt{3}}$ in $\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 12

Question:

Unless a substitution is given or asked for, use the standard results 7 to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Use the method of partial fractions to find $\int \frac{x^2+4x+10}{x^3+5x}\,dx, x>0$.

Solution:

Setting up the model $\frac{x^2+4x+10}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$

$\Rightarrow x^2+4x+10 \equiv A(x^2+5) + (Bx+C)x$

$x=0 \quad \Rightarrow 10=5A \quad \Rightarrow \quad A=2$

Coefficient of $x \quad \Rightarrow \quad 4=C$

Coefficient of $x^2 \quad \Rightarrow \quad 1=A+B \quad \Rightarrow \quad B=-1$

So $\int \frac{x^2+4x+10}{x^3+5x}\,dx = \int\left(\frac{2}{x}+\frac{-x+4}{x^2+5}\right)dx$

$= \int\left(\frac{2}{x}+\frac{4}{x^2+5}-\frac{1}{2}\frac{2x}{x^2+5}\right)dx$

$= 2\ln x + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) - \frac{1}{2}\ln\left(x^2+5\right) + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 13

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Show that $\int_0^1 \frac{2}{(x+1)(x^2+1)}\,dx = \frac{1}{4}(\pi + 2\ln 2)$.

Solution:

Setting up the model $\frac{2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$$\Rightarrow 2 \equiv A(x^2+1) + (Bx+C)(x+1)$$

$x = -1 \Rightarrow 2 = 2A \Rightarrow A = 1$

Coefficient of $x^2 \Rightarrow 0 = A + B \Rightarrow B = -1$

Coefficient of $x \Rightarrow 0 = B + C \Rightarrow C = 1$

So $\int_0^1 \frac{2}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{1}{(x+1)}dx + \int_0^1 \frac{1-x}{(x^2+1)}dx$

$$= \int_0^1 \frac{1}{(x+1)}dx + \int_0^1 \frac{1}{(x^2+1)}dx - \int_0^1 \frac{x}{(x^2+1)}dx$$

$$= \left[\ln(1+x)\right]_0^1 + \left[\arctan x\right]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \ln 2 + \arctan 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln 2$$

$$= \frac{1}{4}(\pi + 2\ln 2)$$

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Integration
Exercise C, Question 14

Question:

Unless a substitution is given or asked for, use the standard results 7 to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

By using the substitution $u = x^2$ evaluate $\int_2^3 \frac{2x}{\sqrt{x^4 - 1}}\,dx$.

Solution:

With $u = x^2$ and $du = 2x\,dx$,

$$\int_2^3 \frac{2x}{\sqrt{x^4 - 1}}\,dx = \int_4^9 \frac{du}{\sqrt{u^2 - 1}}$$
$$= [ar\cosh u]_4^9$$
$$= ar\cosh 9 - \cosh 4$$
$$= 0.824 \quad (3\text{ s.f.})$$

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Integration
Exercise C, Question 15

Question:

By using the substitution $x = \frac{1}{2}\sin\theta$, show that $\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx = \frac{1}{192}(2\pi - 3\sqrt{3})$.

Solution:

With $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\ d\theta$

$1-4x^2 = 1-\sin^2\theta = \cos^2\theta$ and so $\frac{x^2}{\sqrt{1-4x^2}} = \frac{\sin^2\theta}{4\cos\theta}$

So $\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{4\cos\theta} \times \frac{1}{2}\cos\theta\, d\theta$

$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \sin^2\theta\ d\theta$

$= \frac{1}{16}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\,d\theta$

$= \frac{1}{16}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{16}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$

$= \frac{1}{192}\left(2\pi - 3\sqrt{3}\right)$

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Integration
Exercise C, Question 16

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

a Use the substitution $x = 2\cosh u$ to show that
$$\int \sqrt{x^2-4}\,dx = \frac{1}{2}x\sqrt{x^2-4} - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C.$$

b Find the area enclosed between the hyperbola with equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$ and the line $x = 4$.

Solution:

a Using $x = 2\cosh u$, $dx = 2\sinh u \, du$

$$\int \sqrt{x^2-4}\,dx = \int 2\sqrt{\cosh^2 u - 1} \times 2\sinh u \, du$$

$$= 4\int \sinh^2 u \, du$$

$$= 2\int (\cosh 2u - 1)\,du$$

$$= 2\left\{\frac{\sinh 2u}{2} - u\right\} + C$$

$$= 2\sinh u \cosh u - 2u + C$$

$$= 2\left(\sqrt{\left(\frac{x}{2}\right)^2 - 1}\right)\left(\frac{x}{2}\right) - 2\text{arcosh}\left(\frac{x}{2}\right) + C$$

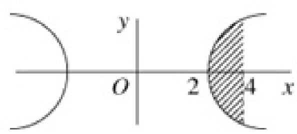
$$= 2\left(\frac{\sqrt{x^2-4}}{2}\right)\left(\frac{x}{2}\right) - 2\text{arcosh}\left(\frac{x}{2}\right) + C$$

$$= \frac{1}{2}x\sqrt{x^2-4} - 2\text{arcosh}\left(\frac{x}{2}\right) + C$$

$\cosh u = \frac{x}{2}$ and $\sinh u = \sqrt{\cosh^2 u - 1}$

b Area $= 2\int_2^4 y \, dx$

Rearranging $\frac{x^2}{4} - \frac{y^2}{9} = 1$ gives $9x^2 - 4y^2 = 36$



$4y^2 = 9x^2 - 36$

$= 9\left(x^2 - 4\right)$

So $y = \frac{3}{2}\sqrt{x^2-4}$, taking the +ve value, representing the part of curve in first quadrant

$$\text{Area} = 3\int_2^4 \sqrt{x^2-4}\,dx = \left[\frac{3}{2}x\sqrt{x^2-4} - 6\text{arcosh}\left(\frac{x}{2}\right)\right]_2^4$$

Using result from **a**

$$= \left[6\sqrt{12} - 6\text{arcosh}\,2\right] - \left[0 - 6\text{arcosh}\,1\right]$$

$$= 12.9 \text{ (3 s.f.)}$$

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Integration
Exercise C, Question 17

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

a Show that $\int \frac{1}{2\cosh x - \sinh x}\,dx$ can be written as $\int \frac{2e^x}{e^{2x}+3}\,dx$.

b Hence, by using the substitution $u = e^x$, find $\int \frac{1}{2\cosh x - \sinh x}\,dx$.

Solution:

a $2\cosh x - \sinh x = 2\left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + 3e^{-x}}{2}$

So $\int \frac{1}{2\cosh x - \sinh x}\,dx = \int \frac{2}{e^x + 3e^{-x}}\,dx$

$= \int \frac{2e^x}{e^{2x}+3}\,dx$

Multiplying numerator and denominator by e^x.

b Using the substitution $u = e^x$, $du = e^x\,dx$ and

$$\int \frac{2e^x}{e^{2x}+3}\,dx = 2\int \frac{du}{u^2+3}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right) + C$$

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Integration
Exercise C, Question 18

Question:

Using the substitution $u = \frac{2}{3}\sinh x$, evaluate $\int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x + 9}} \, dx$.

Solution:

With $u = \frac{2}{3}\sinh x, du = \frac{2}{3}\cosh x dx$ or $\cosh x dx = \frac{3}{2} du$

$$4\sinh^2 x + 9 = 4\left(\frac{3u}{2}\right)^2 + 9 = 9u^2 + 9 = 9\left(u^2 + 1\right)$$

$$\text{so } \int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x + 9}} \, dx = \int_0^{\frac{2}{3}\sinh 1} \frac{1}{3\sqrt{u^2 + 1}} \times \frac{3}{2} \, du$$

$$= \frac{1}{2}\text{arsinh}(u) \quad \text{between the given limits}$$

$$= \frac{1}{2}\text{arsinh}\left(\frac{2}{3}\sinh 1\right)$$

$$= 0.360 \text{ (3 s.f.)}$$

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Integration
Exercise C, Question 19

Question:

a Find $\int \frac{dx}{a^2 - x^2}$ $|x| < a$, by using

i partial fractions,

ii the substitution $x = a \tanh \theta$.

b Deduce the logarithmic form of $\text{artanh}\left(\frac{x}{a}\right)$.

Solution:

a i Using partial fractions $\frac{1}{a^2 - x^2} = \frac{1}{2a}\left\{\frac{1}{a-x} + \frac{1}{a+x}\right\}$

So $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int\left\{\frac{1}{a-x} + \frac{1}{a+x}\right\} dx$

$$= \frac{1}{2a}\left[-\ln|a-x| + \ln|a+x|\right] + C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

ii Using the substitution $x = a \tanh \theta, dx = a \operatorname{sech}^2\theta \, d\theta$

$$\int \frac{dx}{a^2 - x^2} = \int \frac{a \operatorname{sech}^2\theta}{a^2 \operatorname{sech}^2\theta} d\theta$$

$$= \frac{1}{a}\theta + D$$

$$= \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right) + D$$

b Using the result in a $\text{artanh}\left(\frac{x}{a}\right) = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + \text{constant}$

At $x = 0, 0 = 0 + \text{constant}, \Rightarrow \text{constant} = 0$ and so $\text{artanh}\left(\frac{x}{a}\right) = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right|$

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Integration
Exercise C, Question 20

Question:

Using the substitution $x = \sec\theta$, find

a $\int \frac{1}{x\sqrt{x^2-1}}\,dx$

b $\int \frac{\sqrt{x^2-1}}{x}\,dx$.

Solution:

With $x = \sec\theta$,

a
$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta$$
$$= \int 1\,d\theta$$
$$= \theta + C$$
$$= \text{arcsec}\,x + C$$

b
$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta$$
$$= \int \tan^2\theta\,d\theta$$
$$= \int \left(\sec^2\theta-1\right)d\theta$$
$$= \tan\theta - \theta + C$$
$$= \sqrt{\sec^2\theta-1} - \theta + C$$
$$= \sqrt{x^2-1} - \text{arcsec}\,x + C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 1

Question:

Find the following.

a $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$

b $\int \frac{1}{\sqrt{x^2-4x-12}}\,dx$

c $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx$

d $\int \frac{1}{\sqrt{x(x-2)}}\,dx$

e $\int \frac{1}{2x^2+4x+7}\,dx$

f $\int \frac{1}{\sqrt{-4x^2-12x}}\,dx$

g $\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx$

h $\int \frac{1}{\sqrt{9x^2-8x+1}}\,dx$

Solution:

a $5-4x-x^2=-(x^2+4x-5)=-\{(x+2)^2-9\}=9-(x+2)^2$

So $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx=\int \frac{1}{\sqrt{9-(x+2)^2}}\,dx$

Let $u=(x+2)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx=\int \frac{1}{\sqrt{9-u^2}}\,du$

$=\arcsin\left(\frac{u}{3}\right)+C$

$=\arcsin\left(\frac{x+2}{3}\right)+C$

b $x^2-4x-12=\{(x-2)^2-16\}$

So $\int \frac{1}{\sqrt{x^2-4x-12}}\,dx=\int \frac{1}{\sqrt{(x-2)^2-16}}\,dx$

Let $u=(x-2)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{x^2-4x-12}}\,dx=\int \frac{1}{\sqrt{u^2-16}}\,dx$

$=\text{arcosh}\left(\frac{u}{4}\right)+C$

$=\text{arcosh}\left(\frac{x-2}{4}\right)+C$

c $x^2+6x+10=\{(x+3)^2+1\}$

So $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx=\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$

Let $u=(x+3)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx=\int \frac{1}{\sqrt{u^2+1}}\,du$

$=\text{arsinh}(u)+C$

$=\text{arsinh}(x+3)+C$

d $x(x-2)=x^2-2x=\left\{(x-1)^2-1\right\}$

So $\displaystyle\int\frac{1}{\sqrt{x(x-2)}}\,dx=\int\frac{1}{\sqrt{(x-1)^2-1}}\,dx$

Let $u=(x-1)$, so $du=dx$.

Then $\displaystyle\int\frac{1}{\sqrt{x(x-2)}}\,dx=\int\frac{1}{\sqrt{u^2-1}}\,du$

$=\text{arcosh}\,(u)+C$

$=\text{arcosh}\,(x-1)+C$

e $2x^2+4x+7=2\left(x^2+2x+\frac{7}{2}\right)=2\left\{(x+1)^2+\frac{5}{2}\right\}$

Let $u=(x+1)$, so $du=dx$.

Then $\displaystyle\int\frac{1}{2x^2+4x+7}\,dx=\frac{1}{2}\int\frac{1}{u^2+\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2}\,du$

$=\frac{1}{2}\left\{\frac{\sqrt{2}}{\sqrt{5}}\arctan\left(\frac{\sqrt{2}u}{\sqrt{5}}\right)\right\}+C$

$=\frac{\sqrt{10}}{10}\arctan\left(\frac{\sqrt{2}\,(x+1)}{\sqrt{5}}\right)+C$

f $-4x^2-12x=-4\left(x^2+3x\right)=-4\left\{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}\right\}=4\left\{\frac{9}{4}-\left(x+\frac{3}{2}\right)^2\right\}$

So $\displaystyle\int\frac{1}{\sqrt{-4x^2-12x}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{3}{2}\right)^2}}\,dx$

Let $u=\left(x+\frac{3}{2}\right)$, so $du=dx$.

Then $\displaystyle\int\frac{1}{\sqrt{-4x^2-12x}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-u^2}}\,du$

$=\frac{1}{2}\arcsin\left(\frac{2u}{3}\right)+C$

$=\frac{1}{2}\arcsin\left(\frac{2x+3}{3}\right)+C$

g $14-12x-2x^2 = -2\left(x^2+6x-7\right)$

$$= -2\left(\left(x+3\right)^2-16\right)$$

$$= 2\left(16-\left(x+3\right)^2\right)$$

So $\displaystyle\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{4^2-\left(x+3\right)^2}}\,dx$

Let $u = x+3$, so $du = dx$

Then $\displaystyle\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{4^2-u^2}}\,du$

$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{u}{4}\right)+C$$

$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{x+3}{4}\right)+C$$

h $9x^2-8x+1 = 9\left(x^2-\frac{8}{9}x+\frac{1}{9}\right) = 9\left\{\left(x-\frac{4}{9}\right)^2-\frac{7}{81}\right\}$

So $\displaystyle\int \frac{1}{\sqrt{9x^2-8x+1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{\left(x-\frac{4}{9}\right)^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\,dx$

Let $u = \left(x-\frac{4}{9}\right)$, so $du = dx$.

Then $\displaystyle\int \frac{1}{\sqrt{9x^2-8x+1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\,du$

$$= \frac{1}{3}\operatorname{arcosh}\left(\frac{9u}{\sqrt{7}}\right)+C$$

$$= \frac{1}{3}\operatorname{arcosh}\left(\frac{9x-4}{\sqrt{7}}\right)+C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 2

Question:

Find

a $\int \frac{1}{\sqrt{4x^2-12x+10}}\,dx$

b $\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx$.

Solution:

a $4x^2-12x+10=4\left(x^2-3x+\frac{5}{2}\right)=4\left\{\left(x-\frac{3}{2}\right)^2+\frac{1}{4}\right\}$

So $\int \frac{1}{\sqrt{4x^2-12x+10}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{2}\right)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{4x^2-12x+10}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{u^2+\left(\frac{1}{2}\right)^2}}\,du$

$=\frac{1}{2}\text{arsinh}\,(2u)+C$

$=\frac{1}{2}\text{arsinh}\,(2x-3)+C$

b $4x^2-12x+4=4\left(x^2-3x+1\right)=4\left\{\left(x-\frac{3}{2}\right)^2-\frac{5}{4}\right\}$

So $\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{2}\right)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{u^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,du$

$=\frac{1}{2}\text{arcosh}\left(\frac{2u}{\sqrt{5}}\right)+C$

$=\frac{1}{2}\text{arcosh}\left(\frac{2x-3}{\sqrt{5}}\right)+C$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 3

Question:

Evaluate the following, giving answers to 3 significant figures.

a $\int_1^3 \frac{1}{\sqrt{x^2+2x+5}}\,dx$

b $\int_1^3 \frac{1}{x^2+x+1}\,dx$

c $\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx$

Solution:

a $x^2+2x+5=(x+1)^2+4$

So $\int_0^1 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_0^1 \frac{1}{\sqrt{(x+1)^2+4}}\,dx$

Let $u=(x+1)$, so $du=dx$.

Then $\int_0^1 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_1^2 \frac{1}{\sqrt{u^2+2^2}}\,du$

$$=\left[\text{arsinh}\left(\frac{u}{2}\right)\right]_1^2$$

$$=\left[\text{arsinh}1-\text{arsinh}\left(\frac{1}{2}\right)\right]$$

$$=0.400\ (3\text{ s.f.})$$

b $\int_1^3 \frac{1}{x^2+x+1}\,dx=\int_1^3 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

Let $u=\left(x+\frac{1}{2}\right)$, so $du=dx$.

Then $\int_1^3 \frac{1}{x^2+x+1}\,dx=\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$

$$=\left[\frac{2}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right)\right]_{\frac{3}{2}}^{\frac{7}{2}}$$

$$=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{7}{\sqrt{3}}\right)-\arctan\left(\sqrt{3}\right)\right]$$

$$=0.325\ (3\text{ s.f.})$$

c $2+3x-2x^2=-2\left(x^2-\frac{3}{2}x-1\right)=-2\left\{\left(x-\frac{3}{4}\right)^2-\frac{25}{16}\right\}=2\left\{\frac{25}{16}-\left(x-\frac{3}{4}\right)^2\right\}$

So $\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx=\frac{1}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{4}\right)$, so $du=dx$.

Then $\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx=\frac{1}{\sqrt{2}}\int_{-\frac{3}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2-u^2}}\,du$

$$=\frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{4u}{5}\right)\right]_{\frac{-3}{4}}^{\frac{1}{4}}$$

$$=\frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{1}{5}\right)-\arcsin\left(\frac{-3}{5}\right)\right]$$

$$=0.597\ (3\text{ s.f.})$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 4

Question:

Evaluate

a $\int_1^3 \frac{1}{\sqrt{x^2-2x+2}}\,dx$, giving your answer as a single natural logarithm,

b $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx$, giving your answer in the form $k\pi$.

Solution:

a $x^2-2x+2=(x-1)^2+1$

So $\int_1^3 \frac{1}{\sqrt{x^2-2x+2}}\,dx = \int_1^3 \frac{1}{\sqrt{(x-1)^2+1}}\,dx$

$= [\text{arsinh}(x-1)]_1^3$

$= \text{arsinh}\,2$

$= \ln\{2+\sqrt{5}\}$

$ar\sinh x = \ln\{x+\sqrt{x^2+1}\}$

b $1+6x-3x^2 = -3\left(x^2-2x-\frac{1}{3}\right) = -3\left\{(x-1)^2-\frac{4}{3}\right\} = 3\left[\frac{4}{3}-(x-1)^2\right]$

So $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int_1^2 \frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2-(x-1)^2}}\,dx$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{\sqrt{3}(x-1)}{2}\right)\right]_1^2$

$= \frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{3\sqrt{3}}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 5

Question:

Show that $\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}}\,dx = \frac{1}{\sqrt{3}}\ln(2+\sqrt{3})$.

Solution:

$$3x^2-6x+7 = 3\left(x^2-2x+\frac{7}{3}\right) = 3\left\{(x-1)^2+\frac{4}{3}\right\}$$

So $\int \frac{1}{\sqrt{3x^2-6x+7}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{(x-1)^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,dx$

Let $u=(x-1)$, so $du = dx$.

Then $\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}} = \frac{1}{\sqrt{3}}\int_0^2 \frac{1}{\sqrt{u^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,du$

$$= \frac{1}{\sqrt{3}}\left[\operatorname{arsinh}\left(\frac{\sqrt{3}u}{2}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{3}}\operatorname{arsinh}\sqrt{3}$$

$$= \frac{1}{\sqrt{3}}\ln\left\{\sqrt{3}+\sqrt{3+1}\right\} \qquad \operatorname{arsinh}x = \ln\left\{x+\sqrt{x^2+1}\right\}$$

$$= \frac{1}{\sqrt{3}}\ln\left\{2+\sqrt{3}\right\}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 6

Question:

Using a suitable hyperbolic or trigonometric substitution find

a $\int \frac{1}{\sqrt{x^2+4x+5}}\,dx$

b $\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx$.

Solution:

a $x^2+4x+5=(x+2)^2+1$

So let $(x+2)=\sinh u$, then $dx=\cosh u\,du$ and $(x+2)^2+1=\sinh^2 u+1=\cosh^2 u$

Then $\int \frac{1}{\sqrt{x^2+4x+5}}\,dx=\int \frac{1}{\cosh u}\cosh u\,du$

$$=\int 1\,du$$

$$=u+C$$

$$=\text{arsinh}(x+2)+C$$

b $-x^2+4x+5=-(x^2-4x-5)=-\{(x-2)^2-9\}=9-(x-2)^2$

So let $(x-2)=3\sin\theta$, then $dx=3\cos\theta\,d\theta$

and $9-(x-2)^2=9(1-\sin^2\theta)=9\cos^2\theta$

Then $\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx=\int \frac{1}{3\cos\theta}3\cos\theta\,d\theta$

$$=\int 1\,d\theta$$

$$=\theta+C$$

$$=\arcsin\left(\frac{x-2}{3}\right)+C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 7

Question:

Using the substitution $x = \frac{1}{5}(\sqrt{3}\tan\theta - 1)$, obtain $\int_{-0.2}^{0} \frac{1}{25x^2 + 10x + 4}\,dx$, giving your answer in terms of π.

Solution:

Using the substitution $x = \frac{1}{5}\left(\sqrt{3}\tan\theta - 1\right)$, $dx = \frac{\sqrt{3}}{5}\sec^2\theta\,d\theta$ and

$$25x^2 + 10x + 4 = \left(3\tan^2\theta - 2\sqrt{3}\tan\theta + 1\right) + 2\left(\sqrt{3}\tan\theta - 1\right) + 4$$
$$= 3\tan^2\theta + 3$$
$$= 3\left(\tan^2\theta + 1\right) = 3\sec^2\theta$$

Then $\int_{-0.2}^{0} \frac{1}{25x^2 + 10x + 4}\,dx = \frac{\sqrt{3}}{5}\int_{0}^{\frac{\pi}{6}} \frac{1}{3\sec^2\theta}\sec^2\theta\,d\theta$

$$= \frac{\sqrt{3}}{15}\int_{0}^{\frac{\pi}{6}} 1\,d\theta$$
$$= \frac{\pi\sqrt{3}}{90}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 8

Question:

Evaluate $\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,\mathrm{d}x$, giving your answer in the form $\ln(a+b\sqrt{c})$, where a, b and c are integers to be found.

Solution:

$(x-2)(x+4) = x^2+2x-8 = (x+1)^2-9$

So $\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,\mathrm{d}x = \int_3^4 \frac{1}{\sqrt{(x+1)^2-3^2}}\,\mathrm{d}x$

Let $u=(x+1)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,\mathrm{d}x = \int_4^5 \frac{1}{\sqrt{u^2-3^2}}\,\mathrm{d}u$

$= \left[\operatorname{arcosh}\left(\frac{u}{3}\right)\right]_4^5$

$= \operatorname{arcosh}\left(\frac{5}{3}\right) - \operatorname{arcosh}\left(\frac{4}{3}\right)$

$= \ln\left\{\left(\frac{5}{3}\right)+\sqrt{\frac{25}{9}-1}\right\} - \ln\left\{\left(\frac{4}{3}\right)+\sqrt{\frac{16}{9}-1}\right\}$ $\quad$ $\operatorname{ar}\cosh x = \ln\left\{x+\sqrt{x^2-1}\right\}$

$= \ln 3 - \ln\left\{\frac{4+\sqrt{7}}{3}\right\}$

$= \ln\left(\frac{9}{4+\sqrt{7}}\right)$ $\quad$ $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$= \ln\left(\frac{9\left(4-\sqrt{7}\right)}{9}\right)$ $\quad$ Rationalising the denominator

$= \ln\left(4-\sqrt{7}\right)$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 9

Question:

Using the substitution $x = 1 + \sinh\theta$, show that

$$\int \frac{x}{\left(x^2 - 2x + 2\right)^{\frac{3}{2}}}\,dx = \frac{x-1}{\sqrt{x^2 - 2x + 2}} + C.$$

Solution:

Using the substitution $x = 1 + \sinh\theta$, $dx = \cosh\theta\, d\theta$ and

$$x^2 - 2x + 2 = \left(\sinh^2\theta + 2\sinh\theta + 1\right) - 2\left(\sinh\theta + 1\right) + 2 = \sinh^2\theta + 1 = \cosh^2\theta$$

So
$$\int \frac{1}{\left(x^2 - 2x + 2\right)^{\frac{3}{2}}}\,dx = \int \frac{1}{\cosh^3\theta}\cdot\cosh\theta\, d\theta$$
$$= \int \operatorname{sech}^2\theta\, d\theta$$
$$= \tanh\theta + C$$
$$= \frac{x-1}{\sqrt{x^2 - 2x + 2}} + C$$

$$\sinh\theta = x - 1$$
$$\cosh\theta = \sqrt{1 + \sinh^2\theta} = \sqrt{2 - 2x + x^2}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 10

Question:

Use the substitution $x = 2\sin\theta - 1$ to find $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

Solution:

Using the substitution $x = 2\sin\theta - 1, dx = 2\cos\theta\, d\theta$

$$\text{and } 3-2x-x^2 = 3-2(2\sin\theta-1)-(4\sin^2\theta-4\sin\theta+1)$$
$$= 4-4\sin^2\theta$$
$$= 4\cos^2\theta$$

$$\text{So } \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{2\sin\theta-1}{2\cos\theta}\cdot 2\cos\theta\,d\theta$$
$$= \int (2\sin\theta-1)\,d\theta$$
$$= -2\cos\theta - \theta + C$$
$$= -2\sqrt{1-\left(\frac{x+1}{2}\right)^2} - \theta + C$$
$$= -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

$\cos\theta = \sqrt{1-\sin^2\theta}$
and $\sin\theta = \frac{x+1}{2}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 1

Question:

a Show that $\int \text{arsinh}x \, dx = x\text{arsinh}x - \sqrt{1+x^2} + C$.

b Evaluate $\int_0^1 \text{arsinh}x \, dx$, giving your answer to 3 significant figures.

c Using the substitution $u = 2x+1$ and the result in **a**, or otherwise, find $\int \text{arsinh}(2x+1)\, dx$.

Solution:

a $I = \int 1 \cdot \text{arsinh}x \, dx$

Let $u = \text{arsinh}x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{x^2+1}} \quad v = x$

So $I = x\text{arsinh}x - \int \frac{x}{\sqrt{x^2+1}} \, dx$ ← Using integration by parts

$= x\text{arsinh}x - \sqrt{x^2+1} + C$ ← Using $\int f^n(x) f'(x) dx = \frac{1}{n+1} f^{n+1}(x) + C, n \neq -1$

b $\int_0^1 \text{arsinh}x = \left[x\text{arsinh}x - \sqrt{x^2+1}\right]_0^1$

$= \left[\text{arsinh}1 - \sqrt{2}\right] - [-1]$

$= 0.467$ (3 s.f.)

c Let $u = 2x+1$, so $du = dx$

Then $\int ar\sinh(2x+1)\, dx = \frac{1}{2}\int ar\sinh u \, dx$

$= \frac{1}{2} ar\sinh u - \sqrt{1+u^2} + C$ using a

$= \frac{1}{2}(2x+1)\sinh(2x+1) - \sqrt{4x^2+4x+2} + C$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 2

Question:

Show that $\int \arctan 3x \, \mathrm{d}x = x \arctan 3x - \frac{1}{6}\ln(1+9x^2) + C$.

Solution:

Let $u = \arctan 3x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{1+(3x)^2} \quad v = x$

Using $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$, where $u = \arctan t$ and $t = 3x$

Then $\int \arctan 3x \, \mathrm{d}x = x \arctan 3x - \int \frac{3x}{1+9x^2} \mathrm{d}x$

$$= x \arctan 3x - \frac{1}{6}\int \frac{18x}{1+9x^2} \mathrm{d}x$$

$$= x \arctan 3x - \frac{1}{6}\ln\left(1+9x^2\right) + C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 3

Question:

a Show that $\int \text{arcosh}x \, dx = x\text{arcosh}x - \sqrt{x^2 - 1} + C$.

b Hence show that $\int_1^2 \text{arcosh}x = \ln(7 + 4\sqrt{3}) - \sqrt{3}$.

Solution:

a Let $u = \text{arcosh}x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{x^2 - 1}} \quad v = x$

So $\int \text{arcosh}x \, dx = x\text{arcosh}x - \int \frac{x}{\sqrt{x^2 - 1}} dx$

$= x\text{arcosh}x - \sqrt{x^2 - 1} + C$

b Using limits

$\int_1^2 \text{arcosh}x = \left[2\text{arcosh}2 - \sqrt{3}\right] - \left[\text{arcosh}1\right] = \left[2\text{arcosh}2 - \sqrt{3}\right]$ as $\text{arcosh}1 = 0$

As $\text{arcosh}x = \ln\left\{x + \sqrt{x^2 - 1}\right\}$

$\int_1^2 \text{arcosh}x = \left[2\ln\left\{2 + \sqrt{3}\right\} - \sqrt{3}\right]$

$= \left[\ln\left\{2 + \sqrt{3}\right\}^2 - \sqrt{3}\right]$

$= \ln\left(7 + 4\sqrt{3}\right) - \sqrt{3}$

Solutionbank FP3
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Integration
Exercise E, Question 4

Question:

a Show that $\int \arctan x \, dx = x \arctan x - \frac{1}{2}\ln(1+x^2) + C$.

b Hence show that $\int_{-1}^{\sqrt{3}} \arctan x \, dx = \frac{(4\sqrt{3}-3)\pi}{12} - \frac{1}{2}\ln 2$.

The curve C has equation $y = 2\arctan x$. The region R is enclosed by C, the y-axis, the line $y = \pi$ and the line $x = 3$.

c Find the area of R, giving your answer to 3 significant figures.

Solution:

a $I = \int 1 \times \arctan x \, dx$

Let $u = \arctan x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{1+x^2} \quad v = x$

So $I = x\arctan x - \int \frac{x}{1+x^2}\,dx$ — Using integration by parts

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$ — Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$

b $\int_{-1}^{\sqrt{3}} \arctan x = \left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]_{-1}^{\sqrt{3}}$

$= \left[\sqrt{3}\arctan\sqrt{3} - \frac{1}{2}\ln 4\right] - \left[-\arctan(-1) - \frac{1}{2}\ln 2\right]$

$= \frac{\sqrt{3}\pi}{3} - \ln 2 + \left(-\frac{\pi}{4}\right) + \frac{1}{2}\ln 2$

$= \frac{(4\sqrt{3}-3)\pi}{12} - \frac{1}{2}\ln 2$

c Area of R = area of rectangle $- \int_0^3 2\arctan x \, dx$

$= 3\pi - 2\left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]_0^3$ — Using a

$= 3\pi - 6\arctan 3 + \ln 10$

$= 4.23$ (3 s.f.)

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 5

Question:

Evaluate

a $\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \, dx$

b $\int_0^1 x \arctan x \, dx$ giving your answers in terms of π.

Solution:

a Let $u = \arcsin x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad v = x$

Then $\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \, dx = [x \arcsin x]_0^{\frac{\sqrt{2}}{2}} - \int_0^{\frac{\sqrt{2}}{2}} \frac{x}{\sqrt{1-x^2}} \, dx$

$$= \left[x \arcsin x + \sqrt{1-x^2} \right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \left[\frac{\sqrt{2}}{2} \frac{\pi}{4} + \sqrt{\frac{1}{2}} \right] - [+1]$$

$$= \frac{\sqrt{2}}{8}\pi - 1 + \frac{\sqrt{2}}{2} = 0.262 \text{ (3 s.f.)}$$

b Let $u = \arctan x \quad \frac{dv}{dx} = x$

So $\frac{du}{dx} = \frac{1}{1+x^2} \quad v = \frac{x^2}{2}$

Then $\int_0^1 x \arctan x \, dx = \left[\frac{x^2}{2} \arctan x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx$

$$= \left[\frac{1}{2} \arctan 1 \right]_0^1 - \frac{1}{2} \int_0^1 \frac{1+x^2-1}{1+x^2} \, dx$$

$$= \left[\frac{\pi}{8} \right] - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \left[\frac{\pi}{8} \right] - \frac{1}{2} [x - \arctan x]_0^1$$

$$= \left[\frac{\pi}{8} \right] - \frac{1}{2} \left[1 - \frac{\pi}{4} \right]$$

$$= \frac{\pi - 2}{4}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 6

Question:

Using the result that if $y = \text{arcsec}x$, then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$, show that

$\int \text{arcsec}x \, dx = x\,\text{arcsec}x - \ln\{x + \sqrt{x^2-1}\} + C$.

Solution:

Let $u = \text{arcsec}x$ $\quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{x\sqrt{x^2-1}}$ $\quad v = x$

and $\int \text{arcsec}x \, dx = x\,\text{arcsec}x - \int \frac{x}{x\sqrt{x^2-1}} \, dx$

$$= x\,\text{arcsec}x - \text{arcosh}x + C$$

$$= x\,\text{arcsec}x - \ln\left\{x + \sqrt{x^2-1}\right\} + C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 7

Question:

a Show that $\int \text{arsinh}(2x+1)\,dx = X\text{arsinh}(2x+1) - \int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx$.

b Find $\int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx$, using the substitution $2x+1 = \sin\text{H}u$, and hence find $\int \arcsin(2x+1)\,dx$.

Solution:

a Let $u = \text{arsinh}(2x+1)$ $\quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{2}{\sqrt{(2x+1)^2+1}}$ $\quad v = x$

Then $\int \text{arsinh}(2x+1)\,dx = x\text{arsinh}(2x+1) - \int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx$

b Let $2x+1 = \sinh u$ then $2\,dx = \cosh u\,du$

So $\int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx = \frac{1}{2}\int \frac{(\sinh u - 1)}{\cosh u}\cosh u\,du$

$$= \frac{1}{2}\left[\int \sinh u\,du - u\right]$$

$$= \frac{1}{2}\left[\cosh u - u\right] + C$$

$$= \frac{1}{2}\left\{\sqrt{1+(2x+1)^2} - \text{arsinh}(2x+1)\right\} + C$$

$\int \text{arsinh}(2x+1)\,dx = x\text{arsinh}(2x+1) + \frac{1}{2}\text{arsinh}(2x+1) - \frac{1}{2}\sqrt{1+(2x+1)^2} + C$ $\quad$ Using **a** and **b**.

$$= \frac{1}{2}(2x+1)\text{arsinh}(2x+1) - \frac{1}{2}\sqrt{1+(2x+1)^2} + C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 1

Question:

Given that $I_n = \int x^n e^{\frac{x}{2}} dx$,

a show that $I_n = 2x^n e^{\frac{x}{2}} - 2nI_{n-1}, n \ge 1$.

b Hence find $\int x^3 e^{\frac{x}{2}} dx$.

Solution:

a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = e^{\frac{x}{2}}$

so $\frac{du}{dx} = nx^{n-1}, v = 2e^{\frac{x}{2}}$

So $I_n = 2x^n e^{\frac{x}{2}} - \int 2nx^{n-1} e^{\frac{x}{2}} \, dx$

$= 2x^n e^{\frac{x}{2}} - 2n \int x^{n-1} e^{\frac{x}{2}} \, dx$

$= 2x^n e^{\frac{x}{2}} - 2nI_{n-1}$ *

b $I_3 = 2x^3 e^{\frac{x}{2}} - 6I_2$

$= 2x^3 e^{\frac{x}{2}} - 6\left(2x^2 e^{\frac{x}{2}} - 4I_1\right)$

$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 24\left(2xe^{\frac{x}{2}} - 2I_0\right)$, where $I_0 = \int e^{\frac{x}{2}} \, dx = 2e^{\frac{x}{2}} + C$

$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 48I_0$

Substituting $n = 3, 2$ and 1 respectively in *

So $\int x^3 e^{\frac{x}{2}} \, dx = 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 2

Question:

Given that $I_n = \int_1^e x(\ln x)^n \, dx, n \in N$,

a show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}, n \in N$.

b Hence show that $\int_1^e x(\ln x)^4 \, dx = \frac{e^2 - 3}{4}$.

Solution:

a Let $u = (\ln x)^n$ and $\frac{dv}{dx} = x$, so $\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}, v = \frac{x^2}{2}$

Integration by parts:

$$\int_1^e x(\ln x)^n \, dx = \left[\frac{x^2(\ln x)^n}{2}\right]_1^e - \int_1^e \frac{nx^2(\ln x)^{n-1}}{2x} \, dx$$

$$= \left[\frac{e^2}{2} - 0\right] - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$$

So $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ *

b $\int_1^e x(\ln x)^4 \, dx = I_4$

Substituting $n = 4, 3, 2$ and 1 respectively in the reduction formula *

$$I_4 = \frac{e^2}{2} - \frac{4}{2} I_3$$

$$= \frac{e^2}{2} - 2\left(\frac{e^2}{2} - \frac{3}{2} I_2\right)$$

$$= \frac{e^2}{2} - e^2 + 3\left(\frac{e^2}{2} - \frac{2}{2} I_1\right)$$

$$= \frac{e^2}{2} - e^2 + \frac{3e^2}{2} - 3\left(\frac{e^2}{2} - \frac{1}{2} I_0\right), \text{ where } I_0 = \int_1^e x \, dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}$$

So $\int_1^e x(\ln x)^4 \, dx = \frac{e^2}{2} - e^2 + \frac{3e^2}{2} - \frac{3e^2}{2} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$

$$= \frac{e^2}{4} - \frac{3}{4} = \frac{e^2 - 3}{4}$$

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Integration
Exercise F, Question 3

Question:

In Example 21, you saw that, if $I_n = \int_0^1 x^n \sqrt{1-x}\,dx$, then $I_n = \frac{2n}{2n+3} I_{n-1}, n \geq 1$.

Use this reduction formula to evaluate $\int_0^1 (x+1)(x+2)\sqrt{1-x}\,dx$

Solution:

$$\int_0^1 \left[(x+1)(x+2)\sqrt{1-x}\right] dx = \int_0^1 \left[(x^2+3x+2)\sqrt{1-x}\right] dx$$

$$= \int_0^1 \left[x^2\sqrt{1-x}\right] dx + \int_0^1 \left[3x\sqrt{1-x}\right] dx + \int_0^1 \left[2\sqrt{1-x}\right] dx$$

$$= I_2 + 3I_1 + 2I_0$$

Now $I_0 = \int_0^1 \sqrt{1-x}\,dx = \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$

$I_1 = \frac{2}{5} I_0 = \left(\frac{2}{5}\right)\left(\frac{2}{3}\right) = \frac{4}{15}$ ← Using the given formula with $n = 1$

$I_2 = \frac{4}{7} I_1 = \left(\frac{4}{7}\right)\left(\frac{4}{15}\right) = \frac{16}{105}$ ← Using the given formula with $n = 2$

So $\int_0^1 \left[(x+1)(x+2)\sqrt{1-x}\right] dx = \frac{16}{105} + 3\left(\frac{4}{15}\right) + 2\left(\frac{2}{3}\right)$

$$= \frac{16 + 12(7) + 4(35)}{105}$$

$$= \frac{240}{105} = \frac{16}{7}$$

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Integration
Exercise F, Question 4

Question:

Given that $I_n = \int x^n e^{-x}\, dx$, where n is a positive integer,

a show that $I_n = -x^n e^{-x} + nI_{n-1}$, $n \geq 1$.

b Find $\int x^3 e^{-x}\, dx$.

c Evaluate $\int_0^1 x^4 e^{-x} dx$, giving your answer in terms of e.

Solution:

a Using integration by parts with $u = x^n$ and $\frac{dv}{dx} = e^{-x}$

so $\frac{du}{dx} = nx^{n-1}$ and $v = -e^{-x}$

$\int x^n e^{-x}\, dx = -x^n e^{-x} - \int -nx^{n-1} e^{-x}\, dx$, so $I_n = -x^n e^{-x} + nI_{n-1}$

b Repeatedly using the reduction formula to find I_3

$$\begin{aligned} I_3 &= -x^3 e^{-x} + 3I_2 \\ &= -x^3 e^{-x} + 3\left(-x^2 e^{-x} + 2I_1\right) \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6I_1 \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6\left(-xe^{-x} + I_0\right) \end{aligned}$$

But $I_0 = \int e^{-x}\, dx = -e^{-x} + C$

So $I_3 = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + K$

c $I_4 = -x^4 e^{-x} + 4I_3$

$= -x^4 e^{-x} + 4\left(-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + C\right)$ Using the result from **b**

So $\int_0^1 x^4 e^{-x}\, dx = \left[-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^1$

$= \left[-65e^{-1}\right] - \left[-24\right]$

$= 24 - 65e^{-1}$ or $\frac{24e - 65}{e}$

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Integration
Exercise F, Question 5

Question:

$I_n = \int \tanh^n x \, dx$,

a By writing $\tanh^n x = \tanh^{n-2} x \tanh^2 x$, show that for $n \geq 2$,

$I_n = I_{n-2} - \frac{1}{n-1} \tanh^{n-1} x$.

b Find $\int \tanh^5 x \, dx$.

c Show that $\int_0^{\ln 2} \tanh^4 x \, dx = \ln 2 - \frac{84}{125}$.

Solution:

a $I_n = \int \tanh^n x \, dx = \int \tanh^{n-2} x \tanh^2 x \, dx$

$= \int \tanh^{n-2} x \left(1 - \text{sech}^2 x\right) dx$

$= \int \tanh^{n-2} x - \int \tanh^{n-2} \text{sech}^2 x \, dx$

Using $1 - \tan h^2 x = \sec h^2 x$

So $I_n = I_{n-2} - \frac{1}{n-1} \tanh^{n-1} x, \quad n \neq 1$

b $\int \tanh^5 x \, dx = I_5 = I_3 - \frac{1}{4} \tanh^4 x$

$= \left(I_1 - \frac{1}{2} \tanh^2 x \right) - \frac{1}{4} \tanh^4 x$

$= \int \tanh x \, dx - \frac{1}{2} \tanh^2 x - \frac{1}{4} \tanh^4 x$

$= \ln \cosh x - \frac{1}{2} \tanh^2 x - \frac{1}{4} \tanh^4 x + C$

c As $\int \tanh^n x \, dx = \int \tanh^{n-2} x dx - \frac{1}{n-1} \tanh^{n-1} x$, it follows that

$\int_0^{\ln 2} \tanh^n x \, dx = \int_0^{\ln 2} \tanh^{n-2} x \, dx - \left[\frac{1}{n-1} \tanh^{n-1} x \right]_0^{\ln 2} \quad *$

Now $\tanh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} = \frac{2 - \frac{1}{2}}{2 + \frac{1}{2}} = \frac{3}{5}$

Reminder: $e^{-\ln a} = e^{\ln a^{-1}} = a^{-1}$

So $\int_0^{\ln 2} \tanh^4 x \, dx = \int_0^{\ln 2} \tanh^2 x \, dx - \frac{1}{3} \times \left(\frac{3}{5} \right)^3$

Using $*$ with $n = 4$ and $\tanh(\ln 2) = \frac{3}{5}$

$= \left[\int_0^{\ln 2} \tanh^0 x \, dx - 1 \times \left(\frac{3}{5} \right) \right] - \frac{1}{3} \times \frac{27}{125}$

Using $*$ with $n = 2$ and $\tanh(\ln 2) = \frac{3}{5}$

$= \ln 2 - \frac{3}{5} - \frac{9}{125}$

$= \ln 2 - \frac{84}{125}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 6

Question:

Given that $\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$ (derived in Example 23)

a find $\int \tan^4 x\, dx$.

b Evaluate $\int_0^{\frac{\pi}{4}} \tan^5 x\, dx$.

c Show that $\int_0^{\frac{\pi}{3}} \tan^6 x\, dx = \frac{9\sqrt{3}}{5} - \frac{\pi}{3}$.

Solution:

a $\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx$

$= \frac{1}{3}\tan^3 x - \left(\tan x - \int \tan^0 x\,dx\right)$

$= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

b $\int_0^{\frac{\pi}{4}} \tan^n x\,dx = \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx = \frac{1}{n-1} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$

Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, then $I_n = \frac{1}{n-1} - I_{n-2}$

$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left(\frac{1}{2} - I_1\right) = \frac{1}{4} - \frac{1}{2} + \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{1}{4} - \frac{1}{2} + [\ln\sec x]_0^{\frac{\pi}{4}}$

$= -\frac{1}{4} + \left(\ln\sqrt{2} - \ln 1\right)$

So $\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = \ln\sqrt{2} - \frac{1}{4}$

c Defining $J_n = \int_0^{\frac{\pi}{3}} \tan^n x\,dx$,

$J_n = \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\frac{\pi}{3}} - J_{n-2} = \frac{\left(\sqrt{3}\right)^{n-1}}{n-1} - J_{n-2}$

So $J_6 = \frac{\left(\sqrt{3}\right)^5}{5} - J_4 = \frac{\left(\sqrt{3}\right)^5}{5} - \left(\frac{\left(\sqrt{3}\right)^3}{3} - J_2\right) = \frac{\left(\sqrt{3}\right)^5}{5} - \frac{\left(\sqrt{3}\right)^3}{3} + \left(\frac{\sqrt{3}}{1} - J_0\right)$

As $J_0 = \int_0^{\frac{\pi}{3}} 1\,dx = \frac{\pi}{3}$, $\int_0^{\frac{\pi}{3}} \tan^6 x\,dx = \frac{9\sqrt{3}}{5} - \frac{3\sqrt{3}}{3} + \sqrt{3} - \frac{\pi}{3} = \frac{9\sqrt{3}}{5} - \frac{\pi}{3}$

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Integration
Exercise F, Question 7

Question:

Given that $I_n = \int_1^a (\ln x)^n \, dx$, where $a > 1$ is a constant,

a show that, for $n \geq 1, I_n = a(\ln a)^n - nI_{n-1}$.

b Find the exact value of $\int_1^2 (\ln x)^3 \, dx$.

c Show that $\int_1^e (\ln x)^6 \, dx = 5(53e - 144)$.

Solution:

a $I_n = \int_1^a (\ln x)^n \, dx = \int_1^a 1(\ln x)^n \, dx$

Let $u = (\ln x)^n$ and $\frac{dv}{dx} = 1$, so $\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}, v = x$

Integration by parts:

$$\int_1^a (\ln x)^n \, dx = \left[x(\ln x)^n\right]_1^a - \int_1^a \frac{n(\ln x)^{n-1}}{x} x \, dx$$

$$= \left[a(\ln a)^n - 0\right] - n\int_1^a (\ln x)^{n-1} \, dx$$

So $I_n = a(\ln a)^n - nI_{n-1}$

b Putting $a = 2, I_n = \int_1^2 (\ln x)^n \, dx = 2(\ln 2)^n - nI_{n-1}$

$$I_3 = \int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 3I_2$$

$$= 2(\ln 2)^3 - 3\left\{2(\ln 2)^2 - 2I_1\right\}$$

$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6\left\{2(\ln 2) - I_0\right\}$$

$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6I_0$$

As $I_0 = \int_1^2 1 \, dx = [x]_1^2 = 1,$

$$\int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6$$

c Putting $a = e, I_n = \int_1^e (\ln x)^n \, dx = e(\ln e)^n - nI_{n-1} = e - nI_{n-1}$

$$I_6 = \int_1^e (\ln x)^6 \, dx = e - 6I_5$$

$$= e - 6(e - 5I_4)$$

$$= e - 6e + 30(e - 4I_3)$$

$$= e - 6e + 30e - 120(e - 3I_2)$$

$$= e - 6e + 30e - 120e + 360(e - 2I_1)$$

$$= e - 6e + 30e - 120e + 360e - 720(e - I_0)$$

As $I_0 = \int_1^e 1 \, dx = [x]_1^e = e - 1,$

$$\int_1^e (\ln x)^6 \, dx = e - 6e + 30e - 120e + 360e - 720e + 720(e - 1)$$

$$= 265e - 720$$

$$= 5(53e - 144)$$

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Integration
Exercise F, Question 8

Question:

Using the results given in Example 22, evaluate

a $\int_0^{\frac{\pi}{2}} \sin^7 x \, dx$

b $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$

c $\int_0^1 x^5 \sqrt{1-x^2} \, dx$, using the substitution $x = \sin\theta$

d $\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt$, using a suitable substitution.

Solution:

a $I_7 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$

b $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \left(1 - \sin^2 x\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) dx$

$= I_2 - 2I_4 + I_6$

$I_2 = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}; \quad I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}; \quad I_6 = \frac{5}{6} I_4 = \frac{5\pi}{32}$

So $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx = \frac{\pi}{4} - \frac{3\pi}{8} + \frac{5\pi}{32} = \frac{\pi}{32}$

c Using $x = \sin\theta$, $\int_0^1 x^5 \sqrt{1-x^2} \, dx = \int_0^{\frac{\pi}{2}} \sin^5\theta \cos\theta \left(\cos\theta \, d\theta\right)$

$= \int_0^{\frac{\pi}{2}} \sin^5 x \left(1 - \sin^2 x\right) dx = I_5 - I_7$

$I_5 = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15}$ and $I_7 = \frac{16}{35}$ from **a**

So $\int_0^1 x^5 \sqrt{1-x^2} \, dx = \frac{8}{15} - \frac{16}{35} = \frac{56-48}{105} = \frac{8}{105}$

d Using $x = 3t$, $\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt = \int_0^{\frac{\pi}{2}} \sin^8 x \left(\frac{1}{3} dx\right) = \frac{1}{3} I_8$

$= \frac{1}{3} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{35\pi}{768}$

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Integration
Exercise F, Question 9

Question:

Given that $I_n = \int \frac{\sin^{2n} x}{\cos x}\,dx$,

a write down a similar expression for I_{n+1} and hence show that $I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1}$.

b Find $\int \frac{\sin^4 x}{\cos x}\,dx$ and hence show that $\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos x}\,dx = \ln\left(1+\sqrt{2}\right) - \frac{7\sqrt{2}}{12}$.

Solution:

a $I_{n+1} = \int \frac{\sin^{2n+2} x}{\cos x}\,dx$

So $I_n - I_{n+1} = \int \frac{\sin^{2n} x - \sin^{2n+2} x}{\cos x}\,dx$

$= \int \frac{\sin^{2n} x(1-\sin^2 x)}{\cos x}\,dx$

$= \int \sin^{2n} x \cos x\,dx$ — as $1-\sin^2 x = \cos^2 x$

So $I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1}$

or $I_{n+1} = I_n - \frac{\sin^{2n+1} x}{2n+1}$ * — [+C not necessary at this stage]

b i $\int \frac{\sin^4 x}{\cos x}\,dx = I_2$

Substituting $n=1$ in * gives $I_2 = I_1 - \frac{\sin^3 x}{3}$

$= \left(I_0 - \frac{\sin x}{1}\right) - \frac{\sin^3 x}{3}$ using $n = 0$ in *

$I_0 = \int \frac{1}{\cos x}\,dx = \int \sec x\,dx = \ln|(\sec x + \tan x)| + C$

So $\int \frac{\sin^4 x}{\cos x}\,dx = \ln|(\sec x + \tan x)| - \sin x - \frac{\sin^3 x}{3} + C$

Applying the given limits gives

$$\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos x}\,dx = \left[\ln|(\sec x + \tan x)| - \sin x - \frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{4}}$$

$$= \ln(1+\sqrt{2}) - \frac{\sqrt{2}}{2} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3}$$

$$= \ln(1+\sqrt{2}) - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}$$

$$= \ln(1+\sqrt{2}) - \frac{7\sqrt{2}}{12}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 10

Question:

a Given that $I_n = \int_0^1 x(1-x^3)^n \, dx$, show that $I_n = \frac{3n}{3n+2} I_{n-1}, n \geq 1$.

b Use your reduction formula to evaluate I_4.

Hint: After integrating by parts, write x^4 as $x(1-(1-x^3))$

Solution:

a Let $u = (1-x^3)^n$ and $\frac{dv}{dx} = x$, so $\frac{du}{dx} = n(1-x^3)^{n-1}(-3x^2), v = \frac{x^2}{2}$

Integration by parts gives

$$\int_0^1 x(1-x^3)^n \, dx = \left[\frac{x^2}{2}(1-x^3)^n\right]_0^1 - \int_0^1 -3nx^2(1-x^3)^{n-1} \frac{x^2}{2} \, dx$$

$$= [0-0] + \frac{3n}{2}\int_0^1 x^4(1-x^3)^{n-1} \, dx \quad \text{providing } n \geq 0$$

Writing $x^4 = x.x^3 = x\{1-(1-x^3)\}$ and $I_n = \int_0^1 x(1-x^3)^n \, dx$

we have $I_n = \frac{3n}{2}\int_0^1 x\{1-(1-x^3)\}(1-x^3)^{n-1} \, dx$

$$= \frac{3n}{2}\int_0^1 x(1-x^3)^{n-1} \, dx - \frac{3n}{2}\int_0^1 x(1-x^3)^n \, dx$$

$$= \frac{3n}{2} I_{n-1} - \frac{3n}{2} I_n$$

$\Rightarrow (3n+2)I_n = 3nI_{n-1}$, so $I_n = \frac{3n}{3n+2} I_{n-1}, n \geq 1$

b $I_4 = \frac{12}{14} I_3 = \frac{12}{14} \times \frac{9}{11} I_2 = \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} I_1 = \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} I_0 = \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} \int_0^1 x \, dx$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} \times \frac{1}{2} = \frac{243}{1540}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 11

Question:

Given that $I_n = \int_0^a (a^2 - x^2)^n \, dx$, where a is a positive constant,

a show that, for $n > 0$, $I_n = \dfrac{2na^2}{2n+1} I_{n-1}$.

b Use the reduction formula to evaluate

i $\int_0^1 (1 - x^2)^4 \, dx$

ii $\int_0^3 (9 - x^2)^3 \, dx$

iii $\int_0^2 \sqrt{4 - x^2} \, dx$.

c Check your answer to part **b iii** by using another method.

Solution:

a Integrating by parts with $u=\left(a^2-x^2\right)^n$ and $\dfrac{dv}{dx}=1$

$\dfrac{du}{dx}=-2nx\left(a^2-x^2\right)^{n-1} \quad v=x$

So $\displaystyle\int_0^a\left(a^2-x^2\right)^n dx=\left[x\left(a^2-x^2\right)^n\right]_0^a-\int_0^a x\left\{-2nx\left(a^2-x^2\right)^{n-1}\right\}dx$

$\displaystyle=\left[0-0\right]+2n\int_0^a x^2\left(a^2-x^2\right)^{n-1}dx=2n\int_0^a x^2\left(a^2-x^2\right)^{n-1}dx$ (if $n>0$)

Writing x^2 as $\left\{a^2-\left(a^2-x^2\right)\right\}$ and defining $I_n=\displaystyle\int_0^a\left(a^2-x^2\right)^n dx$,

we have

$\displaystyle I_n=2n\int_0^a\left\{a^2\left(a^2-x^2\right)^{n-1}-\left(a^2-x^2\right)^n\right\}dx$

$=2na^2I_{n-1}-2nI_n$

So $(2n+1)I_n=2na^2I_{n-1}$

b i With $a=1$, $I_n=\displaystyle\int_0^1\left(1-x^2\right)^n dx$ and $I_n=\dfrac{2n}{2n+1}I_{n-1}$

So $I_4=\dfrac{8}{9}I_3=\dfrac{8}{9}\times\dfrac{6}{7}I_2=\dfrac{8}{9}\times\dfrac{6}{7}\times\dfrac{4}{5}I_1=\dfrac{8}{9}\times\dfrac{6}{7}\times\dfrac{4}{5}\times\dfrac{2}{3}\times1=\dfrac{128}{315}$

$\mathrm{I}_0=\displaystyle\int_0^a dx=a$

ii With $a=3, I_n=\displaystyle\int_0^3\left(9-x^2\right)^n dx$ and $I_n=\dfrac{18n}{2n+1}I_{n-1}$

So $I_3=\dfrac{54}{7}I_2=\dfrac{54}{7}\times\dfrac{36}{5}I_1=\dfrac{54}{7}\times\dfrac{36}{5}\times\dfrac{18}{3}I_0=\dfrac{54}{7}\times\dfrac{36}{5}\times\dfrac{18}{3}\times3=\dfrac{34\,992}{35}$

iii With $a=2, I_n=\displaystyle\int_0^2\left(4-x^2\right)^n dx$ and $I_n=\dfrac{8n}{2n+1}I_{n-1}$

So $I_{\frac{1}{2}}=\dfrac{4}{2}I_{o\frac{1}{2}}=2\displaystyle\int_0^2\frac{dx}{\sqrt{4-x^2}}=2\left[\arcsin\left(\frac{x}{2}\right)\right]_0^2=2\arcsin1=2\times\frac{\pi}{2}=\pi$

c Using the substitution $x=2\sin\theta$,

$\displaystyle\int_0^2\left(4-x^2\right)^{\frac{1}{2}}dx=\int_0^{\frac{\pi}{2}}(2\cos\theta)(2\cos\theta\, d\theta)$

$\displaystyle=2\int_0^{\frac{\pi}{2}}(1+\cos2\theta)d\theta$

$=\left[2\theta+\sin2\theta\right]_0^{\frac{\pi}{2}}=\pi$

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Integration
Exercise F, Question 12

Question:

Given that $I_n = \int_0^4 x^n \sqrt{4-x}\,dx$,

a establish the reduction formula $I_n = \frac{8n}{2n+3} I_{n-1}, n \geq 1$.

b Evaluate $\int_0^4 x^3 \sqrt{4-x}$, giving your answer correct to 3 significant figures.

Solution:

a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \sqrt{4-x}$

$\frac{du}{dx} = nx^{n-1}, \quad v = -\frac{2}{3}(4-x)^{\frac{3}{2}}$

So $\int_0^4 x^n \sqrt{4-x}\,dx = \left[-\frac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_0^4 - \int_0^4 -\frac{2}{3}nx^{n-1}(4-x)^{\frac{3}{2}}\,dx$

$$= [0-0] + \frac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,dx \ (n>0)$$

$$= \frac{2}{3}n\int_0^4 x^{n-1}\left\{(4-x)\sqrt{4-x}\right\}dx$$

You need to write $(4-x)^{\frac{3}{2}}$ as $(4-x)\sqrt{4-x}$

$$= \frac{2}{3}n\int_0^4 x^{n-1}4\sqrt{4-x}\,dx + \frac{2}{3}n\int_0^4 x^{n-1}\left\{-x\sqrt{4-x}\right\}dx$$

$$= \frac{8}{3}n\int_0^4 x^{n-1}\sqrt{4-x}\,dx - \frac{2}{3}n\int_0^4 x^n\sqrt{4-x}\,dx$$

So $I_n = \frac{8}{3}nI_{n-1} - \frac{2}{3}nI_n$

$\Rightarrow (2n+3)I_n = 8nI_{n-1} \leq I_n = \frac{8n}{2n+3}I_{n-1}, n \geq 1$

b $\int_0^4 x^3\sqrt{4-x}\,dx = I_3 = \frac{24}{9}I_2 = \frac{24}{9}\times\frac{16}{7}I_1 = \frac{24}{9}\times\frac{16}{7}\times\frac{8}{5}I_0 = \frac{1024}{105}I_0$

As $I_0 = \int_0^4 \sqrt{4-x}\,dx = \left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_0^4 = \left[0 - \left\{-\frac{2}{3}(4)^{\frac{3}{2}}\right\}\right] = \frac{16}{3}$,

$\int_0^4 x^3\sqrt{4-x}\,dx = \frac{1024}{105}\times\frac{16}{3} = 52.0\ (3\text{ s.f.})$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 13

Question:

Given that $I_n = \int \cos^n x \, dx$,

a establish, for $n \geq 2$, the reduction formula $nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$.

Defining $J_n = \int_0^{2\pi} \cos^n x \, dx$,

b write down a reduction formula relating J_n and J_{n-2}, for $n \geq 2$.

c Hence evaluate

 i J_4

 ii J_8.

d Show that if n is odd, J_n is always equal to zero.

Solution:

a $I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$

Integrating by parts with $u = \cos^{n-1} x$ and $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = (n-1)\cos^{n-2} x(-\sin x)$, $\quad v = \sin x$

So $I_n = \int \cos^n x \, dx = \cos^{n-1} x \sin x - \int -(n-1)\cos^{n-2} x \sin^2 x \, dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x \, dx$

Giving $I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n$

So $nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$

b It follows that $n\int_0^{2\pi} \cos^n x dx = \left[\cos^{n-1} x \sin x\right]_0^{2\pi} + (n-1)\int_0^{2\pi} \cos^{n-2} x \, dx$

So $nJ_n = (n-1)J_{n-2}$, as $\left[\cos^{n-1} x \sin x\right]_0^{2\pi} = 0$

c **i** $J_4 = \int_0^{2\pi} \cos^4 x dx = \frac{3}{4}J_2 = \frac{3}{4} \times \frac{1}{2}J_0 = \frac{3}{8}\int_0^{2\pi} 1 \, dx = \frac{3}{8} \times 2\pi = \frac{3\pi}{4}$

ii $J_8 = \int_0^{2\pi} \cos^8 x \, dx = \frac{7}{8}J_6 = \frac{7}{8} \times \frac{5}{6}J_4 = \frac{35}{48}J_4 = \frac{35}{48} \times \frac{3\pi}{4} = \frac{35\pi}{64}$ using **c i**

d If n is odd, J_n always reduces to a multiple of J_1,

but $J_1 = \int_0^{2\pi} \cos x dx = \left[\sin x\right]_0^{2\pi} = 0$.

(You could also consider the graphical representation.)

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 14

Question:

Given $I_n = \int_0^1 x^n \sqrt{(1-x^2)}\,dx, n \geq 0$,

a show that $(n+2)I_n = (n-1)I_{n-2}, n \geq 2$.

b Hence evaluate $\int_0^1 x^7 \sqrt{(1-x^2)}\,dx$.

Hint: Write $x^n\sqrt{1-x^2}$ as $x^{n-1}\left\{x\sqrt{1-x^2}\right\}$ before integrating by parts.

Solution:

a Integrating by parts with $u = x^{n-1}$ and $\frac{dv}{dx} = x\sqrt{1-x^2}$

Using the hint.

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}$

So $I_n = \int_0^1 x^{n-1}\left\{x\sqrt{1-x^2}\right\}dx = \left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 + \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx$

$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx$ as $\left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 = 0$

$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$

$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} \left\{x^{n-2}\sqrt{1-x^2} - x^n\sqrt{1-x^2}\right\}dx$

So $I_n = \frac{(n-1)}{3}I_{n-2} - \frac{(n-1)}{3}I_n$

$\Rightarrow \{3+(n-1)\}I_n = (n-1)I_{n-2}$

$\Rightarrow (n+)I_n = (n-1)I_{n-2}$ *

b Using * $I_7 = \frac{6}{9}I_5 = \frac{6}{9}\times\frac{4}{7}I_3 = \frac{6}{9}\times\frac{4}{7}\times\frac{2}{5}I_1 = \frac{48}{315}\int_0^1 x\sqrt{1-x^2}\,dx$

$= \frac{48}{315}\left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= \frac{48}{315}\left[\frac{1}{3}\right] = \frac{16}{315}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 15

Question:

Given $I_n = \int x^n \cosh x \, dx$

a show that for $n \geq 2$, $I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1)I_{n-2}$

b Find $I_4 = \int x^4 \cosh x \, dx$.

c Evaluate $\int_0^1 x^3 \cosh x$, giving your answer in terms of e.

Solution:

a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \cosh x$

$\frac{du}{dx} = nx^{n-1}, \quad v = \sinh x$

So $\int x^n \cosh x dx = x^n \sinh x - \int nx^{n-1} \sinh x dx$

Integrating by parts again with $u = x^{n-1}$ and $\frac{dv}{dx} = \sinh x$

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = \cosh x$

So $I_n = x^n \sinh x - n\left\{x^{n-1} \cosh x - \int (n-1)x^{n-2} \cosh x dx\right\}$

$= x^n \sinh x - nx^{n-1} \cosh x + n(n-1)I_{n-2}, \quad n \geq 2$ *

b $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2$, ← Substituting $n = 4$ in *

$= x^4 \sinh x - 4x^3 \cosh x + 12\{x^2 \sinh x - 2x \cosh x + 2I_0\}$

$= x^4 \sinh x - 4x^3 \cosh x + 12\{x^2 \sinh x - 2x \cosh x\} + 24\int \cosh x dx$

$= x^4 \sinh x - 4x^3 \cosh x + 12\{x^2 \sinh x - 2x \cosh x\} + 24 \sinh x + C$

$= (x^4 + 12x^2 + 24)\sinh x - (4x^3 + 24x)\cosh x + C$

c $\int_0^1 x^3 \cosh x dx = \left[x^3 \sinh x - 3x^2 \cosh x\right]_0^1 + 6\int_0^1 x \cosh x dx$ — Using a

$= \{\sinh 1 - 3\cosh 1\} + 6\left\{\left[x \sinh x\right]_0^1 - \int_0^1 1 \sinh x dx\right\}$ — Integrating by parts

$= \{\sinh 1 - 3\cosh 1\} + 6\{\sinh 1 - [\cosh 1 - 1]\}$

$= 7\sinh 1 - 9\cosh 1 + 6$

$= 7\left(\frac{e^1 - e^{-1}}{2}\right) - 9\left(\frac{e^1 + e^{-1}}{2}\right) + 6$

$= 6 - e - 8e^{-1}$ or $\frac{6e - e^2 - 8}{e}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 16

Question:

Given that $I_n = \int \frac{\sin nx}{\sin x}\,dx, n > 0$,

a write down a similar expression for I_{n-2}, and hence show that

$$I_n - I_{n-2} = \frac{2\sin(n-1)x}{n-1}.$$

b Find

i $\int \frac{\sin 4x}{\sin x}\,dx$

ii the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x}\,dx$.

Solution:

a $I_{n-2} = \int \frac{\sin(n-2)x}{\sin x}\,dx$

So $I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x}\,dx$

$$= \int \frac{2\cos\left\{\frac{n+(n-2)}{2}\right\}x \sin\left\{\frac{n-(n-2)}{2}\right\}x}{\sin x}\,dx$$

Using Edexcel formula booklet

$$= \int \frac{2\cos(n-1)x\sin x}{\sin x}\,dx$$

$$= \int 2\cos(n-1)x\,dx$$

$$= \frac{2\sin(n-1)x}{n-1}, n \geq 2$$

It is not necessary to have $+C$.

b **i** $\int \frac{\sin 4x}{\sin x}\,dx = I_4$

Using a with $n = 4$: $I_4 = I_2 + \frac{2\sin 3x}{3}$

$I_2 = \int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx$

$$= \int 2\cos x\,dx + \frac{2\sin 3x}{3}$$

$$= 2\sin x + \frac{2\sin 3x}{3} + C$$

ii Using a with $n = 5$: $I_5 = I_3 + \frac{2\sin 4x}{4}$

$$= \left\{I_1 + \frac{2\sin 2x}{2}\right\} + \frac{2\sin 4x}{4}$$

$$= \int 1\,dx + \sin 2x + \frac{\sin 4x}{2}$$

$$= x + \sin 2x + \frac{\sin 4x}{2}$$

It follows that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x}\,dx = \left[x + \sin 2x + \frac{\sin 4x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$$= \left[\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right] - \left[\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi - 3\sqrt{3}}{6}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 17

Question:

Given that $I_n = \int \sinh^n x \, dx, n \in N$,

a derive the reduction formula $nI_n = \sinh^{n-1} x \cosh x - (n-1)I_{n-2}, n \geq 2$.

b Hence

i evaluate $\int_0^{\ln 3} \sinh^5 x \, dx$,

ii show that $\int_0^{\text{arsinh} 1} \sinh^4 x \, dx = \frac{1}{8}\{3\ln(1+\sqrt{2}) - \sqrt{2}\}$.

Solution:

a $I_n = \int \sinh^n x\,dx = \int \sinh^{n-1} x \sinh x\,dx$

Integrating by parts with $u = \sinh^{n-1} x$ and $\frac{dv}{dx} = \sinh x$

$\frac{du}{dx} = (n-1)\sinh^{n-2} x \cosh x, \quad v = \cosh x$

So $I_n = \int \sinh^n x\,dx = \sinh^{n-1} x \cosh x - \int (n-1)\sinh^{n-2} x \cosh^2 x\,dx$

$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\left(1+\sinh^2 x\right)dx$

$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\,dx - (n-1)\int \sinh^n x\,dx$

Giving $I_n = \sinh^{n-1} x \cosh x - (n-1)I_{n-2} - (n-1)I_n$

So $nI_n = \sinh^{n-1} x \cosh x - (n-1)I, \quad n \geq 2 \qquad *$

b i $I_5 = \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{5}I_3$ ← using * with $n = 5$

$= \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{5}\left\{\frac{1}{3}\sinh^2 x \cosh x - \frac{2}{3}I_1\right\}$

$= \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{15}\sinh^2 x \cosh x + \frac{8}{15}\int \sinh x\,dx$

$= \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{15}\sinh^2 x \cosh x + \frac{8}{15}\cosh x + C$

When $x = \ln 3, \sinh x = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3-\frac{1}{3}}{2} = \frac{4}{3}, \cosh x = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3+\frac{1}{3}}{2} = \frac{5}{3}$

When $x = 0$, $\sinh x = 0, \cosh x = 1$

Applying the limits 0 and ln 3 to the result in **b**

$$\int_0^{\ln 3} \sinh^5 x\,dx = \left[\frac{1}{5}\left(\frac{4}{3}\right)^4\left(\frac{5}{3}\right) - \frac{4}{15}\left(\frac{4}{3}\right)^2\left(\frac{5}{3}\right) + \frac{8}{15}\left(\frac{5}{3}\right)\right] - \left[0 + 0 + \frac{8}{15}\right]$$

$$= \frac{752}{1215} = 0.619 \text{ (3 s.f.)}$$

ii $\int \sinh^4 x\,dx = I_4 = \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{4}I_2$ ← Using * with $n = 4$

$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{4}\left\{\frac{1}{2}\sinh x \cosh x - \frac{1}{2}I_0\right\}$

$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}\int 1\,dx$

$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}x + C$

When $x = \text{arsinh}1 \quad \sinh x = 1, \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{2}$

When $x = 0 \quad \sinh x = 0 \quad \cosh x = 1$

Applying the limits 0 and arsinh 1 gives

$$\int_0^{\text{arsinh}1} \sinh^4 x \, dx = \frac{1}{4}(1)^3(\sqrt{2}) - \frac{3}{8}(1)(\sqrt{2}) + \frac{3}{8}\text{arsinh}1$$

$$= \frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{8} + \frac{3}{8}\ln\left(1+\sqrt{1^2+1}\right)$$

$$= -\frac{\sqrt{2}}{8} + \frac{3}{8}\ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{8}\left\{3\ln\left(1+\sqrt{2}\right) - \sqrt{2}\right\}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 1

Question:

Find the length of the arc of the curve with equation $y = \frac{1}{3}x^{\frac{3}{2}}$, from the origin to the point with x-coordinate 12.

Solution:

$y = \frac{1}{3}x^{\frac{3}{2}}$, so $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}}$

$$\text{Arc length} = \int_0^{12} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{12} \sqrt{1 + \frac{x}{4}}\, dx$$

$$= \frac{1}{2}\int_0^{12} \sqrt{4 + x}\, dx$$

$$= \frac{1}{2}\left[\frac{2}{3}(4 + x)^{\frac{3}{2}}\right]_0^{12}$$

$$= \frac{1}{3}\left[16^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}[64 - 8]$$

$$= \frac{56}{3} \text{ or } 18\frac{2}{3}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 2

Question:

The curve C has equation $y = \ln \cos x$. Find the length of the arc of C between the points with x-coordinates 0 and $\frac{\pi}{3}$.

Solution:

$y = \ln \cos x$, so $\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

$$\text{Arc length} = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\frac{\pi}{3}} \sec x\, dx$$

$$= \left[\ln\left(\sec x + \tan x\right)\right]_0^{\frac{\pi}{3}}$$

$$= \ln\left(2 + \sqrt{3}\right)$$

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Integration
Exercise G, Question 3

Question:

Find the length of the arc on the catenary, with equation $y = 2\cosh\left(\frac{x}{2}\right)$, between the points with x-coordinates 0 and $\ln 4$.

Solution:

$y = 2\cosh\left(\frac{x}{2}\right)$, so $\frac{dy}{dx} = \sinh\left(\frac{x}{2}\right)$

$$\text{arc length} = \int_0^{\ln 4} \sqrt{1+\sinh^2\left(\frac{x}{2}\right)}\,dx$$
$$= \int_0^{\ln 4} \cosh\left(\frac{x}{2}\right) dx$$
$$= \left[2\sinh\left(\frac{x}{2}\right)\right]_0^{\ln 4}$$
$$= 2\frac{e^{\frac{\ln 4}{2}} - e^{-\frac{\ln 4}{2}}}{2}$$
$$= e^{\ln 2} - e^{-\ln 2}$$
$$= 2 - \frac{1}{2} = \frac{3}{2}$$

As $\ln 4 = \ln 2^2 = 2\ln 2$

As $e^{\ln k} = k$; $e^{-\ln k} = e^{\ln k^{-1}} = k^{-1}$

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Integration
Exercise G, Question 4

Question:

Find the length of the arc of the curve with equation $y^2 = \frac{4}{9}x^3$, from the origin to the point $(3, 2\sqrt{3})$.

Solution:

$y^2 = \frac{4}{9}x^3$, so $2y\frac{dy}{dx} = \frac{4}{3}x^2 \Rightarrow \frac{dy}{dx} = \frac{2x^2}{3y} = \pm\frac{x^2}{x^{\frac{3}{2}}} = \pm\sqrt{x}$

The arc in question is above the x-axis.

$$\text{arc length} = \int_0^3 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3$$

$$= \frac{2}{3}[8-1] = 4\frac{2}{3}$$

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Integration
Exercise G, Question 5

Question:

The curve C has equation $y=\frac{1}{2}\sinh^2 2x$. Find the length of the arc on C from the origin to the point whose x-coordinate is 1, giving your answer to 3 significant figures.

Solution:

$y=\frac{1}{2}\sinh^2 2x$, so $\frac{dy}{dx}=2\sinh 2x\cosh 2x=\sinh 4x$

$$\text{So arc length }=\int_0^1\sqrt{1+\sinh^2 4x}\,dx$$

$$=\int_0^1\cosh 4x\,dx$$

$$=\frac{1}{4}\left[\sinh 4x\right]_0^1$$

$$=\frac{1}{4}\sinh 4=6.82\quad(3\text{ s.f.})$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 6

Question:

The curve C has equation $y=\frac{1}{4}(2x^2-\ln x)$, $x>0$. The points A and B on C have x-coordinates 1 and 2 respectively. Show that the length of the arc from A to B is $\frac{1}{4}(6+\ln 2)$.

Solution:

$y=\frac{1}{4}\left(2x^2-\ln x\right)$, so $\frac{dy}{dx}=x-\frac{1}{4x}$

$$1+\left(\frac{dy}{dx}\right)^2=1+x^2-\frac{1}{2}+\frac{1}{16x^2}=x^2+\frac{1}{2}+\frac{1}{16x^2}=\left(x+\frac{1}{4x}\right)^2$$

$$\text{So arc length }=\int_1^2\left(x+\frac{1}{4x}\right)dx$$

$$=\left[\frac{x^2}{2}+\frac{1}{4}\ln x\right]_1^2$$

$$=\left[2+\frac{1}{4}\ln 2\right]-\left[\frac{1}{2}\right]$$

$$=\frac{1}{4}(6+\ln 2)$$

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Integration
Exercise G, Question 7

Question:

Find the length of the arc on the curve $y = 2\text{arcosh}\left(\frac{x}{2}\right)$, from the point at which the curve crosses the x-axis to the point with x-coordinate $\frac{5}{2}$. Compare your answer with that in Example 25 and explain the relationship.

Solution:

$$y = 2\text{arcosh}\left(\frac{x}{2}\right), \text{ so } \frac{dy}{dx} = 2 \times \frac{1}{2}\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - 1}} = \frac{2}{\sqrt{x^2-4}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4}{x^2-4} = \frac{x^2}{x^2-4}$$

The curve crosses the x-axis at $x = 2$,

So arc length $= \int_2^{\frac{5}{2}} x\left(x^2-4\right)^{-\frac{1}{2}}\,dx$

$= \left[\sqrt{x^2-4}\right]_2^{2.5}$

$= 1.5$

Eliminating t from the two equations in Example 25, you find that the Cartesian equation is $\frac{x}{2} = \cosh\left(\frac{y}{2}\right)$. For $t \geq 1$, the curve is $y = 2\text{arcosh}\left(\frac{x}{2}\right)$. The limits in both questions correspond, and so they are essentially the same question.

[For $0 < t < 1$, the reflection of $y = 2\text{arcosh}\left(\frac{x}{2}\right)$ in the x-axis is generated.]

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Integration
Exercise G, Question 8

Question:

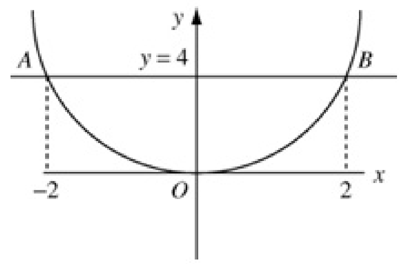
The line $y=4$ intersects the parabola with equation $y=x^2$ at the points A and B. Find the length of the arc of the parabola from A to B.

Solution:

The line $y=4$ intersects the parabola with equation $y=x^2$ where $x=-2$ and $x=+2$.

Using symmetry arc length $= 2\int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= 2\int_0^2 \sqrt{1+4x^2}\,dx$



Using the substitution $2x=\sinh u$, so that $2dx=\cosh u\,du$,

$$\text{arc length} = \int_0^{\operatorname{arsinh}4} \sqrt{1+\sinh^2 u}\,\cosh u\,du$$
$$= \int_0^{\operatorname{arsinh}4} \cosh^2 u\,du$$
$$= \int_0^{\operatorname{arsinh}4} \frac{(1+\cosh 2u)}{2}\,du$$
$$= \frac{1}{2}\left[u+\frac{1}{2}\sinh 2u\right]_0^{\operatorname{arsinh}4}$$
$$= \frac{1}{2}\left[u+\sinh u\cosh u\right]_0^{\operatorname{arsinh}4}$$
$$= \frac{1}{2}\operatorname{arsinh}4+\frac{1}{2}\left(4\sqrt{1+16}\right)$$

← Using $\cosh u=\sqrt{1+\sinh^2 u}$ and $\sinh u=4$

$$= \frac{1}{2}\operatorname{arsinh}4+2\sqrt{17}$$
$$= \frac{1}{2}\ln\left(4+\sqrt{17}\right)+2\sqrt{17}$$

← Using $\operatorname{arsinh}x=\ln\left\{x+\sqrt{(1+x^2)}\right\}$

$$= 9.29 \text{ (3 s.f.)}$$

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Integration
Exercise G, Question 9

Question:

The circle C has parametric equations $x = r\cos\theta, y = r\sin\theta$. Use the formula for arc length on page 79 for to show that the length of the circumference is $2\pi r$.

Solution:

As $x = r\cos\theta, y = r\sin\theta, \frac{dx}{d\theta} = -r\sin\theta, \frac{dy}{d\theta} = r\cos\theta$

So $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

The circumference of the circle $= 4\int_0^{\frac{\pi}{2}} r\, d\theta$

Using symmetry.

$= 4r[\theta]_0^{\frac{\pi}{2}}$

$= 2\pi r$

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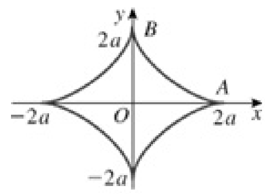
Integration
Exercise G, Question 10

Question:

The diagram shows the astroid, with parametric equations $x = 2a\cos^3 t, y = 2a\sin^3 t, 0 \le t < 2\pi$.
Find the length of the arc of the curve AB, and hence find the total length of the curve.



Solution:

$x = 2a\cos^3 t, y = 2a\sin^3 t$, so $\frac{dx}{dt} = -6a\cos^2 t\sin t, \frac{dy}{dt} = 6a\sin^2 t\cos t$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t) = 36a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)$$
$$= 36a^2\sin^2 t\cos^2 t$$

At A, $t = 0$, at B, $t = \frac{\pi}{2}$,

so arc length $AB = \int_0^{\frac{\pi}{2}} 6a\sin t\cos t\,dt$

$$= 3a\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$
$$= \frac{3}{2}a\left[-\cos 2t\right]_0^{\frac{\pi}{2}}$$
$$= \frac{3}{2}a\left[1 - (-1)\right]$$
$$= 3a$$

Total length of curve $= 4 \times 3a = 12a$ (symmetry)

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Integration
Exercise G, Question 11

Question:

Calculate the length of the arc on the curve with parametric equations $x = \tanh u$, $y = \text{sech}\, u$, between the points with parameters $u = 0$ and $u = 1$.

Solution:

$x = \tanh u, y = \text{sech}\, u$, so $\frac{dx}{du} = \text{sech}^2 u, \frac{dy}{du} = -\text{sech}\, u \tanh u$,

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 = \text{sech}^4 u + \text{sech}^2 u \tanh^2 u = \text{sech}^2 u \left(\text{sech}^2 u + \tanh^2 u\right) = \text{sech}^2 u$$

So arc length $= \int_0^1 \text{sech}\, u \, du$ ← See Example 7.

$$= \int_0^1 \frac{2}{e^u + e^{-u}} \, du$$

$$= \int_0^1 \frac{2e^u}{(e^u)^2 + 1} \, du$$

$$= 2\left[\arctan\left(e^u\right)\right]_0^1$$

$$= 2\arctan(e) - \frac{\pi}{2} \text{ or } 0.866 \quad (3 \text{ s.f.})$$

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Integration
Exercise G, Question 12

Question:

The cycloid has parametric equations $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$. Find the length of the arc from $\theta = 0$ to $\theta = \pi$.

Solution:

As $x = a(\theta + \sin\theta), y = a(1-\cos\theta), \frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a\sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\left(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\right)$$
$$= a^2(2 + 2\cos\theta)$$
$$= 4a^2\cos^2\left(\frac{\theta}{2}\right)$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = \left(\frac{\theta}{2}\right)$

So arc length $= 2a\int_0^{\pi}\cos\left(\frac{\theta}{2}\right)d\theta$
$$= 4a\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi}$$
$$= 4a$$

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Integration
Exercise G, Question 13

Question:

Show that the length of the arc, between the points with parameters $t = 0$ and $t = \frac{\pi}{3}$ on the curve defined by the equations $x = t + \sin t, y = 1 - \cos t$, is 2.

Solution:

$x = t + \sin t, y = 1 - \cos t$

$\frac{dx}{dt} = 1 + \cos t, \frac{dy}{dt} = \sin t$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left\{(1 + 2\cos t + \cos^2 t) + (\sin^2 t)\right\}$

$= 2(1 + \cos t) = 4\cos^2\left(\frac{t}{2}\right)$

Using $s = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

arc length $= \int_0^{\frac{\pi}{3}} \sqrt{4\cos^2\left(\frac{t}{2}\right)}\, dt$

$= 2\int_0^{\frac{\pi}{3}} \cos\left(\frac{t}{2}\right) dt$

$= 4\left[\sin\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{3}}$

$= 2$

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Integration
Exercise G, Question 14

Question:

Find the length of the arc of the curve given by the equations $x = e^t \cos t, y = e^t \sin t$, between the points with parameters $t = 0$ and $t = \frac{\pi}{4}$.

Solution:

$x = e^t \cos t, y = e^t \sin t$

$\frac{dx}{dt} = e^t(\cos t - \sin t), \frac{dy}{dt} = e^t(\sin t + \cos t)$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t)^2\{(\cos^2 t - 2\sin t\cos t + \sin^2 t) + (\sin^2 t + 2\sin t\cos t + \cos^2 t)\}$,

$= 2(e^t)^2(\sin^2 t + \cos^2 t)$

$= 2(e^t)^2$

$$\text{arc length} = \int_0^{\frac{\pi}{4}} \sqrt{2(e^t)^2}\, dt$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} e^t\, dt$$

$$= \sqrt{2}\left[e^t\right]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}\left[e^{\frac{\pi}{4}} - 1\right] \text{ or } 1.69 \quad (3 \text{ s.f.})$$

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Integration
Exercise G, Question 15

Question:

a Denoting the length of one complete wave of the sine curve with equation $y=\sqrt{3}\sin x$ by L, show that $L=4\int_0^{\frac{\pi}{2}}\sqrt{1+3\cos^2 x}\,dx$.

b The ellipse has parametric equations $x=\cos t, y=2\sin t$. Show that the length of its circumference is equal to that of the wave in **a**.

Solution:

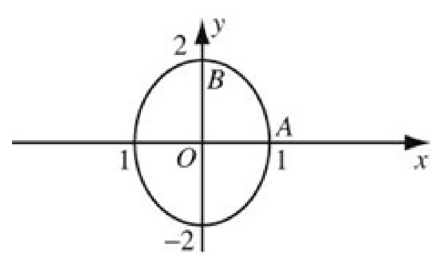
a $y=\sqrt{3}\sin x$, so $\frac{dy}{dx}=\sqrt{3}\cos x$

Using the symmetry of the sine curve $s=4\int_0^{\frac{\pi}{2}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{1+3\cos^2 x}\,dx$$

b $x=\cos t, y=2\sin t$

$\frac{dx}{dt}=-\sin t, \frac{dy}{dt}=2\cos t$



$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\sin^2 t+4\cos^2 t$

$=1-\cos^2 t+4\cos^2 t$

$=1+3\cos^2 t$

From the diagram, at A, $t=0$,

at B, $t=\frac{\pi}{2}$,

so using the symmetry of the ellipse, the length of the circumference is

$4\int_0^{\frac{\pi}{2}}\sqrt{1+3\cos^2 t}\,dt$, equal to that of the sine curve in **a**

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Integration
Exercise H, Question 1

Question:

a The section of the line $y=\frac{3}{4}x$ between points with x-coordinates 4 and 8 is rotated completely about the x-axis. Use integration to find the area of the surface generated.

b The same section of line is rotated completely about the y-axis. Show that the area of the surface generated is 60π.

Solution:

a $y=\frac{3}{4}x \Rightarrow \frac{dy}{dx}=\frac{3}{4} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2=\frac{25}{16}$

Surface area $=\int_4^8 2\pi\left(\frac{3}{4}x\right)\left(\frac{5}{4}\right)dx$ ← Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\frac{15}{8}\pi\int_4^8 x\,dx$

$=\frac{15}{8}\pi\left[\frac{x^2}{2}\right]_4^8=45\pi$

b Rotating about the y-axis:

From the work in **a** $1+\left(\frac{dx}{dy}\right)^2=1+\frac{16}{9}=\frac{25}{9}$

As integration is w.r.t. y, the integrand must be in terms of y
The limits for y are 3 (when $x=4$) and 6 (when $x=8$),

so area of surface is $\int_3^6 2\pi\left(\frac{4}{3}y\right)\left(\frac{5}{3}\right)dy$,

$=\frac{40}{9}\pi\left[\frac{y^2}{2}\right]_3^6$

$=\frac{40\times27}{9\times2}\pi=60\pi$

Although it is quicker to use $\int_a^8 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, here $\int_{y_1}^{y_2} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ is used to give an example of its use.

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Integration
Exercise H, Question 2

Question:

The arc of the curve $y = x^3$, between the origin and the point (1, 1), is rotated through 4 right-angles about the x-axis. Find the area of the surface generated.

Solution:

$y = x^3$ so $\frac{dy}{dx} = 3x^2$

Using $\int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\, dx$

$$= \frac{2\pi}{36}\int_0^1 36x^3\sqrt{1+9x^4}\, dx$$

$$= \frac{2\pi}{36}\left[\frac{2}{3}\left(1+9x^4\right)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{\pi}{27}\left[10\sqrt{10} - 1\right] \quad (3.56,\ 3\text{ s.f.})$$

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Integration
Exercise H, Question 3

Question:

The arc of the curve $y=\frac{1}{2}x^2$, between the origin and the point (2, 2), is rotated through 4 right-angles about the y-axis. Find the area of the surface generated.

Solution:

$y=\frac{1}{2}x^2$, so $\frac{dy}{dx}=x$

Using $\int_{x_1}^{x_2} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $\int_0^2 2\pi x\sqrt{1+x^2}\,dx$

$$=\pi\int_0^2 2x\sqrt{1+x^2}\,dx$$

$$=\pi\left[\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]_0^2$$

$$=\frac{2\pi}{3}\left[5\sqrt{5}-1\right]$$

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Integration
Exercise H, Question 4

Question:

The points A and B, in the first quadrant, on the curve $y^2 = 16x$ have x-coordinates 5 and 12 respectively. Find, in terms π, the area of the surface generated when the arc AB is rotated completely about the x-axis.

Solution:

$y^2 = 16x$ so $2y\dfrac{dy}{dx} = 16 \Rightarrow \dfrac{dy}{dx} = \dfrac{8}{y}$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{64}{y^2} = 1 + \dfrac{4}{x} = \dfrac{x+4}{x}$

Using $\displaystyle\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $\displaystyle\int_5^{12} 2\pi 4\sqrt{x}\sqrt{\frac{4+x}{x}}\,dx$

$$= 8\pi\int_5^{12}\sqrt{4+x}\,dx$$

$$= 8\pi\left[\frac{2}{3}(4+x)^{\frac{3}{2}}\right]_5^{12}$$

$$= \frac{16\pi}{3}[37]$$

$$= \frac{592\pi}{x}$$

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Integration
Exercise H, Question 5

Question:

The curve C has equation $y = \cosh x$. The arc s on C, has end points (0, 1) and (1, cosh 1).

a Find the area of the surface generated when s is rotated completely about the x-axis.

b Show that the area of the surface generated when s is rotated completely about the y-axis is $2\pi\left(\frac{e-1}{e}\right)$.

Solution:

$y = \cosh x$, so $\dfrac{dy}{dx} = \sinh x$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$

a Using $\displaystyle\int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\displaystyle\int_0^1 2\pi \cosh^2 x \, dx$

$$= \pi \int_0^1 (\cosh 2x + 1)\,dx$$

$$= \pi \left[\frac{\sinh 2x}{2} + x\right]_0^1$$

$$= \pi \left[\sinh x \cosh x + x\right]_0^1$$

$$= \pi \left[\sinh 1 \cosh 1 + 1\right]$$

$$= 8.84 \ (3 \text{ s.f.})$$

b Using $\displaystyle\int_{x_1}^{x_2} 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\displaystyle\int_0^1 2\pi x \cosh x \, dx$

$$= 2\pi \left\{\left[x \sinh x\right]_0^1 - \int_0^1 \sinh x \, dx\right\}$$

Using integration by parts

$$= 2\pi \left\{\sinh 1 - \left[\cosh x\right]_0^1\right\}$$

$$= 2\pi \left\{\sinh 1 - \cosh 1 + 1\right\}$$

$$= 2\pi \left\{\frac{1}{2}\left(e - \frac{1}{e} - e - \frac{1}{e}\right) + 1\right\}$$

$$= 2\pi \left(1 - \frac{1}{e}\right)$$

$$= 2\pi \left(\frac{e-1}{e}\right)$$

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Integration
Exercise H, Question 6

Question:

The curve C has equation $y = \frac{1}{2x} + \frac{x^3}{6}$.

a Show that $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$.

The arc of the curve between points with x-coordinates 1 and 3 is rotated completely about the x-axis.

b Find the area of the surface generated.

Solution:

a $y = \frac{1}{2x} + \frac{x^3}{6}$, so $\frac{dy}{dx} = -\frac{1}{2x^2} + \frac{x^2}{2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2$$

So $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$

b Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $\pi\int_1^3 \left(\frac{1}{2x} + \frac{x^3}{6}\right)\left(x^2 + \frac{1}{x^2}\right)dx$

$$= \pi\int_1^3 \left(\frac{2x}{3} + \frac{x^5}{6} + \frac{1}{2x^3}\right)dx$$

$$= \pi\left[\frac{x^2}{3} + \frac{x^6}{36} - \frac{1}{4x^2}\right]_1^3$$

$$= 23\frac{1}{9}\pi = 72.6 \text{ (3 s.f.)}$$

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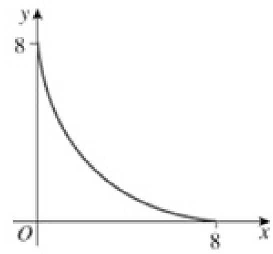
Integration
Exercise H, Question 7

Question:

The diagram shows part of the curve with equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$. Find the area of the surface generated when this arc is rotated completely about the y-axis.



Solution:

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, so $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

So $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{4}{x^{\frac{2}{3}}}$

Using $\int_{x_1}^{x_2} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $2\pi \int_0^8 x\left(\frac{2}{x^{\frac{1}{3}}}\right) dx$

$$= 2\pi \int_0^8 2x^{\frac{2}{3}}\, dx$$

$$= 2\pi \left[\frac{6}{5}x^{\frac{5}{3}}\right]_0^8$$

$$= \frac{12\pi}{5}[32]$$

$$= \frac{384\pi}{5} = 241\ (3\text{ s.f.})$$

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Integration
Exercise H, Question 8

Question:

a The arc of the circle with equation $x^2 + y^2 = R^2$, between the points $(-R, 0)$ and $(R, 0)$, is rotated through 2π radians about the x-axis. Use integration to find the surface area of the sphere S formed.

b The axis of a cylinder C of radius R is the x-axis. Show that the areas of the surface of S and C, contained between planes with equations $x = a$ and $x = b$, where $a < b < R$, are equal.

Solution:

a $x^2 + y^2 = R^2$, so $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

So $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{x^2 + y^2}{y^2} = \frac{R^2}{y^2}$

Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface of the sphere is $2\pi \int_{-R}^{R} y\left(\frac{R}{y}\right) dx$

$= 4\pi \int_0^R R\, dx$ ← Using the symmetry

$= 4\pi R[x]_0^R$

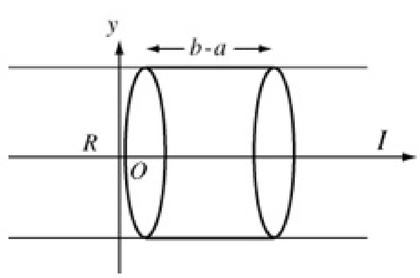
$= 4\pi R^2$

b The required area is $2\pi \int_a^b y\left(\frac{R}{y}\right) dx$ ← see diagram

$= 2\pi \int_a^b R\, dx$

$= 2\pi R(b - a)$

This is the same area as that of a cylinder of radius R and height $(b - a)$.



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Integration
Exercise H, Question 9

Question:

The finite arc of the parabola with parametric equations $x = at^2, y = 2at$, where a is a positive constant, cut off by the line $x = 4a$, is rotated through $180°$ about the x-axis. Show that the area of the surface generated is $\frac{8}{3}\pi a^2(5\sqrt{5}-1)$.

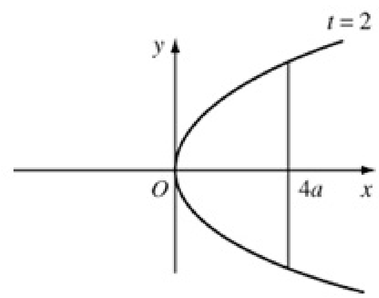
Solution:

$x = at^2, y = 2at$, so $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4a^2t^2 + 4a^2 = 4a^2(1+t^2)$

$x = 4a$ when $t = \pm 2$ (See diagram.)

A rotation of π radians gives a surface which would be found by rotating the section $y \geq 0$, i.e. $t = 0$ to $t = 2$ through 2π radians.



Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_0^2 4a^2 t\sqrt{1+t^2}\, dt$

$= 8\pi a^2 \left[\frac{1}{3}(1+t^2)^{\frac{3}{2}}\right]_0^2$

$= \frac{8}{3}\pi a^2 \left[5^{\frac{3}{2}} - 1\right]$

$= \frac{8}{3}\pi a^2 (5\sqrt{5} - 1)$

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Integration
Exercise H, Question 10

Question:

The arc, in the first quadrant, of the curve with parametric equations $x = \text{sech}\, t, y = \tanh t$, between the points where $t = 0$ and $t = \ln 2$, is rotated completely about the x-axis. Show that the area of the surface generated is $\frac{2\pi}{5}$.

Solution:

$x = \text{sech}\, t, y = \tanh t$, so $\frac{dx}{dt} = -\text{sech}\, t \tanh t, \frac{dy}{dt} = \text{sech}^2 t$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \text{sech}^2 t \tanh^2 t + \text{sech}^4 t = \text{sech}^2 t \left(\tanh^2 t + \text{sech}^2 t\right) = \text{sech}^2 t$

Using $\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_0^{\ln 2} \tanh t\, \text{sech}\, t\, dt$

$= 2\pi \left[-\text{sech}\, t\right]_0^{\ln 2}$

$= 2\pi \left[-\frac{2}{e^t + e^{-t}}\right]_0^{\ln 2}$

$= \frac{2\pi}{5}\left[\frac{-2}{2.5} + 1\right]$

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Integration
Exercise H, Question 11

Question:

The arc of the curve given by $x = 3t^2, y = 2t^3$, from $t = 0$ and $t = 2$, is completely rotated about the y-axis.

a Show that the area of the surface generated can be expressed as $36\pi \int_0^2 t^3 \sqrt{1+t^2}\, dt$.

b Using integration by parts, find the exact value of this area.

Solution:

a $x = 3t^2, y = 2t^3$, so $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2\left(t^2+1\right)$$

Using $\int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_0^2 3t^2 \times 6t\sqrt{1+t^2}\, dt$

$$= 36\pi \int_0^2 t^3\sqrt{1+t^2}\, dt$$

b Let $u = t^2, \frac{dv}{dt} = t\sqrt{1+t^2}$

So $\frac{du}{dt} = 2t, v = \frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}$

$$36\pi \int_0^2 t^2\left(t\sqrt{1+t^2}\right)dt = 36\pi\left\{\left[\frac{1}{3}t^2\left(1+t^2\right)^{\frac{3}{2}}\right]_0^2 - \int_0^2 \frac{2}{3}t\left(1+t^2\right)^{\frac{3}{2}} dt\right\}$$

$$= 12\pi\left[t^2\left(1+t^2\right)^{\frac{3}{2}} - \frac{2}{5}\left(1+t^2\right)^{\frac{5}{2}}\right]_0^2$$

$$= 12\pi\left[4\left(5\sqrt{5}\right) - \frac{2}{5}\left(25\sqrt{5}\right) + \frac{2}{5}\right]$$

$$= 12\pi\left[10\sqrt{5} + \frac{2}{5}\right]$$

$$= \frac{24\pi}{5}\left[25\sqrt{5}+1\right]$$

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Integration
Exercise H, Question 12

Question:

The arc of the curve with parametric equations $x = t^2, y = t - \frac{1}{3}t^3$, between the points where $t = 0$ and $t = 1$, is rotated through $360°$ about the x-axis. Calculate the area of the surface generated.

Solution:

$x = t^2, y = t - \frac{1}{3}t^3$, so $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 1 - t^2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 1 - 2t^2 + t^4 = (1 + t^2)^2$

Using $\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_0^1 \left(t - \frac{1}{3}t^3\right)(1 + t^2)\, dt$

$= 2\pi \int_0^1 \left(t + \frac{2}{3}t^3 - \frac{1}{3}t^5\right) dt$

$= 2\pi \left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^1$

$= \frac{11\pi}{9}$

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Integration
Exercise H, Question 13

Question:

The astroid C has parametric equations $x = a\cos^3 t, y = a\sin^3 t$, where a is a positive constant. The arc of C, between $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis. Find the area of the surface of revolution formed.

Solution:

$x = a\cos^3 t, y = a\sin^3 t$, so $\frac{dx}{dt} = -3a\cos^2 t\sin t$, $\frac{dy}{dt} = 3a\sin^2 t\cos t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)$$
$$= 9a^2\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right)$$
$$= 9a^2\sin^2 t\cos^2 t$$

Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a\sin^3 t\left(3a\sin t\cos t\right)dt$

$$= 6\pi a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^4 t\cos t\, dt$$

$$= 6\pi a^2\left[\frac{1}{5}\sin^5 t\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{6\pi a^2}{5}\left[1 - \frac{1}{32}\right]$$

$$= \frac{93\pi a^2}{80}$$

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Integration
Exercise H, Question 14

Question:

The part of the curve $y = e^x$, between $(0, 1)$ and $(\ln 2, 2)$, is rotated completely about the x-axis. Show that the area of the surface generated is $\pi(\text{arsinh}\,2 - \text{arsinh}\,1 + 2\sqrt{5} - \sqrt{2})$.

Solution:

$y = e^x, \frac{dy}{dx} = e^x$

Using $\int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $2\pi \int_0^{\ln 2} e^x \sqrt{1 + e^{2x}}\, dx$

Make the substitution $e^x = \sinh u$, so $e^x dx = \cosh u\, du$

Limits: when $x = \ln 2, u = \text{arsinh}\, e^{\ln 2} = \text{arsinh}\, 2$

when $x = 0, u = \text{arsinh}\, e^0 = \text{arsinh}\, 1$

Then the area of the surface is $2\pi \int_{\text{arsinh}\,1}^{\text{arsinh}\,2} \cosh^2 u\, du$

$= \pi \int_{\text{arsinh}\,1}^{\text{arsinh}\,2} (1 + \cosh 2u)\, du$

$= \pi \left[u + \frac{\sinh 2u}{2} \right]_{\text{arsinh}\,1}^{\text{arsinh}\,2}$

$= \pi \left[u + \sinh u \cosh u \right]_{\text{arsinh}\,1}^{\text{arsinh}\,2}$

$\cosh u = \sqrt{1 + \sinh^2 u}$

$= \pi \left[\text{arsinh}\,2 + 2\sqrt{5} - \left(\text{arsinh}\,1 + (1)\left(\sqrt{2}\right) \right) \right]_{\text{arsinh}\,1}^{\text{arsinh}\,2}$

$= \pi \left(\text{arsinh}\,2 - \text{arsinh}\,1 + 2\sqrt{5} - \sqrt{2} \right)$

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Integration
Exercise I, Question 1

Question:

Show that the volume of the solid generated when the finite region enclosed by the curve with equation $y = \tanh x$, the line $x = 1$ and the x-axis is rotated through 2π radians about the x-axis is $\frac{2\pi}{1+e^2}$. **[E]**

Solution:

$$\text{Volume} = \pi\int_0^1 y^2\,dx = \pi\int_0^1 \tanh^2 x\,dx$$
$$= \pi\int_0^1 (1-\operatorname{sech}^2 x)\,dx$$
$$= \pi\left[x - \tanh x\right]_0^1$$
$$= \pi(1-\tanh 1)$$
$$= \pi\left(1-\frac{e^2-1}{e^2+1}\right)$$
$$= \frac{2\pi}{1+e^2}$$

$$\tanh 1 = \frac{e-e^{-1}}{e+e^{-1}} = \frac{e^2-1}{e^2+1}$$

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Integration
Exercise I, Question 2

Question:

$4x^2+4x+17 \equiv (ax+b)^2+c, a>0$.

a Find the values of a, b and c.

b Find the exact value of $\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,dx$ **[E]**

Solution:

$4x^2+4x+17 \equiv (ax+b)^2+c, \quad a>0$

a $4x^2+4x+17 \equiv (2x+b)^2+c \quad a=2$

$\equiv 4x^2+4bx+b^2+c$

Comparing coefficient of x: $b=1$

Comparing constant term: $17=1+c \Rightarrow c=16$

b Using **a**, $\int \frac{1}{4x^2+4x+17}\,dx = \int \frac{1}{(2x+1)^2+16}\,dx$

Let $2x+1=4\tan\theta$, then $2dx=4\sec^2\theta d\theta$

and $\int \frac{1}{(2x+1)^2+16}\,dx = \int \frac{2\sec^2\theta}{16\tan^2\theta+16}\,d\theta$

$= \int \frac{2\sec^2\theta}{16\sec^2\theta}\,d\theta$

$= \frac{1}{8}\theta + C$

$= \frac{1}{8}\arctan\left(\frac{2x+1}{4}\right)+C$

So $\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,dx = \frac{1}{8}\left[\arctan 1 - \arctan 0\right]$

$= \frac{\pi}{32}$

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Integration
Exercise I, Question 3

Question:

Find the following.

a $\int \sinh 4x \cosh 6x \, dx$

b $\int \frac{\text{sech}\, x \tanh x}{1 + 2\text{sech}\, x} dx$

c $\int e^x \sinh x \, dx$

Solution:

a Using the definitions of $\sinh 4x$ and $\cosh 6x$

$$\int \sinh 4x \cosh 6x \, dx = \int \left(\frac{e^{4x} - e^{-4x}}{2}\right)\left(\frac{e^{6x} + e^{-6x}}{2}\right) dx$$

$$= \frac{1}{4}\int \left(e^{10x} + e^{-2x} - e^{2x} - e^{-10x}\right) dx$$

You could use hyperbolic identities to split up into a difference of two sinhs.

$$= \frac{1}{4}\left\{\frac{e^{10x}}{10} + \frac{e^{-2x}}{-2} - \frac{e^{2x}}{2} - \frac{e^{-10x}}{-10}\right\} + C$$

$$= \frac{1}{4}\left\{\frac{e^{10x}}{10} + \frac{e^{-10x}}{10} - \frac{e^{2x}}{2} - \frac{e^{-2x}}{2}\right\} + C$$

$$= \frac{1}{20}\cosh 10x - \frac{1}{4}\cosh 2x + C$$

as $\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

b $$\int \frac{\text{sech}x \tanh x}{1 + 2\text{sech}x} dx = -\frac{1}{2}\int \frac{-2\text{sech}x \tanh x}{1 + 2\text{sech}x} dx = -\frac{1}{2}\ln\left(1 + 2\text{sech}x\right) + C$$

c You cannot use by parts for $\int e^x \sinh x dx$

Using the definition of $\sinh x$

$$\int e^x \sinh x dx = \int e^x \left(\frac{e^x - e^{-x}}{2}\right) dx$$

$$= \frac{1}{2}\int \left(e^{2x} - 1\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2}e^{2x} - x\right) + C$$

$$= \frac{1}{4}e^{2x} - \frac{1}{2}x + C$$

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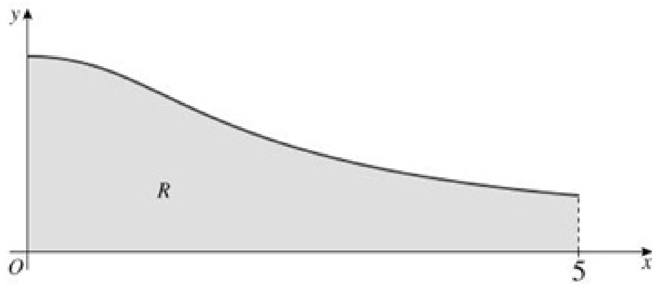
Integration
Exercise I, Question 4

Question:



The diagram shows the cross-section R of an artificial ski slope. The slope is modelled by the curve with equation

$$y = \frac{10}{\sqrt{(4x^2+9)}}, 0 \le x \le 5.$$

Given that 1 unit on each axis represents 10 metres, use integration to calculate the area R. Show your method clearly and give your answer to 2 significant figures. **[E]**

Solution:

$$\text{Area under curve} = \int_0^5 y\,dx = \int_0^5 \frac{10}{\sqrt{4x^2+9}}\,dx$$

$$= 5\int_0^5 \frac{1}{\sqrt{x^2+\frac{9}{4}}}\,dx$$

$$= 5\left[\operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_0^5$$

$$= 5\operatorname{arsinh}\left(\frac{10}{3}\right)\text{(sq. units)}$$

Using $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right)$

'Real' area $= 5\operatorname{arsinh}\left(\frac{10}{3}\right) \times 100\ \text{m}^2 = 960$ (2 s.f.)

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Integration

Exercise I, Question 5

Question:

a Find $\int \frac{1+2x}{1+4x^2} \, dx$.

b Find the exact value of

$$\int_0^{0.5} \frac{1+2x}{1+4x^2} \, dx.$$

Solution:

a
$$\int \frac{1+2x}{1+4x^2} \, dx = \int \frac{1}{1+4x^2} \, dx + \int \frac{2x}{1+4x^2} \, dx$$
$$= \int \frac{1}{4(\frac{1}{4}+x^2)} \, dx + \frac{1}{4}\int \frac{8x}{1+4x^2} \, dx$$
$$= \frac{1}{2}\arctan 2x + \frac{1}{4}\ln(1+4x^2) + C$$

b
$$\int_0^{0.5} \frac{1+2x}{1+4x^2} \, dx = \frac{1}{2}\arctan 1 + \frac{1}{4}\ln 2$$

Using the result from **a**

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Integration
Exercise I, Question 6

Question:

A rope is hung from points two points on the same horizontal level. The curve formed by the rope is modelled by the equation

$y = 4\cosh\left(\frac{x}{4}\right), -20 \le x \le 20,$

Find the length of the rope, giving your answer to 3 significant figures.

Solution:

$y = 4\cosh\left(\frac{x}{4}\right)$, so $\frac{dy}{dx} = \frac{4}{4}\sinh\left(\frac{x}{4}\right) = \sinh\left(\frac{x}{4}\right)$

$$\text{arc length} = \int_{-20}^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_0^{20} \sqrt{1 + \sinh^2\left(\frac{x}{4}\right)}\, dx$$

Using the symmetry of the catenary

$$= 2\int_0^{20} \cosh\left(\frac{x}{4}\right) dx$$

$$= 2\left[4\sinh\left(\frac{x}{4}\right)\right]_0^{20}$$

$$= 8\sinh 5$$

$$= 594 \text{ (3 s.f.)}$$

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Integration
Exercise I, Question 7

Question:

Show that $\int_0^{\frac{1}{2}} \text{artanh}x \, dx = \frac{1}{4}\ln\left(\frac{a}{b}\right)$, where a and b are positive integers to be found.

Solution:

Let $u = \text{artanh}x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{1-x^2} \quad v = x$

Then $\int_0^{\frac{1}{2}} \text{artanh}\, x dx = [x \text{ artanh } x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1-x^2} dx$

$= [x \text{ artanh } x]_0^{\frac{1}{2}} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x}{1-x^2} dx$

$= \left[x \text{ artanh } x + \frac{1}{2}\ln(1-x^2)\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\text{artanh}\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$

$= \frac{1}{2}\left\{\frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)\right\} + \frac{1}{2}\ln\left(\frac{3}{4}\right)$

Using $\text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$= \frac{1}{4}\ln 3 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$

$= \frac{1}{4}\left\{\ln 3 + 2\ln\left(\frac{3}{4}\right)^2\right\}$

$= \frac{1}{4}\left\{\ln 3 + \ln\left(\frac{9}{16}\right)\right\}$

$= \frac{1}{4}\ln\left(\frac{27}{16}\right)^2$ so $a = 27$ and $b = 16$

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Integration
Exercise I, Question 8

Question:

Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$,

a find the values of

i I_0 and

ii I_1,

b show, by using integration by parts twice, that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}, n \geq 2$.

c Hence show that $\int_0^{\frac{\pi}{2}} x^3 \cos x \, dx = \frac{1}{8}(\pi^3 - 24\pi + 48)$.

d Evaluate $\int_0^{\frac{\pi}{2}} x^4 \cos x dx$, leaving your answer in terms of π.

Solution:

a **i** $I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} = 1$

ii $I_1 = \int_0^{\frac{\pi}{2}} x \cos x dx = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dxc$

Using integration by parts

$= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + [0 - 1] = \frac{\pi}{2} - 1$

b Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = nx^{n-1}, \quad v = \sin x$

So $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx = \left[x^n \sin x \right]_0^{\frac{\pi}{2}} - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$

$= \left(\frac{\pi}{2} \right)^n - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx \quad *$

Integrating by parts on $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$ with $u = x^{n-1}$ and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = -\cos x$

gives $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx = \left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$

$= (n-1) I_{n-2} \quad \text{as} \quad \left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} = 0$

Substituting in *

$I_n = \left(\frac{\pi}{2} \right)^n - n(n-1) I_{n-2}$

c $\int_0^{\frac{\pi}{2}} x^3 \cos x \, dx = I_3 = \left(\frac{\pi}{2} \right)^3 - 3(2) I_1$

$= \left(\frac{\pi}{2} \right)^3 - 6 \left(\frac{\pi}{2} - 1 \right)$

Using a **ii**

$= \frac{\pi^3}{8} - 3\pi + 6$

$= \frac{1}{8} \left(\pi^3 - 24\pi + 48 \right)$

d $\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx = I_4 = \left(\frac{\pi}{2} \right)^4 - 4(3) I_2$

$= \left(\frac{\pi}{2} \right)^4 - 12 \left\{ \left(\frac{\pi}{2} \right)^2 - 2(1) I_0 \right\}$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$

as $I_0 = 1$ from **a i**

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Integration
Exercise I, Question 9

Question:

a Find $\int \frac{dx}{\sqrt{x^2 - 2x + 10}}$.

b Find $\int \frac{dx}{x^2 - 2x + 10}$.

c By using the substitution $x = \sin\theta$, show that $\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{(1-x^2)}} = \frac{(4\pi - 7\sqrt{3})}{64}$ **[E]**

Solution:

a $x^2-2x+10=(x-1)^2+9$

So $\displaystyle\int\frac{dx}{\sqrt{x^2-2x+10}}=\int\frac{dx}{\sqrt{(x-1)^2+9}}$

Let $x-1=3\sinh u$, then $dx=3\cosh u\,du$

$$\text{so }\int\frac{dx}{\sqrt{x^2-2x+10}}=\int\frac{3\cosh u}{3\cosh u}\,du$$
$$=u+C$$
$$=\text{arsinh}\left(\frac{x-1}{3}\right)+C$$

b $\displaystyle\int\frac{dx}{x^2-2x+10}=\int\frac{dx}{(x-1)^2+9}$

Let $x-1=3\tan\theta$, then $dx=3\sec^2\theta\,d\theta$

$$\text{so }\int\frac{dx}{x^2-2x+10}=\int\frac{3\sec^2\theta}{9\tan^2\theta+9}\,d\theta$$
$$=\int\frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta$$
$$=\frac{1}{3}\theta+C$$
$$=\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)+C$$

c Using the substitution $x=\sin\theta$, so $dx=\cos\theta\,d\theta$

$$\int_0^{\frac{1}{2}}\frac{x^4\,dx}{\sqrt{(1-x^2)}}=\int_0^{\frac{\pi}{6}}\frac{\sin^4\theta\cos\theta\,d\theta}{\cos\theta}$$
$$=\int_0^{\frac{\pi}{6}}\sin^4\theta\,d\theta$$

$\sin^4\theta=(\sin^2\theta)^2=\frac{1}{4}(1-\cos 2\theta)^2$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}(1-2\cos 2\theta+\cos^2 2\theta)\,d\theta$$
$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$
$$=\frac{1}{4}\left[\frac{3\theta}{2}-\sin 2\theta+\frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{6}}$$
$$=\frac{1}{4}\left(\frac{\pi}{4}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{16}\right)$$
$$=\frac{(4\pi-7\sqrt{3})}{64}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 10

Question:

Given that $I_n = \int_0^1 x^n (1-x)^{\frac{1}{3}}\,\mathrm{d}x, n \ge 0$,

a show that $I_n = \frac{3n}{3n+4} I_{n-1}, n \ge 1$

b Hence find the exact value of $\int_0^1 (x+1)(1-x)^{\frac{1}{3}}$. **[E]**

Solution:

a Using integration by parts on I_n, with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = (1-x)^{\frac{1}{3}}$

so $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$ and $v = -\frac{3}{4}(1-x)^{\frac{4}{3}}$

$$I_n = -\frac{3}{4}\left[x^n (1-x)^{\frac{4}{3}}\right]_0^8 + \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{4}{3}}\,\mathrm{d}x$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{4}{3}}\,\mathrm{d}x$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)(1-x)^{\frac{1}{3}}\,\mathrm{d}x$$

$$= 6n\int_0^8 x^{n-1}(1-x)^{\frac{1}{3}}\,\mathrm{d}x - \frac{3n}{4}\int_0^8 x^n (1-x)^{\frac{1}{3}}\,\mathrm{d}x$$

$$\Rightarrow 4I_n = 6nI_{n-1} - \frac{3n}{4} I_n \Rightarrow I_n = \frac{24n}{3n+4} I_{n-1}$$

b $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,\mathrm{d}x = \int_0^1 (1+x^2)(1-x)^{\frac{1}{3}}\,\mathrm{d}x = I_0 - I_2$

$I_0 = \int_0^1 (1+x)^{\frac{1}{3}}\,\mathrm{d}x = \left[-\frac{3}{4}(1-x)^{\frac{4}{3}}\right]_0^1 = \frac{3}{4}$

Using **a** $I_2 = \frac{3}{5} I_1 = \frac{3}{5}\left(\frac{3}{7} I_0\right)\left(= \frac{27}{140}\right)$

So $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,\mathrm{d}x = \frac{3}{4} - \frac{27}{140} = \frac{78}{140} = \frac{39}{70}$

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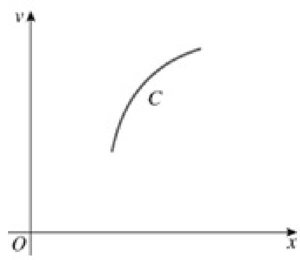
Integration
Exercise I, Question 11

Question:



The curve C has parametric equations
$x = t - \ln t,$
$y = 4\sqrt{t}, 1 \le t \le 4.$
a Show that the length of C is $3 + \ln 4$.
The curve is rotated through 2π radians about the x-axis.
b Find the exact area of the curved surface generated. **[E]**

Solution:

$x = t - \ln t$, so $\frac{dx}{dt} = 1 - \frac{1}{t}$

$y = 4\sqrt{t}$, so $\frac{dy}{dt} = \frac{2}{\sqrt{t}}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t} = 1 + \frac{2}{t} + \frac{1}{t^2} = \left(1 + \frac{1}{t}\right)^2$$

a Arc length $= \int_1^4 \sqrt{\left(1 + \frac{1}{t}\right)^2}\, dt = \int_1^4 \left(1 + \frac{1}{t}\right) dt = [t + \ln t]_1^4 = (4 + \ln 4) - 1 = 3 + \ln 4$

b Using $\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_1^4 4\sqrt{t}\left(1 + \frac{1}{t}\right) dt$

$$= 8\pi \int_1^4 \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) dt$$

$$= 8\pi \left[\frac{2}{3} t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right]_1^4$$

$$= 8\pi \left[\left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 2\right)\right]$$

$$= \frac{160\pi}{3}$$

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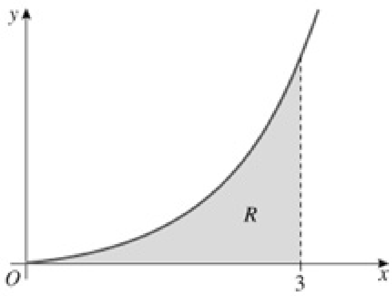
Integration
Exercise I, Question 12

Question:



Above is a sketch of part of the curve with equation
$y = x^2 \text{arsinh} x$.
The region R, shown shaded, is bounded by the curve, the x-axis and the line $x = 3$.
Show that the area of R is
$9\ln(3+\sqrt{10}) - \frac{1}{9}(2+7\sqrt{10})$. **[E]**

Solution:

$$\text{Area} = \int_0^3 y\,dx = \int_0^3 x^2 \text{arsinh}x\,dx$$

Using integration by parts on I_n, with $u = \text{arsinh}x$ and $\frac{dv}{dx} = x^2$

so $\frac{du}{dx} = \frac{1}{\sqrt{1+x^2}}$ and $v = \frac{x^3}{3}$

$$\int x^2 \text{arsinh}x\,dx = \frac{1}{3}x^3 \text{arsinh}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1+x^2}}\,dx$$

Let $x = \sinh u$ so $dx = \cosh u\,du$

$$\int_0^3 x^2 \text{ arsinh}x\,dx = 9\text{arsinh}3 - \frac{1}{3}\int_0^{\text{arsinh}3} \frac{\sinh^3 u}{\cosh u}\cosh u\,du$$

$$= 9\text{arsinh}3 - \frac{1}{3}\int_0^{\text{arsinh}3} \sinh^3 u\,du$$

$$= 9\text{arsinh}3 - \frac{1}{3}\int_0^{\text{arsinh}3} \sinh u\left(\cosh^2 u - 1\right)$$

$$= 9\text{arsinh}3 - \frac{1}{3}\left[\frac{1}{3}\cosh^3 u - \cosh u\right]_0^{\text{arsinh}3}$$

You could use integration by parts with $u = x^2$ and $\frac{du}{dx} = \frac{x}{\sqrt{1+x^2}}$

When $x = 3, \sinh u = 3$ so $\cosh u = \sqrt{1+\sinh^2 u} = \sqrt{10}$

$\text{arsinh}x = \ln\left\{x + \sqrt{1+x^2}\right\}$

$$\text{So } \int_0^3 x^2 \text{arsinh}x\,dx = 9\ln\left\{3+\sqrt{10}\right\} - \frac{1}{3}\left[\frac{10}{3}\sqrt{10} - \sqrt{10} - \left(\frac{1}{3} - 1\right)\right]$$

$$= 9\ln\left\{3+\sqrt{10}\right\} - \frac{1}{9}\left[7\sqrt{10} + 2\right]$$

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Integration
Exercise I, Question 13

Question:

a Use the substitution $u = x^2$ to find $\int_0^1 \frac{x}{1+x^4}\,dx$

b Find

i $\int \frac{1}{\sqrt{4x-x^2}}\,dx$

ii $\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx$.

Hence, or otherwise, evaluate

iii $\int_3^4 \frac{5-2x}{\sqrt{4x-x^2}}\,dx$.

Solution:

a Using $x^2 = u$ '$2x\,\mathrm{d}x$' becomes '$\mathrm{d}u$'

So $\int_0^1 \frac{x}{1+x^4}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{\mathrm{d}u}{1+u^2}$

$= \frac{1}{2}[\arctan u]_0^1$

$= \frac{\pi}{8}$

b **i** $4x - x^2 = -(x^2 - 4x) = -\left[(x-2)^2 - 4\right]$

$= 4 - (x-2)^2$

$\int \frac{1}{\sqrt{4x - x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4 - (x-2)^2}}\,\mathrm{d}x$

$= \arcsin\left(\frac{x-2}{2}\right) + C$

Using $\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

ii $\int \frac{4-2x}{\sqrt{4x - x^2}}\,\mathrm{d}x$

$= 2(4x - x^2)^{\frac{1}{2}} + C$

Notice that $\frac{\mathrm{d}}{\mathrm{d}x}(4x - x^2) = 4 - 2x$

iii $\int_3^4 \frac{5-2x}{\sqrt{4x - x^2}}\,\mathrm{d}x = \int_3^4 \left\{\frac{1}{\sqrt{4x - x^2}} + \frac{4-2x}{\sqrt{4x - x^2}}\right\}\mathrm{d}x$

$= \int_3^4 \frac{1}{\sqrt{4x - x^2}}\,\mathrm{d}x + \int_3^4 \frac{4-2x}{\sqrt{4x - x^2}}\,\mathrm{d}x$

$= \left[\arcsin\left(\frac{x-2}{2}\right) + 2(4x - x^2)^{\frac{1}{2}}\right]_3^4$

Using **i** and **ii**

$= \left(\frac{\pi}{2}\right) - \left(\frac{\pi}{6} + 2\sqrt{3}\right) = \frac{\pi}{3} - 2\sqrt{3}$

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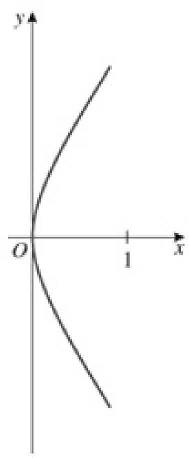
Integration
Exercise I, Question 14

Question:

The curve C shown in the diagram has equation $y^2 = 4x, 0 \leq x \leq 1$.
The part of the curve in the first quadrant is rotated through 2π radians about the x-axis.



a Show that the surface area of the solid generated is given by $4\pi \int_0^1 \sqrt{(1+x)}\,dx$.

b Find the exact value of this surface area.

c Show also that the length of the curve C, between the points $(1, -2)$ and $(1, 2)$, is given by $2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\,dx$.

d Use the substitution $x = \sinh^2\theta$ to show that the exact value of this length is $2[\sqrt{2} + \ln(1+\sqrt{2})]$. **[E]**

Solution:

$y = 2\sqrt{x}$ represents the section of curve for $x \geq 0, y \geq 0$, so $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

a Using $2\pi \int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$\text{area of surface} = 2\pi \int_0^1 2\sqrt{x}\sqrt{1+\frac{1}{x}}\, dx$$
$$= 4\pi \int_0^1 \sqrt{x}\sqrt{\frac{x+1}{x}}\, dx$$
$$= 4\pi \int_0^1 \sqrt{1+x}\, dx$$

b $4\pi \int_0^1 \sqrt{1+x}\, dx = 4\pi \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1$

$$= \frac{8\pi}{3}\left(2\sqrt{2}-1\right)$$

c Using the symmetry of the parabola, arc length is $2\times$ the length of arc from origin to (1, 2)

$$\text{so arc length} = 2\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}dx$$

d Using $x = \sinh^2\theta, dx = 2\sinh\theta\cosh\theta d\theta$

$$2\int \sqrt{\left(\frac{x+1}{x}\right)}\, dx = 2\int \sqrt{\left(\frac{\sinh^2\theta+1}{\sinh^2\theta}\right)}2\sinh\theta\cosh\theta d\theta$$
$$= 4\int \cosh^2\theta d\theta$$
$$= 2\int (1+\cosh 2\theta)d\theta$$
$$= 2\left(\theta + \frac{\sinh 2\theta}{2}\right)+C$$
$$= 2(\theta+\sinh\theta\cosh\theta)+C$$
$$= 2\left\{\text{arsinh}\sqrt{x}+\sqrt{x}\sqrt{1+x}\right\}+C$$

$$\text{So arc length} = 2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\, dx = 2\left(\text{arsinh}1+\sqrt{2}\right)$$
$$= 2\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

$\text{arsinh}x = \ln\left\{x+\sqrt{1+x^2}\right\}$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 15

Question:

a Show that $\int x\text{arcosh}x\,\text{d}x = \frac{1}{4}(2x^2-1)\text{arcosh}x - \frac{1}{4}x\sqrt{x^2-1}+C$

b Hence, using the substitution $x=u^2$, find $\int \text{arcosh}(\sqrt{x})\text{d}x$.

Solution:

a Using integration by parts with $u=\text{arcosh}x$ and $\frac{\text{d}v}{\text{d}x}=x$,

$\frac{\text{d}u}{\text{d}x}=\frac{1}{\sqrt{x^2-1}}$ and $v=\frac{x^2}{2}$

So $\int x\text{arcosh}x\text{d}x = \frac{x^2}{2}\text{arcosh}x - \int\frac{x^2}{2\sqrt{x^2-1}}\text{d}x$ *

You could use integration by parts with $u=x$ and $\frac{\text{d}y}{\text{d}x}=\frac{x}{\sqrt{x^2-1}}$

Substitute $x=\cosh u$ in $\int\frac{x^2}{\sqrt{x^2-1}}\text{d}x$ gives

$$\int\frac{x^2}{\sqrt{x^2-1}}\text{d}x = \int\frac{\cosh^2 u}{\sinh u}\sinh u\,\text{d}u$$
$$= \int\cosh^2 u\,\text{d}u$$
$$= \frac{1}{2}\int(1+\cosh 2u)\text{d}u$$
$$= \frac{1}{2}[u+\sinh u\cosh u]+C$$
$$= \frac{1}{2}\left[\text{arcosh}x + x\sqrt{x^2-1}\right]+C$$

So $\int x\text{arcosh}x\text{d}x = \frac{x^2}{2}\text{arcosh}x - \frac{1}{4}\left[\text{arcosh}x+x\sqrt{x^2-1}\right]+C$ from *

$$= \frac{1}{4}(2x^2-1)\text{arcosh}x - \frac{1}{4}x\sqrt{x^2-1}+C$$

b Let $x=u^2$, so $\text{d}x=2u\text{d}u$,

then $\int\text{arcosh}(\sqrt{x})\text{d}x = 2\int u\text{arcosh}u\,\text{d}u$

$$= \frac{1}{2}(2u^2-1)\text{arcosh}u - \frac{1}{2}u\sqrt{u^2-1}+C$$

Using a

$$= \frac{1}{2}(2x-1)\text{arcosh}\sqrt{x} - \frac{1}{2}\sqrt{x}\sqrt{x-1}+C$$

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Integration
Exercise I, Question 16

Question:

Given that $I_n = \int \frac{\sin(2n+1)x}{\sin x}\,dx$,

a show that $I_n - I_{n-1} = \frac{\sin 2nx}{n}$.

b Hence find I_5.

c Show that, for all positive integers n, $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx$ always has the same value, which should be found.

Solution:

a $I_n - I_{n-1} = \int \frac{[\sin(2n+1)x - \sin(2n-1)x]}{\sin x}\,dx$

$= \int \frac{2\cos 2nx \sin x}{\sin x}\,dx$

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$= \int 2\cos 2nx\,dx$

$= \frac{\sin 2nx}{n}$

b $I_5 - I_4 = \frac{\sin 10x}{5}, I_4 - I_3 = \frac{\sin 8x}{4}, I_3 - I_2 = \frac{\sin 6x}{3}, I_2 - I_1 = \frac{\sin 4x}{2}$

$I_1 - I_0 = \sin 2x$

Adding: $I_5 = \frac{\sin 10x}{5} + \frac{\sin 8x}{4} + \frac{\sin 6x}{3} + \frac{\sin 4x}{2} + \sin 2x + I_0$

where $I_0 = \int 1\,dx = x + C$

$= \frac{\sin 10x}{5} + \frac{\sin 8x}{4} + \frac{\sin 6x}{3} + \frac{\sin 4x}{2} + \sin 2x + x + C$

c $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}dx = \left[\frac{\sin 2nx}{n}\right]_0^{\frac{\pi}{2}} = \frac{\sin(n\pi)}{n}$

So, if n is any a positive integer $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}dx = 0$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}dx = \ldots = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x}dx = \frac{\pi}{2}$

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Integration
Exercise I, Question 17

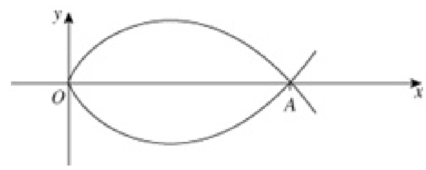
Question:

The diagram shows part of the graph of the curve with equation $y^2 = \frac{1}{3}x(x-1)^2$.

a Show that the length of the loop is $\frac{4\sqrt{3}}{3}$.

The arc OA (in boys) is rotated completely about the x-axis.

b Find the area of the surface generated.



Solution:

a The point A on the curve has coordinates (1, 0).

Using symmetry, the length of the loop is $2\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

As $y^2 = \frac{1}{3}x(x-1)^2 = \frac{1}{3}(x^3 - 2x^2 + x)$

$$2y\frac{dy}{dx} = \frac{1}{3}(3x^2 - 4x + 1) = \frac{1}{3}(3x-1)(x-1)$$

So $\frac{dy}{dx} = \dfrac{\frac{1}{3}(3x-1)(x-1)}{\pm 2\sqrt{\frac{x}{3}}(x-1)} = \pm\frac{1}{2\sqrt{3}}\frac{(3x-1)}{\sqrt{x}}$

and $1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{9x^2-6x+1}{12x} = \frac{9x^2+6x+1}{12x} = \frac{(3x+1)^2}{12x}$

$$\text{Therefore, arc length} = 2\int_0^1 \frac{3x+1}{2\sqrt{3}\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^1\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$$

$$= \frac{1}{\sqrt{3}}\left[2x^{\frac{3}{2}} + 2\sqrt{x}\right]_0^1$$

$$= \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

b Using $2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ for area of surface generated about the x-axis

$$\text{Area of surface} = 2\pi\int_0^1 \frac{1}{\sqrt{3}}\sqrt{x}(1-x)\frac{(3x+1)}{\sqrt{12x}}\,dx$$

$$= \frac{\pi}{3}\int_0^1 (1-x)(3x+1)\,dx$$

$$= \frac{\pi}{3}\int_0^1 (1+2x-3x^2)\,dx$$

$$= \frac{\pi}{3}\left[x + x^2 - x^3\right]_0^1$$

$$= \frac{\pi}{3}$$

Note: y is +ve for OA, so you need to take $y = -\frac{\sqrt{x}(x-1)}{\sqrt{3}} = \frac{\sqrt{x}(1-x)}{\sqrt{3}}$

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Integration
Exercise I, Question 18

Question:

a Find $\int \frac{1}{\sinh x + 2\cosh x}\,dx$.

b Show that $\int_1^4 \frac{3x-1}{\sqrt{x^2-2x+10}}\,dx = 9(\sqrt{2}-1) + 2\text{arsinh}\,1$. **[E]**

Solution:

a Using the exponential forms

$$\int \frac{1}{\sinh x + 2\cosh x}\,dx = \int \frac{1}{\left(\frac{e^x - e^{-x}}{2}\right) + 2\left(\frac{e^x + e^{-x}}{2}\right)}\,dx$$

$$= \int \frac{2}{3e^x + e^{-x}}\,dx$$

$$= \int \frac{2e^x}{3e^{2x}+1}\,dx$$

Using the substitution $u = e^x$, then $\frac{du}{dx} = e^x$ so '$e^x\,dx$' can be replaced by 'du',

So $\int \frac{1}{\sinh x + 2\cosh x}\,dx = \int \frac{2}{3u^2+1}\,du$

$$= \frac{2}{3}\int \frac{1}{u^2 + \frac{1}{3}}\,du$$

$$= \frac{2}{3}\left(\sqrt{3}\right)\arctan\left(\sqrt{3}u\right) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}e^x\right) + C$$

b $x^2 - 2x + 10 = (x-1)^2 + 9$

So let $x - 1 = 3\sinh u$, then $dx = 3\cosh u\,du$

and $\int \frac{3x-1}{\sqrt{x^2-2x+10}}\,dx = \int \frac{9\sinh u + 2}{\sqrt{9\sinh^2 u + 9}}3\cosh u\,du$

$$= \int \frac{9\sinh u + 2}{3\cosh u}3\cosh u\,du$$

$$= 9\cosh u + 2u + C$$

$$= 9\sqrt{1 + \left(\frac{x-1}{3}\right)^2} + 2\text{arsinh}\left(\frac{x-1}{3}\right) + C$$

So $\int_1^4 \frac{3x-1}{\sqrt{x^2-2x+10}} = \left[9\sqrt{2} + 2\text{arsinh}1\right] - [9]$

$$= 9\left(\sqrt{2} - 1\right) + 2\text{arsinh}1$$

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Integration
Exercise I, Question 19

Question:

Given that $I_n = \int \sec^n x \, dx$;

a by writing $\sec^n x = \sec^{n-2} x \sec^2 x$, show that, for $n \geq 2$,
$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$.

b Find I_5.

c Hence show that $\int_0^{\frac{\pi}{4}} \sec^5 x \, dx = \frac{1}{8}\{7\sqrt{2} + 3\ln(1+\sqrt{2})\}$

Solution:

a $\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$

Let $u = \sec^{n-2} x$ and $\frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = (n-2)\sec^{n-3} x(\sec x \tan x) = (n-2)\sec^{n-2} x \tan x$ and $v = \tan x$

Integrating by parts

$$\int \sec^n x dx = I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x \, dx + (n-2)\int \sec^{n-2} x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

So $(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}, n \geq 2,$ *

b $\int \sec^5 x \, dx = I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}I_3$ ← Substituting $n = 5$ in *

$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left(\frac{1}{2}\sec x \tan x + \frac{1}{2}I_1\right)$ ← Substituting $n = 3$ in *

But $I_1 = \int \sec x \, dx = \ln|\sec x + \tan x| + C$ ← On Edexcel formula sheet

So $\int \sec^5 x \, dx = I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

c $\int_0^{\frac{\pi}{4}} \sec^5 x dx = \frac{1}{4}(\sqrt{2})^3 + \frac{3}{8}(\sqrt{2}) + \frac{3}{8}\ln(\sqrt{2}+1)$

$= \frac{1}{8}\left\{7\sqrt{2} + 3\ln(\sqrt{2}+1)\right\}$

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Integration
Exercise I, Question 20

Question:

a Show by using a suitable substitution for x, that

$$\int \sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

b Hence show that the area of the region enclosed by the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

Solution:

a Let $x = a\sin\theta$, then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta$

So $\int \sqrt{a^2 - x^2}\,\mathrm{d}x = \int a^2\cos^2\theta\,\mathrm{d}\theta$

$$= \frac{a^2}{2}\int (1 + \cos 2\theta)\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2}\right) + C$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

b Area enclosed by the ellipse $= 4\times$ area enclosed by arc in first quadrant and the positive coordinate axes (symmetry)

$$= 4\int_0^a y\,\mathrm{d}x$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

+ve square root required

$$\text{So area} = 4\frac{b}{a}\left[\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_0^a$$

from a

$$= 2ab\arcsin 1$$

$$= \pi ab$$

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Integration
Exercise I, Question 21

Question:

a Show by using a suitable substitution for x, that

$$\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

b Hence show that the area of the region enclosed by the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

Solution:

a Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$

So $\int \sqrt{a^2 - x^2}\,dx = \int a^2\cos^2\theta\,d\theta$

$$= \frac{a^2}{2}\int (1 + \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2}\right) + C$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

b Area enclosed by the ellipse $= 4 \times$ area enclosed by arc in first quadrant (symmetry)

$$= 4\int_0^a y\,dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

(+ve square root required)

So area $= 4\frac{b}{a}\left[\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_0^a$ (from a)

$$= 2ab\arcsin 1$$

$$= \pi ab$$

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Vectors
Exercise A, Question 1

Question:

Simplify

a $5\mathbf{j}\times\mathbf{k}$

b $3\mathbf{i}\times\mathbf{k}$

c $\mathbf{k}\times3\mathbf{i}$

d $3\mathbf{i}\times(9\mathbf{i}-\mathbf{j}+\mathbf{k})$

e $2\mathbf{j}\times(3\mathbf{i}+\mathbf{j}-\mathbf{k})$

f $(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times2\mathbf{j}$

g $(5\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})$

h $(2\mathbf{i}-\mathbf{j}+6\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

i $(\mathbf{i}+5\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}-\mathbf{j}-\mathbf{k})$

j $(3\mathbf{i}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Solution:

a $5\mathbf{j}\times\mathbf{k}=5(\mathbf{j}\times\mathbf{k})=5\mathbf{i}$

Use the results $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$
$\mathbf{i}\times\mathbf{j}=\mathbf{k}, \mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$
and $\mathbf{j}\times\mathbf{i}=-\mathbf{k}, \mathbf{k}\times\mathbf{j}=-\mathbf{i}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

b $3\mathbf{i}\times\mathbf{k}=3(\mathbf{i}\times\mathbf{k})=-3\mathbf{j}$

c $\mathbf{k}\times3\mathbf{i}=3(\mathbf{k}\times\mathbf{i})=3\mathbf{j}$

d
$$\begin{aligned}3\mathbf{i}\times(9\mathbf{i}-\mathbf{j}+\mathbf{k}) &= 3\mathbf{i}\times9\mathbf{i}-3\mathbf{i}\times\mathbf{j}+3\mathbf{i}\times\mathbf{k}\\ &=27(\mathbf{i}\times\mathbf{i})-3(\mathbf{i}\times\mathbf{j})+3(\mathbf{i}\times\mathbf{k})\\ &=0-3\mathbf{k}-3\mathbf{j}\end{aligned}$$

e
$$\begin{aligned}2\mathbf{j}\times(3\mathbf{i}+\mathbf{j}-\mathbf{k}) &= 2\mathbf{j}\times3\mathbf{i}+2\mathbf{j}\times\mathbf{j}-2\mathbf{j}\times\mathbf{k}\\ &=6(\mathbf{j}\times\mathbf{i})+2(\mathbf{j}\times\mathbf{j})-2(\mathbf{j}\times\mathbf{k})\\ &=-6\mathbf{k}-2\mathbf{i}\end{aligned}$$

f
$$\begin{aligned}(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times2\mathbf{j} &= 3\mathbf{i}\times2\mathbf{j}+\mathbf{j}\times2\mathbf{j}-\mathbf{k}\times2\mathbf{j}\\ &=6(\mathbf{i}\times\mathbf{j})+2(\mathbf{j}\times\mathbf{j})-2(\mathbf{k}\times\mathbf{j})\\ &=6\mathbf{k}+2\mathbf{i}\end{aligned}$$

g
$$(5\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&2&-1\\1&-1&3\end{vmatrix}$$
$$\begin{aligned}&=(2\times3-(-1)\times(-1))\mathbf{i}-(5\times3-(-1)\times1)\mathbf{j}+(5\times(-1)-2\times1)\mathbf{k}\\ &=5\mathbf{i}-16\mathbf{j}-7\mathbf{k}\end{aligned}$$

h
$$(2\mathbf{i}-\mathbf{j}+6\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&6\\1&-2&3\end{vmatrix}$$
$$\begin{aligned}&=((-1)\times3-6\times(-2)\mathbf{i}-(2\times3-6\times1)\mathbf{j}+(2\times-2-(-1)\times1)\mathbf{k}\\ &=9\mathbf{i}-0\mathbf{j}-3\mathbf{k}\\ &=9\mathbf{i}-3\mathbf{k}\end{aligned}$$

i
$$(\mathbf{i}+5\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}-\mathbf{j}-\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&5&-4\\2&-1&-1\end{vmatrix}$$
$$\begin{aligned}&=(5\times(-1)-(-4)\times(-1))\mathbf{i}-(1\times(-1)-(-4)\times2)\mathbf{j}+(1\times-1-5\times2)\mathbf{k}\\ &=-9\mathbf{i}-7\mathbf{j}-11\mathbf{k}\end{aligned}$$

j
$$(3\mathbf{i}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&0&1\\1&-1&2\end{vmatrix}$$
$$\begin{aligned}&=(0\times2-1\times(-1))\mathbf{i}-(3\times2-1\times1)\mathbf{j}+(3\times-1-0\times1)\mathbf{k}\\ &=\mathbf{i}-5\mathbf{j}-3\mathbf{k}\end{aligned}$$

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Vectors
Exercise A, Question 2

Question:

Find the vector product of the vectors **a** and **b**, leaving your answers in terms of λ in each case.

a $\mathbf{a} = (\lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ $\mathbf{b} = (\mathbf{i} - 3\mathbf{k})$

b $\mathbf{a} = (2\mathbf{i} - \mathbf{j} + 7\mathbf{k})$ $\mathbf{b} = (\mathbf{i} - \lambda\mathbf{j} + 3\mathbf{k})$

Solution:

a $\mathbf{a} = (\lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \mathbf{b} = (\mathbf{i} - 3\mathbf{k})$

Use the determinant method to find the vector product

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & 2 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$

$$= (2\times(-3) - 1\times 0)\mathbf{i} - (\lambda\times(-3) - 1\times 1)\mathbf{j} + (\lambda\times 0 - 2\times 1)\mathbf{k}$$

$$= -6\mathbf{i} + (3\lambda + 1)\mathbf{j} - 2\mathbf{k}$$

b $\mathbf{a} = (2\mathbf{i} - \mathbf{j} + 7\mathbf{k}), \mathbf{b} = (\mathbf{i} - \lambda\mathbf{j} + 3\mathbf{k})$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 7 \\ 1 & -\lambda & 3 \end{vmatrix}$$

$$= (-1\times 3 - 7\times(-\lambda))\mathbf{i} - (2\times 3 - 7\times 1)\mathbf{j} + (2\times(-\lambda) - (-1)\times 1)\mathbf{k}$$

$$= (7\lambda - 3)\mathbf{i} + \mathbf{j} + (1 - 2\lambda)\mathbf{k}$$

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Vectors
Exercise A, Question 3

Question:

Find a unit vector that is perpendicular to both $2\mathbf{i}-\mathbf{j}$ and to $4\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

Solution:

Let $\mathbf{a}=2\mathbf{i}-\mathbf{j}$ and $\mathbf{b}=(4\mathbf{i}+\mathbf{j}+3\mathbf{k})$

Find the vector product of the two given vectors – then divide by its modulus.

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 0\\ 4 & 1 & 3\end{vmatrix}$

$=-3\mathbf{i}-6\mathbf{j}+6\mathbf{k}$

$\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$|\mathbf{a}\times\mathbf{b}|=\sqrt{(-3)^2+(-6)^2+6^2}$

$=\sqrt{81}$

$=9$

So $\frac{1}{9}(\mathbf{a}\times\mathbf{b})$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$\therefore$ Required vector is $\frac{1}{9}(-3\mathbf{i}-6\mathbf{j}+6\mathbf{k})$

$=\frac{-1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

Another possible answer is $\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$.

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Vectors
Exercise A, Question 4

Question:

Find a unit vector that is perpendicular to both of $4\mathbf{i}+\mathbf{k}$ and $\mathbf{j}-\sqrt{2}\mathbf{k}$.

Solution:

Let $\mathbf{a}=4\mathbf{i}+\mathbf{k}$ and $\mathbf{b}=\mathbf{j}-\sqrt{2}\mathbf{k}$

Find the vector product of the two given vectors, then find its modulus.

Then $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{vmatrix}$

$= -\mathbf{i}+4\sqrt{2}\mathbf{j}+4\mathbf{k}$

Now $|\mathbf{a}\times\mathbf{b}| = \sqrt{(-1)^2+\left(4\sqrt{2}\right)^2+4^2}$

$= \sqrt{1+32+16}$

$= \sqrt{49}$

$= 7$

So $\frac{1}{7}\left(-\mathbf{i}+4\sqrt{2}\mathbf{j}+4\mathbf{k}\right)$ is a unit vector, which is perpendicular to $4\mathbf{i}+\mathbf{k}$ and to $\mathbf{j}-\sqrt{2}\mathbf{k}$.

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Vectors
Exercise A, Question 5

Question:

Find a unit vector that is perpendicular to both $\mathbf{i}-\mathbf{j}$ and $3\mathbf{i}+4\mathbf{j}-6\mathbf{k}$.

Solution:

Let $\mathbf{a}=\mathbf{i}-\mathbf{j}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-6\mathbf{k}$

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 0\\ 3 & 4 & -6\end{vmatrix}$

$=6\mathbf{i}+6\mathbf{j}+7\mathbf{k}$

Find the vector product of the two given vectors then divide by its modulus.

Also $|\mathbf{a}\times\mathbf{b}|=\sqrt{6^2+6^2+7^2}$

$=\sqrt{36+36+49}$

$=\sqrt{121}$

$=11$

So $\frac{1}{11}(6\mathbf{i}+6\mathbf{j}+7\mathbf{k})$ is the required unit vector.

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Vectors
Exercise A, Question 6

Question:

Find a unit vector that is perpendicular to both $\mathbf{i}+6\mathbf{j}+4\mathbf{k}$ and to $5\mathbf{i}+9\mathbf{j}+8\mathbf{k}$.

Solution:

Let $\mathbf{a}=\mathbf{i}+6\mathbf{j}+4\mathbf{k}$ and $\mathbf{b}=5\mathbf{i}+9\mathbf{j}+8\mathbf{k}$.

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 6 & 4\\ 5 & 9 & 8\end{vmatrix}$

$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 6 & 4\\ 5 & 9 & 8\end{vmatrix}=(6\times8-4\times9)\mathbf{i}-(1\times8-4\times5)\mathbf{j}+(1\times9-6\times5)\mathbf{k}$

$=+12\mathbf{i}+12\mathbf{j}-21\mathbf{k}$

Also $|\mathbf{a}\times\mathbf{b}|=\sqrt{12^2+12^2+(-21)^2}$

$=\sqrt{144+144+441}$

$=\sqrt{729}$

$=27$

$\therefore \frac{1}{27}(12\mathbf{i}+12\mathbf{j}-21\mathbf{k})=\frac{1}{9}(4\mathbf{i}+4\mathbf{j}-7\mathbf{k})$ is the required unit vector.

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Vectors
Exercise A, Question 7

Question:

Find a vector of magnitude 5 which is perpendicular to both $4\mathbf{i}+\mathbf{k}$ and $\sqrt{2}\mathbf{j}+\mathbf{k}$.

Solution:

Let $\mathbf{a}=4\mathbf{i}+\mathbf{k}$ and $\mathbf{b}=\sqrt{2}\mathbf{j}+\mathbf{k}$

Find the vector product of the two given vectors and compare its magnitude with 5.

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 0 & 1\\ 0 & \sqrt{2} & 1\end{vmatrix}$

$=-\sqrt{2}\mathbf{i}-4\mathbf{j}+4\sqrt{2}\mathbf{k}$

But $|\mathbf{a}\times\mathbf{b}|=\sqrt{\left[\left(-\sqrt{2}\right)^2+(-4)^2+\left(4\sqrt{2}\right)^2\right]}$

$=\sqrt{(2+16+32)}$

$=\sqrt{50}$

$=5\sqrt{2}$

So $\frac{1}{\sqrt{2}}(\mathbf{a}\times\mathbf{b})$ has magnitude 5

$\therefore \frac{1}{\sqrt{2}}\left(-\sqrt{2}\mathbf{i}-4\mathbf{j}+4\sqrt{2}\mathbf{k}\right)=-\mathbf{i}-2\sqrt{2}\mathbf{j}+4\mathbf{k}$ is the required vector.

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Vectors
Exercise A, Question 8

Question:

Find the magnitude of $(\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})$. **[E]**

Solution:

Let $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$

Then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}$

$=0\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

$=-2\mathbf{j}-2\mathbf{k}$

So $|\mathbf{a}\times\mathbf{b}|=\sqrt{(-2)^2+(-2)^2}$

$=\sqrt{4+4}$

$=\sqrt{8}$ or $2\sqrt{2}$ or 2.83 (to 3 s.f.)

Given an exact answer as well as a decimal answer correct to 3 s.f.

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Vectors
Exercise A, Question 9

Question:

Given that $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}, \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ find

a $\mathbf{a} \cdot \mathbf{b}$

b $\mathbf{a} \times \mathbf{b}$

c the unit vector in the direction $\mathbf{a} \times \mathbf{b}$. **[E]**

Solution:

$\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}, \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

a
$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (-1) \times 5 + 2 \times (-2) + (-5) \times 1 \\ &= -5 - 4 - 5 \\ &= -14 \end{aligned}$$

b
$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -5 \\ 5 & -2 & 1 \end{vmatrix} \\ &= -8\mathbf{i} - 24\mathbf{j} - 8\mathbf{k} \end{aligned}$$

c
$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= \sqrt{(-8)^2 + (-24)^2 + (-8)^2} \\ &= 8\sqrt{(-1)^2 + (-3)^2 + (-1)^2} \\ &= 8\sqrt{11} \end{aligned}$$

$\therefore$ unit vector in direction $\mathbf{a} \times \mathbf{b}$ is

$$\frac{1}{8\sqrt{11}}(-8\mathbf{i} - 24\mathbf{j} - 8\mathbf{k}) = \frac{1}{\sqrt{11}}(-\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

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Vectors
Exercise A, Question 10

Question:

Find the sine of the angle between **a** and **b** in each of the following. You may leave your answers as surds, in their simplest form.

a $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

b $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

c $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

Solution:

a $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}, \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$|\mathbf{a}| = \sqrt{(3)^2 + (-4)^2}, |\mathbf{b}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$= 5 \qquad = 3$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 0 \\ 2 & 2 & 1 \end{vmatrix} = -4\mathbf{i} - 3\mathbf{j} + 14\mathbf{k}$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = \sqrt{(-4)^2 + (-3)^2 + 14^2} = \sqrt{221}$$

If θ is the angle between **a** and **b** then

$$\sin\theta = \frac{\sqrt{221}}{5 \times 3} = \frac{\sqrt{221}}{15}$$

Use $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$

b $\mathbf{a} = \mathbf{j} + 2\mathbf{k}, \mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$$|\mathbf{a}| = \sqrt{1^2 + 2^2}, |\mathbf{b}| = \sqrt{5^2 + 4^2 + (-2)^2}$$
$$= \sqrt{5} \qquad = \sqrt{45}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = \sqrt{(-10)^2 + (10)^2 + (-5)^2} = 5\sqrt{(-2)^2 + 2^2 + (-1)^2}$$
$$= 15$$

If θ is the angle between **a** and **b** then

$$\sin\theta = \frac{15}{\sqrt{5} \times \sqrt{45}} = \frac{15}{15} = 1$$

c $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

$$|\mathbf{a}| = \sqrt{5^2 + 2^2 + 2^2}, |\mathbf{b}| = \sqrt{4^2 + 4^2 + 1^2}$$
$$= \sqrt{33} \qquad = \sqrt{33}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & 2 \\ 4 & 4 & 1 \end{vmatrix}$$
$$= -6\mathbf{i} + 3\mathbf{j} + 12\mathbf{k} = 3(-2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = 3\sqrt{(-2)^2 + 1^2 + 4^2}$$
$$= 3\sqrt{21}$$

If θ is the angle between **a** and **b** then

$$\sin\theta = \frac{3\sqrt{21}}{\sqrt{33}\sqrt{33}} = \frac{\sqrt{21}}{11}$$

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Vectors
Exercise A, Question 11

Question:

The line l_1 has equation $\mathbf{r} = (\mathbf{i} - \mathbf{j}) + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and the line l_2 has equation $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. Find a vector that is perpendicular to both l_1 and l_2.

Solution:

The direction of line l_1 is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
The direction of line l_2 is $2\mathbf{i} - \mathbf{j} + \mathbf{k}$

l_1 is in the direction $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and l_2 is in the direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

A vector perpendicular to both l_1 and l_2 is in the direction:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

Any multiple of $(\mathbf{i} + \mathbf{j} - \mathbf{k})$ is perpendicular to lines l_1 and l_2.

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Vectors
Exercise A, Question 12

Question:

It is given that $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + u\mathbf{j} + v\mathbf{k}$ and that $\mathbf{a} \times \mathbf{b} = w\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}$, where u, v and w are scalar constants. Find the values of u, v and w.

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & u & v \end{vmatrix}$$

Calculate the vector product of **a** and **b**, then equate coefficients of **i**, **j** and **k**.

$$= (3v + u)\mathbf{i} - (v + 2)\mathbf{j} + (u - 6)\mathbf{k}$$

But $\mathbf{a} \times \mathbf{b} = w\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}$

So equating **i**, **j** and **k** components gives

$3v + u = w$ ①

$v + 2 = 6$ ②

$u - 6 = -7$ ③

From ② $v = 4$

From ③ $u = -1$

From ① $w = 12 - 1$ i.e. $w = 11$

So $u = -1, v = 4$ and $w = 11$.

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Vectors
Exercise A, Question 13

Question:

Given that $\mathbf{p} = a\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, that $\mathbf{q} = \mathbf{j} - \mathbf{k}$ and that their vector product $\mathbf{q} \times \mathbf{p} = 3\mathbf{i} - \mathbf{j} + b\mathbf{k}$ where a and b are scalar constants,

a find the values of a and b,

b find the value of the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$.

Solution:

a $\mathbf{q} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ a & -1 & 4 \end{vmatrix}$

$= 3\mathbf{i} - a\mathbf{j} - a\mathbf{k}$

But $\mathbf{q} \times \mathbf{p} = 3\mathbf{i} - \mathbf{j} + b\mathbf{k}$ so equate components of **i**, **j** and **k**.

$\therefore a = 1$ − from **j** component.

$-a = b$ − from **k** component.

$\therefore b = -1$

So $a = 1$ and $b = -1$

b Use $\cos\theta = \dfrac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$

$\mathbf{p} \cdot \mathbf{q} = a \times 0 + (-1) \times 1 + 4 \times (-1) = -5$

$|\mathbf{p}| = \sqrt{a^2 + (-1)^2 + 4^2} = \sqrt{18}$ as $a = 1$

$|\mathbf{q}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\therefore \cos\theta = \dfrac{-5}{\sqrt{18}\sqrt{2}} = -\dfrac{5}{6}$

Use scalar product and the definition $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Note that this gives the obtuse angle between the vectors. The cosine of the corresponding acute angle will be $\dfrac{5}{6}$.

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Vectors
Exercise A, Question 14

Question:

If $\mathbf{a}\times\mathbf{b}=0$, and $\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$, and $\mathbf{b}=3\mathbf{i}+\lambda\mathbf{j}+\mu\mathbf{k}$, where λ and μ are scalar constants, find the values of λ and μ.

Solution:

$\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}, \mathbf{b}=3\mathbf{i}+\lambda\mathbf{j}+\mu\mathbf{k}$

Given $\mathbf{a}\times\mathbf{b}=0$

This implies that $\mathbf{a}$ is parallel to $\mathbf{b}$

i.e. $\mathbf{a}=c\mathbf{b}$ where c is a scalar constant.

If the vector product of two vectors is zero, then one is a multiple of the other.

$$\therefore \begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}3c\\\lambda c\\\mu c\end{pmatrix}$$

So as $3c=2$,

$$c=\frac{2}{3}$$

$$\therefore 1=\frac{2}{3}\lambda \Rightarrow \lambda=\frac{3}{2}$$

$$\text{Also } -1=\frac{2}{3}\mu \Rightarrow \mu=-\frac{3}{2}$$

Alternative method

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-1\\3&\lambda&\mu\end{vmatrix}=(\mu+\lambda)\mathbf{i}-(2\mu+3)\mathbf{j}+(2\lambda-3)\mathbf{k}$$

But $\mathbf{a}\times\mathbf{b}=0 \therefore \mu+\lambda=0, 2\mu+3=0, 2\lambda-3=0$

$$\Rightarrow \lambda=\frac{3}{2} \text{ and } \mu=-\frac{3}{2}.$$

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Vectors
Exercise A, Question 15

Question:

If three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, show that
$\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

Solution:

Given $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ *
Take the vector product of this with $\mathbf{a}$

$$\mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{0}$$

Multiply $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, first by $\mathbf{a}$ and then by $\mathbf{b}$.

i.e. $\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$
But $\mathbf{a}\times\mathbf{a}=0$ and $\mathbf{a}\times\mathbf{c}=-\mathbf{c}\times\mathbf{a}$
$\therefore \mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}=\mathbf{0}$
i.e. $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$
This time multiply equation * by $\mathbf{b}$, using vector product.

$$\mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{b}\times\mathbf{0}$$

$\therefore \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}=\mathbf{0}$
But $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{b}=0$
$\therefore -\mathbf{a}\times\mathbf{b}+\mathbf{0}+\mathbf{b}\times\mathbf{c}=\mathbf{0}$
$\therefore \mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}$
So $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

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Vectors
Exercise B, Question 1

Question:

Find the area of triangle OAB, where O is the origin, A is the point with position vector $\mathbf{a}$ and B is the point with position vector $\mathbf{b}$, when
$\mathbf{a} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$ $\quad \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

Solution:

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix}$$

$$= -6\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$$

$$\therefore |\mathbf{a} \times \mathbf{b}| = \sqrt{(-6)^2 + (-6)^2 + (-3)^2}$$

$$= \sqrt{81}$$

$$= 9$$

$$\therefore \text{Area of triangle} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

$$= 4.5$$

Find the vector product of $\mathbf{a}$ and $\mathbf{b}$ and use the formula area $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

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Vectors
Exercise B, Question 2

Question:

Find the area of triangle OAB, where O is the origin, A is the point with position vector $\mathbf{a}$ and B is the point with position vector $\mathbf{b}$, when
$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \qquad \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Solution:

$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\therefore \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -5 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-3)^2 + (-4)^2 + (-5)^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

$$\therefore \text{Area of triangle} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

$$= \frac{5\sqrt{2}}{2}$$

Use the formula that area of triangle $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

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Vectors
Exercise B, Question 3

Question:

Find the area of triangle OAB, where O is the origin, A is the point with position vector **a** and B is the point with position vector **b**, when

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ -9 \end{pmatrix}$$

Solution:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ -9 \end{pmatrix}$$

$$\text{So } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 2 & 6 & -9 \end{vmatrix}$$

$$= -27\mathbf{i} + 18\mathbf{j} + 6\mathbf{k} = 3(-9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$$

$$|\mathbf{a} \times \mathbf{b}| = 3\sqrt{(-9)^2 + 6^2 + 2^2}$$

$$= 3\sqrt{121}$$

$$= 33$$

$$\therefore \text{Area of triangle} = \frac{1}{2} \times 33 = 16.5$$

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Vectors
Exercise B, Question 4

Question:

Find the area of the triangle with vertices $A(0,0,0), B(1,-2,1)$ and $C(2,-1,-1)$.

Solution:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

Use area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$\text{Area of triangle} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\therefore \text{Area of triangle} = \frac{1}{2}\sqrt{3^2 + 3^2 + 3^2} = \frac{1}{2}\sqrt{27} = \frac{3}{2}\sqrt{3}$$

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Vectors
Exercise B, Question 5

Question:

Find the area of triangle ABC, where the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, in the following cases:

i $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ $\mathbf{b} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

ii $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix}$

Solution:

i $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
$= 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$
$= 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

First find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then calculate their vector product.

Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 3 & -2 & 2 \end{matrix}\right\|$$

$$= \frac{1}{2}|[8\mathbf{i} + 0\mathbf{j} - 12\mathbf{k}]|$$

$$= |4\mathbf{i} - 6\mathbf{k}|$$

$$= \sqrt{4^2 + (-6)^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

ii $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix}$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then use area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -12 \end{pmatrix}$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & -1 & -12 \end{vmatrix}$$

$$= 12\mathbf{i} + 12\mathbf{j} + \mathbf{k}$$

So area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= |6\mathbf{i} + 6\mathbf{j} + \frac{1}{2}\mathbf{k}|$$

$$= \sqrt{6^2 + 6^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{72.25}$$

$$= 8.5$$

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Vectors
Exercise B, Question 6

Question:

Find the area of the triangle with vertices $A(1,0,2)$, $B(2,-2,0)$ and $C(3,-1,1)$.

Solution:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then use area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2}|-3\mathbf{j} + 3\mathbf{k}|$$

$$= \frac{1}{2}\sqrt{(-3)^2 + (3)^2}$$

$$= \frac{1}{2}\sqrt{18}$$

$$= \frac{3}{2}\sqrt{2}$$

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Vectors
Exercise B, Question 7

Question:

Find the area of the triangle with vertices $A(-1,1,1)$, $B(1,0,2)$ and $C(0,3,4)$.

Solution:

$$\mathbf{a}=\begin{pmatrix}-1\\1\\1\end{pmatrix}\mathbf{b}=\begin{pmatrix}1\\0\\2\end{pmatrix}\text{ and }\mathbf{c}=\begin{pmatrix}0\\3\\4\end{pmatrix}$$

$$\therefore \overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}2\\-1\\1\end{pmatrix}\text{ and }\overrightarrow{AC}=\mathbf{c}-\mathbf{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&1\\1&2&3\end{vmatrix}$$

$$=-5\mathbf{i}-5\mathbf{j}+5\mathbf{k}$$

$$\therefore \text{Area of triangle } ABC=\frac{1}{2}\left|-5\mathbf{i}-5\mathbf{j}+5\mathbf{k}\right|$$

$$=\frac{5}{2}\left|-\mathbf{i}-\mathbf{j}+\mathbf{k}\right|$$

$$=\frac{5}{2}\sqrt{(-1)^2+(-1)^2+1^2}$$

$$=\frac{5}{2}\sqrt{3}$$

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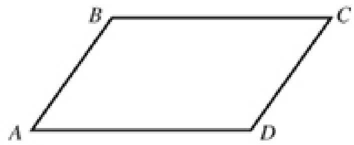
Vectors
Exercise B, Question 8

Question:

Find the area of the parallelogram $ABCD$, shown in the figure, where the position vectors of A, B and D are $\mathbf{i}+\mathbf{j}+\mathbf{k}, -3\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}$ respectively.



Solution:

$$\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}=\begin{pmatrix}-3\\4\\1\end{pmatrix} \text{ and } \mathbf{d}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and calculate $|\overrightarrow{AB}\times\overrightarrow{AD}|$.

$$\therefore \overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}-4\\3\\0\end{pmatrix} \text{ and } \overrightarrow{AD}=\mathbf{d}-\mathbf{a}=\begin{pmatrix}1\\-2\\-1\end{pmatrix}$$

$$\text{So } \overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-4 & 3 & 0\\1 & -2 & -1\end{vmatrix}$$

$$=-3\mathbf{i}-4\mathbf{j}+5\mathbf{k}$$

$$\therefore |\overrightarrow{AB}\times\overrightarrow{AD}|=\sqrt{(-3)^2+(-4)^2+5^2}$$

$$=\sqrt{50}$$

$$=5\sqrt{2}$$

So area of parallelogram $ABCD=5\sqrt{2}$.

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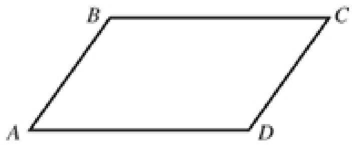
Vectors
Exercise B, Question 9

Question:

Find the area of the parallelogram $ABCD$, shown in the figure, in which the vertices A, B and D have coordinates $(0, 5, 3)$, $(2, 1, -1)$ and $(1, 6, 6)$ respectively.



Solution:

$$\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and use area $= |\overrightarrow{AB} \times \overrightarrow{AD}|$.

$$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} \text{ and } \overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -4 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= -8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$$

$$\therefore |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-8)^2 + (-10)^2 + 6^2}$$

$$= \sqrt{200}$$

$$= 10\sqrt{2}$$

$\therefore$ Area of parallelogram $= 10\sqrt{2}$

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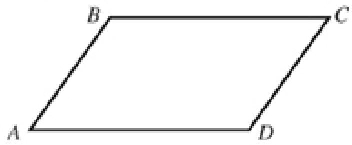
Vectors
Exercise B, Question 10

Question:

Find the area of the parallelogram $ABCD$, shown in the figure, where the position vectors of A, B and D are $\mathbf{j}, \mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+6\mathbf{j}+3\mathbf{k}$ respectively.



Solution:

$$\mathbf{a}=\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{b}=\begin{pmatrix}1\\4\\1\end{pmatrix} \text{ and } \mathbf{d}=\begin{pmatrix}2\\6\\3\end{pmatrix}$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and then use area of parallelogram $=|\overrightarrow{AB}\times\overrightarrow{AD}|$.

$$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\3\\1\end{pmatrix} \text{ and } \overrightarrow{AD}=\mathbf{d}-\mathbf{a}=\begin{pmatrix}2\\5\\3\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & 1\\ 2 & 5 & 3\end{vmatrix}=4\mathbf{i}-\mathbf{j}-\mathbf{k}$$

The area of $ABCD=|\overrightarrow{AB}\times\overrightarrow{AD}|=\sqrt{4^2+(-1)^2+(-1)^2}$

$$=\sqrt{18}$$

$$=3\sqrt{2}$$

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Vectors
Exercise B, Question 11

Question:

Relative to an origin O, the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where $\mathbf{p} = a(\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and $\mathbf{q} = a(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$ and $a > 0$. Find the area of triangle OPQ. **[E]**

Solution:

$$\mathbf{p} = a(\mathbf{i}+\mathbf{j}+2\mathbf{k}), \mathbf{q} = a(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

$$\mathbf{p}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & a & 2a \\ 2a & a & 3a \end{vmatrix}$$

$$= a^2\mathbf{i} + a^2\mathbf{j} - a^2\mathbf{k}$$

$$\therefore \text{Area of triangle } OPQ = \frac{1}{2}|a^2\mathbf{i} + a^2\mathbf{j} - a^2\mathbf{k}|$$

$$= \frac{1}{2}a^2\sqrt{1^2+1^2+(-1)^2}$$

$$= \frac{\sqrt{3}}{2}a^2$$

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Vectors
Exercise B, Question 12

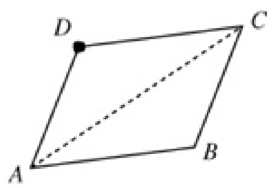
Question:

a Show that the area of the parallelogram $ABCD$ is also given by the formula $|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$.

b Show that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$ implies that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$ and explain the geometrical significance of this vector product.

Solution:

a



Draw a diagram and divide the parallelogram into two triangles by drawing the line AC. Find the area of triangle ABC and deduce the area of the parallelogram.

$$\text{Area of parallelogram } ABCD = 2\times\text{area of triangle } ABC$$
$$= 2\times\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$
$$= |\overrightarrow{AB}\times\overrightarrow{AC}|$$

As $\overrightarrow{AB}=(\mathbf{b}-\mathbf{a})$ and $\overrightarrow{AC}=(\mathbf{c}-\mathbf{a})$

Area $=|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$

b

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$$
$$\therefore (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})-(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})=0$$
$$\therefore (\mathbf{b}-\mathbf{a})\times\left[(\mathbf{c}-\mathbf{a})-(\mathbf{d}-\mathbf{a})\right]=0$$
$$\text{i.e. } (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$$

This implies $\overrightarrow{AB}\times\overrightarrow{DC}=0$

i.e. $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

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Vectors
Exercise B, Question 13

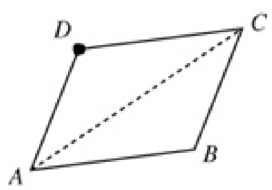
Question:

a Show that the area of the parallelogram $ABCD$ is also given by the formula $|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$.

b Show that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$ implies that $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$ and explain the geometrical significance of this vector product.

Solution:

a



Draw a diagram and divide the parallelogram into two triangles by drawing the line AC. Find the area of triangle ABC and deduce the area of the parallelogram.

$$\text{Area of parallelogram } ABCD = 2\times\text{area of triangle } ABC$$
$$= 2\times\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$
$$= |\overrightarrow{AB}\times\overrightarrow{AC}|$$

As $\overrightarrow{AB}=(\mathbf{b}-\mathbf{a})$ and $\overrightarrow{AC}=(\mathbf{c}-\mathbf{a})$

$\text{Area}=|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$

b

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$$
$$\therefore (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})-(\mathbf{b}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})=0$$
$$\therefore (\mathbf{b}-\mathbf{a})\times\left[(\mathbf{c}-\mathbf{a})-(\mathbf{d}-\mathbf{a})\right]=0$$
$$\text{i.e. } (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{d})=0$$

This implies $\overrightarrow{AB}\times\overrightarrow{DC}=0$

i.e. $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

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Vectors
Exercise C, Question 1

Question:

Given that $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{k}$
find

a $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

b $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

c $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

Solution:

a $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{c} = 3\mathbf{i} + 4\mathbf{k}$

Calculate the vector product in the bracket first, then perform the scalar product on the answer.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

$$\therefore \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} - 3\mathbf{k})$$
$$= 20 - 2 + 3$$
$$= 21$$

b

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 4 \\ 5 & 2 & -1 \end{vmatrix} = -8\mathbf{i} + 23\mathbf{j} + 6\mathbf{k}$$

$$\therefore \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (-8\mathbf{i} + 23\mathbf{j} + 6\mathbf{k})$$
$$= -8 + 23 + 6$$
$$= 21$$

c

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

$$\therefore \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (3\mathbf{i} + 4\mathbf{k}) \cdot (3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k})$$
$$= 9 + 12$$
$$= 21$$

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Vectors
Exercise C, Question 2

Question:

Given that $\mathbf{a} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$
find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. What can you deduce about the vectors **a**, **b** and **c**?

Solution:

$$\mathbf{a} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{c} = 2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & -3 & -5 \end{vmatrix} = -8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$$

$$\begin{aligned} \therefore \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \cdot (-8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}) \\ &= -8 - 8 + 16 \\ &= 0 \end{aligned}$$

If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ then **a** is perpendicular to $\mathbf{b} \times \mathbf{c}$.

a is parallel to the plane containing **b** and **c** (in fact $\mathbf{a} = \frac{1}{8}\mathbf{b} + \frac{3}{8}\mathbf{c}$).

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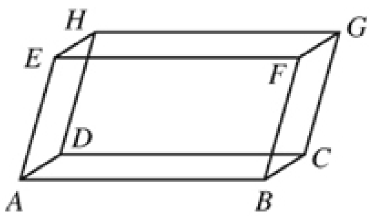
Vectors
Exercise C, Question 3

Question:

Find the volume of the parallelepiped $ABCDEFGH$ where the vertices A, B, D and E have coordinates (0, 0, 0), (3, 0, 1), (1, 2, 0) and (1, 1, 3) respectively.



Solution:

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

Then $\overrightarrow{AE} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AD}\right) = (\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (-2\mathbf{i} + \mathbf{j} + 6\mathbf{k})$

$= -2 + 1 + 18$

$= 17$

Use volume $= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$\therefore$ The volume of the parallelepiped is 17.

Alternative method:

$$\text{Volume} = \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix}$$

$= 1(0-2) - 1(0-1) + 3(6-0)$

$= 17$

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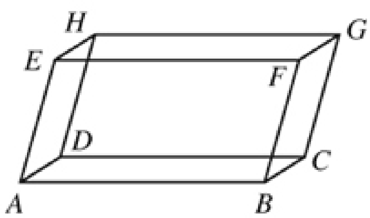
Vectors
Exercise C, Question 4

Question:

Find the volume of the parallelepiped $ABCDEFGH$ where the vertices A, B, D and E have coordinates $(-1,0,1),(3,0,-1),(2,2,0)$ and $(2, 1, 2)$ respectively.



Solution:

$\mathbf{a} = -\mathbf{i}+\mathbf{k}, \mathbf{b} = 3\mathbf{i}-\mathbf{k}, \mathbf{d} = 2\mathbf{i}+2\mathbf{j}$ and $\mathbf{e} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} = 4\mathbf{i}-2\mathbf{k}, \overrightarrow{AD} = \mathbf{d}-\mathbf{a} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$

and $\overrightarrow{AE} = \mathbf{e}-\mathbf{a} = 3\mathbf{i}+\mathbf{j}+\mathbf{k}$

Find the vectors in the directions $\overrightarrow{AE}$, $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and use these in the triple scalar product.

$$\therefore \text{Volume} = \begin{vmatrix} 3 & 1 & 1 \\ 4 & 0 & -2 \\ 3 & 2 & -1 \end{vmatrix}$$

$$= 3(0+4)-1(-4+6)+1(8-0)$$

$$= 12-2+8$$

$$= 18$$

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Vectors
Exercise C, Question 5

Question:

A tetrahedron has vertices at $A(1, 2, 3)$, $B(4, 3, 4)$, $C(1, 3, 1)$ and $D(3, 1, 4)$. Find the volume of the tetrahedron.

Solution:

$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \mathbf{c} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$
and $\mathbf{d} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

Use volume of tetrahedron $= \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \mathbf{j} - 2\mathbf{k}$

and $\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\text{Volume of tetrahedron} = \left|\frac{1}{6}\begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{6}\{3(1-2) - 1(0+4) + 1(0-2)\}\right|$$

$$= \left|\frac{1}{6}\{-3-4-2\}\right|$$

$$= \left|-\frac{9}{6}\right|$$

$$= \left|-\frac{3}{2}\right|$$

$$= \frac{3}{2}$$

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Vectors
Exercise C, Question 6

Question:

A tetrahedron has vertices at $A(2, 2, 1)$, $B(3, -1, 2)$, $C(1, 1, 3)$ and $D(3, 1, 4)$.

a Find the area of base BCD.

b Find a unit vector normal to the face BCD.

c Find the volume of the tetrahedron.

Solution:

a $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

Find $\overrightarrow{BC} \times \overrightarrow{BD}$ and use this for parts **a** and **b**

$\therefore \overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{BD} = \mathbf{d} - \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

But area of $\Delta BCD = \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{BD}|$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix}$$

$$= 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\therefore \text{Area of } \Delta BCD = \frac{1}{2}\sqrt{2^2 + 4^2 + (-4)^2}$$

$$= 3$$

b The normal to the face BCD is in the direction of $\overrightarrow{BC} \times \overrightarrow{BD}$, i.e. in the direction $2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

As $|2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}| = \sqrt{2^2 + 4^2 + (-4)^2}$
$= 6$

The unit vector normal to the face is $\frac{1}{6}(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$

$= \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

c Given also that $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, the volume of the tetrahedron $ABCD$ is $\frac{1}{6}\left|\overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD})\right|$

Find $\overrightarrow{BA}$ and use $\overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD})$ to answer part **c**.

$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$

$$\therefore \text{Volume} = \frac{1}{6}\{-2 + 12 + 4\} = \frac{14}{6} = 2\frac{1}{3}$$

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Vectors
Exercise C, Question 7

Question:

A tetrahedron has vertices at $A(0, 0, 0)$, $B(2, 0, 0)$, $C(1, \sqrt{3}, 0)$ and $D\left(1, \frac{\sqrt{3}}{3}, \frac{2\sqrt{6}}{3}\right)$.

a Show that the tetrahedron is regular.
b Find the volume of the tetrahedron.

Solution:

a $|\overrightarrow{AB}| = 2 |\overrightarrow{AC}| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ ← A tetrahedron is regular if all of its edges are the same length.

$|\overrightarrow{AD}| = \sqrt{1^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = \sqrt{1 + \frac{1}{3} + \frac{4\times 6}{9}} = 2$

$\overrightarrow{BC} = \begin{pmatrix} -1 \\ \sqrt{3} \\ 0 \end{pmatrix}$ and $|\overrightarrow{BC}| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$

$\overrightarrow{BD} = \begin{pmatrix} -1 \\ \frac{\sqrt{3}}{3} \\ \frac{2\sqrt{6}}{3} \end{pmatrix}$ and $|\overrightarrow{BD}| = \sqrt{(-1)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = 2$

$\overrightarrow{CD} = \begin{pmatrix} 0 \\ -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{6}}{3} \end{pmatrix}$ and $|\overrightarrow{CD}| = \sqrt{\left(\frac{-2\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2}$

$= \sqrt{\frac{4}{3} + \frac{8}{3}} = 2$

All 6 edges have the same length and the tetrahedron is regular.

b Volume $= \frac{1}{6}\begin{vmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 1 & \frac{\sqrt{3}}{3} & \frac{2\sqrt{6}}{3} \end{vmatrix}$

$= \frac{1}{6} \times 2 \times \left[\frac{2\sqrt{18}}{3}\right]$

$= \frac{4}{18} \times 3\sqrt{2}$

$= \frac{2}{3}\sqrt{2}$

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Vectors
Exercise C, Question 8

Question:

A tetrahedron $OABC$ has its vertices at the points $O(0,0,0), A(1,2,-1), B(-1,1,2)$ and $C(2,-1,1)$.

a Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of **i**, **j** and **k** and find $\overrightarrow{AB}\times\overrightarrow{AC}$.

b Deduce the area of triangle ABC.

c Find the volume of the tetrahedron. **[E]**

Solution:

a $\mathbf{a}=\mathbf{i}+2\mathbf{j}-\mathbf{k}, \mathbf{b}=-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \mathbf{c}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$

$$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=-2\mathbf{i}-\mathbf{j}+3\mathbf{k}$$

$$\overrightarrow{AC}=\mathbf{c}-\mathbf{a}=\mathbf{i}-3\mathbf{j}+2\mathbf{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & -1 & 3\\ 1 & -3 & 2\end{vmatrix}$$

$$=7\mathbf{i}+7\mathbf{j}+7\mathbf{k}$$

b Area of triangle $ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$=\frac{1}{2}\sqrt{7^2+7^2+7^2}$$

$$=\frac{1}{2}\times7\sqrt{3}$$

$$=\frac{7\sqrt{3}}{2}$$

c Volume of tetrahedron is $|\frac{1}{6}\overrightarrow{AO}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)|$

$$=\frac{1}{6}|(-\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(7\mathbf{i}+7\mathbf{j}+7\mathbf{k})|$$

$$=\frac{14}{6}$$

$$=\frac{7}{3}$$

You may use your answer to part **a** and form the triple scalar product $-\mathbf{a}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$ or $-\mathbf{b}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$ or $-\mathbf{c}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$

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Vectors
Exercise C, Question 9

Question:

The points A, B, C and D have position vectors

$\mathbf{a} = (2\mathbf{i}+\mathbf{j})$ $\mathbf{b} = (3\mathbf{i}-\mathbf{j}+\mathbf{k})$ $\mathbf{c} = (-2\mathbf{j}-\mathbf{k})$ $\mathbf{d} = (2\mathbf{i}-\mathbf{j}+3\mathbf{k})$ respectively.

a Find $\overrightarrow{AB}\times\overrightarrow{BC}$ and $\overrightarrow{BD}\times\overrightarrow{DC}$.

b Hence find

i the area of triangle ABC

ii the volume of the tetrahedron $ABCD$ **[E]**

Solution:

a $\mathbf{a} = (2\mathbf{i}+\mathbf{j}), \mathbf{b} = (3\mathbf{i}-\mathbf{j}+\mathbf{k}), \mathbf{c} = (-2\mathbf{j}-\mathbf{k}), \mathbf{d} = (2\mathbf{i}-\mathbf{j}+3\mathbf{k})$

$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} = \mathbf{i}-2\mathbf{j}+\mathbf{k}, \overrightarrow{BC} = \mathbf{c}-\mathbf{b} = (-3\mathbf{i}-\mathbf{j}-2\mathbf{k})$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ -3 & -1 & -2 \end{vmatrix}$$

$$= 5\mathbf{i}-\mathbf{j}-7\mathbf{k}$$

Also $\overrightarrow{BD} = \mathbf{d}-\mathbf{b} = -\mathbf{i}+2\mathbf{k}, \overrightarrow{DC} = \mathbf{c}-\mathbf{d} = (-2\mathbf{i}-\mathbf{j}-4\mathbf{k})$

$$\therefore \overrightarrow{BD}\times\overrightarrow{DC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ -2 & -1 & -4 \end{vmatrix}$$

$$= 2\mathbf{i}-8\mathbf{j}+\mathbf{k}$$

b i Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{BA}\times\overrightarrow{BC}|$

($\overrightarrow{AB}\times\overrightarrow{BC} = -\overrightarrow{BA}\times\overrightarrow{BC}$ and $|\overrightarrow{AB}\times\overrightarrow{BC}| = |\overrightarrow{BA}\times\overrightarrow{BC}|$)

$$= \frac{1}{2}|-5\mathbf{i}+\mathbf{j}+7\mathbf{k}|$$

$$= \frac{1}{2}\sqrt{25+1+49}$$

$$= \frac{1}{2}\sqrt{75}$$

$$= \frac{5}{2}\sqrt{3}$$

ii Volume of tetrahedron $ABCD = \frac{1}{6}\left|\overrightarrow{BD}\cdot\left(\overrightarrow{BA}\times\overrightarrow{BC}\right)\right|$

$$= \frac{1}{6}\left|(-\mathbf{i}+2\mathbf{k})\cdot(-5\mathbf{i}+\mathbf{j}+7\mathbf{k})\right|$$

$$= \frac{19}{6}$$

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Vectors
Exercise C, Question 10

Question:

The edges OP, OQ, OR of a tetrahedron $OPQR$ are the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where

$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$

$\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

$\mathbf{c} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

a Evaluate $(\mathbf{b} \times \mathbf{c})$ and deduce that OP is perpendicular to the plane OQR.

b Write down the length of OP and the area of triangle OQR and hence the volume of the tetrahedron.

c Verify your result by evaluating $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. **[E]**

Solution:

a $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \mathbf{c} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= \mathbf{i} + 2\mathbf{j}$$

As $\mathbf{a} = 2(\mathbf{b} \times \mathbf{c}), \overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$ and to $\overrightarrow{OR}$, i.e. $\overrightarrow{OP}$ is perpendicular to the plane OQR.

b $|\overrightarrow{OP}| = |\mathbf{a}| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

$$\text{Area of } OQR = \frac{1}{2}|\mathbf{b} \times \mathbf{c}|$$
$$= \frac{1}{2}\sqrt{1^2 + 2^2}$$
$$= \frac{\sqrt{5}}{2}$$

$$\therefore \text{Volume of tetrahedron} = \frac{1}{3} \times \text{base} \times \text{height}$$
$$= \frac{1}{3} \times \frac{\sqrt{5}}{2} \times 2\sqrt{5}$$
$$= \frac{5}{3}$$

Use volume of tetrahedron $= \frac{1}{3}$ base $\times$ height.

c $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 4 & 0 \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = 2 - (4 \times -2) = 10$$

or $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (2\mathbf{i} + 4\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) = 2 + 8 = 10$

This is $6 \times$ volume of tetrahedron so verified.

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Vectors
Exercise D, Question 1

Question:

Find an equation of the straight line passing through the point with position vector **a** which is parallel to the vector **b**, giving your answer in the form $\mathbf{r}\times\mathbf{b}=\mathbf{c}$, where **c** is evaluated:

a $\mathbf{a}=2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ $\quad\mathbf{b}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$

b $\mathbf{a}=2\mathbf{i}-3\mathbf{k}$ $\quad\mathbf{b}=\mathbf{i}+\mathbf{j}+5\mathbf{k}$

c $\mathbf{a}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ $\quad\mathbf{b}=-\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

Solution:

a $[\mathbf{r}-(2\mathbf{i}+\mathbf{j}+2\mathbf{k})]\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=0$

$$\therefore \mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 2\\ 3 & 1 & -2\end{vmatrix}$$

In each case **c** is obtained by calculating $\mathbf{a}\times\mathbf{b}$.

i.e. $\mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=-4\mathbf{i}+10\mathbf{j}-\mathbf{k}$

b $[\mathbf{r}-(2\mathbf{i}-3\mathbf{k})]\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=0$

$$\therefore \mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -3\\ 1 & 1 & 5\end{vmatrix}$$

$\therefore \mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=3\mathbf{i}-13\mathbf{j}+2\mathbf{k}$

c $[\mathbf{r}-(4\mathbf{i}-2\mathbf{j}+\mathbf{k})]\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=0$

$$\therefore \mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -2 & 1\\ -1 & -2 & 3\end{vmatrix}$$

$$= -4\mathbf{i}-13\mathbf{j}-10\mathbf{k}$$

i.e. $\mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=-4\mathbf{i}-13\mathbf{j}-10\mathbf{k}$

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Vectors
Exercise D, Question 2

Question:

Find a Cartesian equation for each of the lines given in question 1.

Solution:

a $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{-2}=\lambda$

b $\frac{x-2}{1}=\frac{y}{1}=\frac{z+3}{5}=\lambda$

c $\frac{x-4}{-1}=\frac{y+2}{-2}=\frac{z-1}{3}=\lambda$

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Vectors
Exercise D, Question 3

Question:

Find, in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, an equation of the straight line passing through the points with coordinates

a (1, 3, 5), (6, 4, 2)
b (3, 4, 12), (4, 3, 5)
c $(-2, 2, 6), (3, 7, 11)$
d $(4, 2, -4), (1, 1, 1)$

Solution:

a The line is in the direction

$$\begin{pmatrix}6\\4\\2\end{pmatrix}-\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$$

In each question one solution is given but there are a number of alternatives. Either given point may be substituted for **a** and any multiple of the direction vector may be used as **b**.

The equation is $\left[\mathbf{r}-\begin{pmatrix}1\\3\\5\end{pmatrix}\right]\times\begin{pmatrix}5\\1\\-3\end{pmatrix}=0$

b The line is in the direction

$$\begin{pmatrix}4\\3\\5\end{pmatrix}-\begin{pmatrix}3\\4\\12\end{pmatrix}=\begin{pmatrix}1\\-1\\-7\end{pmatrix}$$

The equation is $\left[\mathbf{r}-\begin{pmatrix}3\\4\\12\end{pmatrix}\right]\times\begin{pmatrix}1\\-1\\-7\end{pmatrix}=0$

c The line is in the direction

$$\begin{pmatrix}3\\7\\11\end{pmatrix}-\begin{pmatrix}-2\\2\\6\end{pmatrix}=\begin{pmatrix}5\\5\\5\end{pmatrix}$$

The equation is $\left[\mathbf{r}-\begin{pmatrix}-2\\2\\6\end{pmatrix}\right]\times\begin{pmatrix}5\\5\\5\end{pmatrix}=0$

d The line is in the direction

$$\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}4\\2\\-4\end{pmatrix}=\begin{pmatrix}-3\\-1\\5\end{pmatrix}$$

The equation is $\left[\mathbf{r}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\times\begin{pmatrix}-3\\-1\\5\end{pmatrix}=0$

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Vectors
Exercise D, Question 4

Question:

Find a Cartesian equation for each of the lines given in question 3.

Solution:

a $\frac{x-1}{5}=\frac{y-3}{1}=\frac{z-5}{-3}=\lambda$

b $\frac{x-3}{1}=\frac{y-4}{-1}=\frac{z-12}{-7}=\lambda$

c $\frac{x+2}{5}=\frac{y-2}{5}=\frac{z-6}{5}=\lambda$ *or* as $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is also in the direction of the line

$x+2=y-2=z-6=\mu$

d $\frac{x-4}{3}=\frac{y-2}{1}=\frac{z+4}{-5}=\lambda$

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Vectors
Exercise D, Question 5

Question:

Find, in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, an equation of the straight line given by the equation, where λ is scalar

a $\mathbf{r}=\mathbf{i}+\mathbf{j}-2\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{k})$

b $\mathbf{r}=\mathbf{i}+4\mathbf{j}+\lambda(3\mathbf{i}+\mathbf{j}-5\mathbf{k})$

c $\mathbf{r}=3\mathbf{i}+4\mathbf{j}-4\mathbf{k}+\lambda(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})$

Solution:

a $[\mathbf{r}-(\mathbf{i}+\mathbf{j}-2\mathbf{k})]\times(2\mathbf{i}-\mathbf{k})=0$

b $[\mathbf{r}-(\mathbf{i}+4\mathbf{j})]\times(3\mathbf{i}+\mathbf{j}-5\mathbf{k})=0$

c $[\mathbf{r}-(3\mathbf{i}+4\mathbf{j}-4\mathbf{k})]\times(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})=0$

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Vectors
Exercise D, Question 6

Question:

Find, in the form

i $\mathbf{r} \times \mathbf{b} = \mathbf{c}$, and also in the form

ii $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is a scalar parameter, the equation of the straight line with Cartesian equation $\frac{(x-3)}{2} = \frac{(y+1)}{5} = \frac{(2z-3)}{3} = \lambda$.

Solution:

When $\frac{x-3}{2} = \frac{y+1}{5} = \frac{2z-3}{3} = \lambda$

then $\frac{x-3}{2} = \frac{y+1}{5} = \frac{z-\frac{3}{2}}{\frac{3}{2}} = \lambda$

The direction of the line, $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k}$

A point on the line has position vector

$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}$.

a $\therefore \mathbf{r} \times \left(2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k}\right) = \left(3\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}\right) \times \left(2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k}\right)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & \frac{3}{2} \\ 2 & 5 & \frac{3}{2} \end{vmatrix}$$

i.e. $\mathbf{r} \times \left(2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k}\right) = -9\mathbf{i} - \frac{3}{2}\mathbf{j} + 17\mathbf{k}$

b $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k} + t\left(2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k}\right)$

or $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k} + s(4\mathbf{i} + 10\mathbf{j} + 3\mathbf{k})$

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Vectors
Exercise D, Question 7

Question:

Given that the point with coordinates $(p, q, 1)$ lies on the line with equation

$\mathbf{r} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ -3 \end{pmatrix}$, find the values of p and q.

Solution:

As $(p, q, 1)$ lies an the line with equation

$$\mathbf{r} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ -3 \end{pmatrix} \text{ then } \begin{pmatrix} p \\ q \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ -3 \end{pmatrix}$$

Find the vector product of $\begin{pmatrix} p \\ q \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and equate to $\begin{pmatrix} 8 \\ -7 \\ -3 \end{pmatrix}$.

But $\begin{pmatrix} p \\ q \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3q-1 \\ 2-3p \\ p-2q \end{pmatrix}$

$$\therefore \begin{pmatrix} 3q-1 \\ 2-3p \\ p-2q \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ -3 \end{pmatrix}$$

i.e. $3q - 1 = 8 \Rightarrow q = 3$

$2 - 3p = -7 \Rightarrow p = 3$

i.e: $p = 3$ and $q = 3$

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Vectors
Exercise D, Question 8

Question:

Given that the equation of a straight line is $\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$,

Hint: Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and set up simultaneous equations.

find an equation for the line in the form $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, where t is a scalar parameter.

Solution:

The line with equation

$$\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

has direction $\mathbf{i}+\mathbf{j}-\mathbf{k}$, i.e. $\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

It passes through a point (a_1,a_2,a_3) where

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

i.e. $\begin{pmatrix}-a_2-a_3\\a_1+a_3\\a_1-a_2\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and set up simultaneous equations. These equations have an infinite number of solutions so let $a_1=0$ and find a_2 and a_3.

Let $a_1=0$, then as $a_1+a_3=2$ and $a_1-a_2=1$ this implies that $a_3=2$ and $a_2=-1$

$\therefore (0,-1,2)$ lies on the line.

So the line equation may be written as

$\mathbf{r}=-\mathbf{j}+2\mathbf{k}+t(\mathbf{i}+\mathbf{j}-\mathbf{k})$

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Vectors
Exercise E, Question 1

Question:

Find, in the form $\mathbf{r}\cdot\mathbf{n} = p$, an equation of the plane that passes through the point with position vector $\mathbf{a}$ and is perpendicular to the vector $\mathbf{n}$ where

a $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

b $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

c $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

d $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

Solution:

a
$$\mathbf{r}\cdot(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (\mathbf{i} - \mathbf{j} - \mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$= 2 - 1 - 1$$
i.e. $\mathbf{r}\cdot(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$

b
$$\mathbf{r}\cdot(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} + \mathbf{k})\cdot(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$$
$$= 5 - 2 - 3$$
i.e. $\mathbf{r}\cdot(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 0$

c
$$\mathbf{r}\cdot(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = (2\mathbf{i} - 3\mathbf{k})\cdot(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$
$$= 2 - 12$$
i.e. $\mathbf{r}\cdot(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -10$

d
$$\mathbf{r}\cdot(4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k})\cdot(4\mathbf{i} + \mathbf{j} - 5\mathbf{k})$$
$$= 16 - 2 - 5$$
i.e. $\mathbf{r}\cdot(4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$

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Vectors

Exercise E, Question 2

Question:

Find a Cartesian equation for each of the planes in question 1.

Solution:

a $2x + y + z = 0$

b $5x - y - 3z = 0$

c $x + 3y + 4z = -10$

d $4x + y - 5z = 9$

Replace $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in each equation.

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Vectors
Exercise E, Question 3

Question:

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ an equation of the plane that passes through the points

a $(1,2,0),(3,1,-1)$ and $(4,3,2)$

b $(3,4,1),(-1,-2,0)$ and $(2,1,4)$

c $(2,-1,-1),(3,1,2)$ and $(4,0,1)$

d $(-1,1,3),(-1,2,5)$ and $(0,4,4)$.

Solution:

a Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

and $\mathbf{c} = (4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j}) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Choose one of the points to have position vector $\mathbf{a}$ then let the other two points have position vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$ respectively.

b Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \mathbf{b} = -\mathbf{i} - 2\mathbf{j} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$

and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$\therefore \mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

or $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda'(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

c Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k})$

$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

and $\mathbf{c} = 4\mathbf{i} + \mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d Let $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{j} + 2\mathbf{k}$

and $\mathbf{c} = 4\mathbf{j} + 4\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\therefore \mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

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Vectors
Exercise E, Question 4

Question:

Find a Cartesian equation for each of the planes in question 3.

Solution:

a Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{vmatrix} = -\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$

Find the equation in the form $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ using $\mathbf{a}$ from question 3 and finding $\mathbf{n} = \mathbf{b}\times\mathbf{c}$ from question 3.

$\therefore$ Equation is $\mathbf{r}\cdot(-\mathbf{i}-7\mathbf{j}+5\mathbf{k}) = (\mathbf{i}+2\mathbf{j})\cdot(-\mathbf{i}-7\mathbf{j}+5\mathbf{k})$

In Cartesian form: $-x-7y+5z=-15$

or $x+7y-5z=15$

b Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & 1 \\ -1 & -3 & 3 \end{vmatrix} = 21\mathbf{i} - 13\mathbf{j} - 6\mathbf{k}$

$\therefore$ Equation is $\mathbf{r}\cdot(21\mathbf{i}-13\mathbf{j}-6\mathbf{k}) = (3\mathbf{i}+4\mathbf{j}+\mathbf{k})\cdot(21\mathbf{i}-13\mathbf{j}-6\mathbf{k})$

i.e. $21x-13y-6z = 63-52-6$

i.e. $21x-13y-6z = 5$

c Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

$\therefore$ Equation is $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-3\mathbf{k}) = (2\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+4\mathbf{j}-3\mathbf{k})$

$= 2-4+3$

i.e. $x+4y-3z=1$

d Normal to plane is in direction $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = -5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\therefore$ Equation is $\mathbf{r}\cdot(-5\mathbf{i}+2\mathbf{j}-\mathbf{k}) = (-\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(-5\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$= 5+2-3$

i.e. $-5x+2y-z=4$

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Vectors
Exercise E, Question 5

Question:

Find a Cartesian equation of the plane that passes through the points

a $(0, 4, 2)$, $(1, 1, 2)$ and $(-1, 5, 0)$

b $(1, 1, 0)$, $(2, 3, -3)$ and $(3, 7, -2)$

c $(3, 0, 0)$, $(2, 0, -1)$ and $(4, 1, 3)$

d $(1, -1, 6)$, $(3, 1, -2)$ and $(4, 1, 0)$.

Solution:

a $\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}1\\-3\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\5\\0\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\-2\end{pmatrix}$

are two directions in the plane.

Find two directions in the plane and take their vector product to give a normal to the plane.

The normal to the plane is $\mathbf{n}$ where $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & -3 & 0\\-1 & 1 & -2\end{vmatrix}$

i.e: $\mathbf{n}=+6\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ is also normal to plane.

$\therefore$ Equation of plane is $\mathbf{r}\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k})=(4\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k})$

i.e. $3x+y-z=2$

b $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}-(\mathbf{i}+\mathbf{j})=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+7\mathbf{j}-2\mathbf{k}-(\mathbf{i}+\mathbf{j})=2\mathbf{i}+6\mathbf{j}-2\mathbf{k}$ are two directions in the plane.

The normal to the plane is $\mathbf{n}$ where $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 2 & -3\\2 & 6 & -2\end{vmatrix}$

i.e: $\mathbf{n}=14\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $7\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is also a normal.

$\therefore$ Equation of plane is

$\mathbf{r}\cdot(7\mathbf{i}-2\mathbf{j}+\mathbf{k})=(\mathbf{i}+\mathbf{j})\cdot(7\mathbf{i}-2\mathbf{j}+\mathbf{k})$

i.e. $7x-2y+z=5$

c $(2\mathbf{i}-\mathbf{k})-(3\mathbf{i})=-\mathbf{i}-\mathbf{k}$ and $(4\mathbf{i}+\mathbf{j}+3\mathbf{k})-3\mathbf{i}=\mathbf{i}+\mathbf{j}+3\mathbf{k}$ are two directions in the plane.

The normal to the plane is $\mathbf{n}$ where $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-1 & 0 & -1\\1 & 1 & 3\end{vmatrix}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$

The equation an of the plane is

$\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3\mathbf{i}\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$\therefore x+2y-z=3$

d Two directions in the plane are:

$3\mathbf{i}+\mathbf{j}-2\mathbf{k}-(\mathbf{i}-\mathbf{j}+6\mathbf{k})=2\mathbf{i}+2\mathbf{j}-8\mathbf{k}$ and

$4\mathbf{i}+\mathbf{j}-(\mathbf{i}-\mathbf{j}+6\mathbf{k})=3\mathbf{i}+2\mathbf{j}-6\mathbf{k}$

The normal to the plane is $\mathbf{n}$ where

$$\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 2 & -8\\3 & 2 & -6\end{vmatrix}=4\mathbf{i}-12\mathbf{j}-2\mathbf{k}$$

The equation of the plane is

$\mathbf{r}\cdot(4\mathbf{i}-12\mathbf{j}-2\mathbf{k})=(\mathbf{i}-\mathbf{j}+6\mathbf{k})\cdot(4\mathbf{i}-12\mathbf{j}-2\mathbf{k})$

$=4$

i.e. $4x-12y-2z=4$ *or* $2x-6y-z=2$

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Vectors
Exercise E, Question 6

Question:

Find, in the form $\mathbf{r}\cdot\mathbf{n}=p$, an equation of the plane which contains the line l and the point with position vector $\mathbf{a}$ where

a l has equation $\mathbf{r}=\mathbf{i}+\mathbf{j}-2\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{k})$ and $\mathbf{a}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}$

b l has equation $\mathbf{r}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ and $\mathbf{a}=3\mathbf{i}+5\mathbf{j}+\mathbf{k}$

c l has equation $\mathbf{r}=2\mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ and $\mathbf{a}=7\mathbf{i}+8\mathbf{j}+6\mathbf{k}$

Solution:

a The line has direction $2\mathbf{i}-\mathbf{k}$, and this is a direction in the plane.

Another vector in the plane is $4\mathbf{i}+3\mathbf{j}+\mathbf{k}-(\mathbf{i}+\mathbf{j}-2\mathbf{k})$

i.e. $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

The equation of the line includes the position vector of another point on the plane and includes a direction vector in the plane.

The normal to the plane is in direction

$(2\mathbf{i}-\mathbf{k})\times(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

i.e. $\mathbf{n}=2\mathbf{i}-9\mathbf{j}+4\mathbf{k}$

$\therefore$ The plane has equation

$$\mathbf{r}\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=(4\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})$$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=8-27+4$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=-15$

b The line has direction $(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$

Another vector in the plane is $3\mathbf{i}+5\mathbf{j}+\mathbf{k}-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

i.e: $(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$

$\therefore$ the normal to the plane is $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -3\\ 2 & 3 & -1\end{vmatrix}=8\mathbf{i}-4\mathbf{j}+4\mathbf{k}$

$\therefore$ Equation of the plane is

$$\mathbf{r}\cdot(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})=(3\mathbf{i}+5\mathbf{j}+\mathbf{k})\cdot(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})$$
$$=24-20+4$$
$$=8$$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=2$

c $7\mathbf{i}+8\mathbf{j}+6\mathbf{k}$ is the position vector of a point on the plane.

$2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is the position vector of another point on the plane.

You need 2 directions in the plane. One is the direction of the line. The other is the vector joining the two points that are given in the plane i.e. (7, 8, 6) and $(2,-1,1)$.

The vector joining these points is $5\mathbf{i}+9\mathbf{j}+5\mathbf{k}$

This lies in the plane.

A second vector which lies in the plane is $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

The normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 9 & 5\\ 1 & 2 & 2\end{vmatrix}$

i.e: $\mathbf{n}=8\mathbf{i}-5\mathbf{j}+\mathbf{k}$

$\therefore$ The equation of the plane is

$$\mathbf{r}\cdot(8\mathbf{i}-5\mathbf{j}+\mathbf{k})=(7\mathbf{i}+8\mathbf{j}+6\mathbf{k})\cdot(8\mathbf{i}-5\mathbf{j}+\mathbf{k})$$
$$=56-40+6$$

$\therefore\mathbf{r}\cdot(8\mathbf{i}-5\mathbf{j}+\mathbf{k})=22$

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Vectors
Exercise E, Question 7

Question:

Find a Cartesian equation of the plane which passes through the point (1, 1, 1) and contains the line with equation $\frac{x-2}{3}=\frac{y+4}{1}=\frac{z-1}{2}$.

Solution:

The line is in the direction $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. This lies in the plane.

$(2,-4,1)$ is a point on the line. This also lies in the plane, as does the point (1, 1, 1).

$\therefore \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix}$ is a direction in the plane.

First obtain the equation of the plane in the form $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, then convert to Cartesian form.

The normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & 0 \\ 3 & 1 & 2 \end{vmatrix}$

$= -10\mathbf{i}-2\mathbf{j}+16\mathbf{k}$

$\therefore$ The equation of the plane is

$\mathbf{r}\cdot(-10\mathbf{i}-2\mathbf{j}+16\mathbf{k}) = (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(-10\mathbf{i}-2\mathbf{j}+16\mathbf{k})$

i.e: $-10x-2y+16z=4$

This is a Cartesian equation of the plane.

Solutionbank FP3
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Vectors
Exercise F, Question 1

Question:

In each case establish whether lines l_1 and l_2 meet and if they meet find the coordinates of their point of intersection:

a l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b l_1 has equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mu(-\mathbf{i} + \mathbf{j} - \mathbf{k})$

c l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and l_2 has equation $\mathbf{r} = \mathbf{i} + 2\frac{1}{2}\mathbf{j} + 2\frac{1}{2}\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

(In each of the above cases λ and μ are scalars.)

Solution:

a The line l_1 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$$

and the line l_2 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Use column vector form for clarity. Put the two equations equal and compare x, y and z components. Then solve simultaneous equations.

These lines meet when

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

i.e. $1 + \lambda = -1 + \mu$ ①

$3 - \lambda = -3 + \mu$ ②

$5\lambda = 2 + 2\mu$ ③

Add equations ① and ②

$4 = -4 + 2\mu$

$\therefore 2\mu = 8$

i.e: $\mu = 4$

Substitute into equation ①

$\therefore 1 + \lambda = -1 + 4$

i.e: $\lambda = 2$

Substitute $\lambda = 2$ into equation for line l_1

$\therefore (x, y, z) = (3, 1, 10)$

Substitute $\mu = 4$ into equation for line l_2

$\therefore (x, y, z) = (3, 1, 10)$

So the two lines do meet at the point (3, 1, 10)

b l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

These lines meet when $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

i.e. $3+\lambda = 4-\mu$ ①

$2+\lambda = 3+\mu$ ②

$1+2\lambda = -\mu$ ③

Add equations ① and ②

$\therefore 5+2\lambda = 7$

i.e: $\lambda = 1$

Substitute into equation ①

$\therefore 3+1 = 4-\mu$

i.e: $\mu = 0$

Substitute $\lambda = 1$ into equation for line l_1:

$\therefore (x, y, z) = (4, 3, 3)$

Substitute $\mu = 0$ into line l_2:

$\therefore (x, y, z) = (4, 3, 0)$

This is a contradiction and the lines do not meet.

[N.B. $\lambda = 1$ and $\mu = 0$ do not satisfy equation ③ above.]

c l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

l_1 meets l_2 when $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

i.e. $1 + 2\lambda = 1 + \mu$ ①

$3 + 3\lambda = 2\frac{1}{2} + \mu$ ②

$5 + \lambda = 2\frac{1}{2} - 2\mu$ ③

Subtract equation ① from equation ②

$\therefore 2 + \lambda = 1\frac{1}{2}$

i.e. $\lambda = -\frac{1}{2}$

Substitute into equation ①

$\therefore 1 - 1 = 1 + \mu$

i.e. $\mu = -1$

Substitute $\lambda = -\frac{1}{2}$ into equation for line l_1:

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

Substitute $\mu = -1$ into equation for line l_2::

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

So the two lines do meet at the point $\left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$.

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Vectors
Exercise F, Question 2

Question:

In each case establish whether the line l meets the plane Π and, if they meet, find the coordinates of their point of intersection.

a $l : \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
$\Pi : \mathbf{r} \cdot (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 16$

b $l : \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\Pi : \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1$

c $l : \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - 2\mathbf{k})$
$\Pi : \mathbf{r} \cdot (3\mathbf{i} - \mathbf{j} - 6\mathbf{k}) = 1$

(In each of the above cases λ is a scalar.)

Solution:

a The line meets the plane when

$$[(1-2\lambda)\mathbf{i}+(1+\lambda)\mathbf{j}+(1-4\lambda)\mathbf{k}]\cdot(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=16$$

i.e. $3(1-2\lambda)-4(1+\lambda)+2(1-4\lambda)=16$

$\therefore 3-6\lambda-4-4\lambda+2-8\lambda=16$

$\therefore 1-18\lambda=16$

i.e. $-18\lambda=15$

$\therefore \lambda=-\frac{15}{18}$

i.e. $\lambda=\frac{-5}{6}$

Assume that the line meets the plane and perform the scalar product. Solve the resulting equation to find the value of λ. If there is no value for λ, then the line does not meet the plane.

Substitute into the equation of the line

$$\therefore (x,y,z)=\left(1+\frac{10}{6},1-\frac{5}{6},1+\frac{20}{6}\right)$$

$$=\left(2\frac{2}{3},\frac{1}{6},4\frac{1}{3}\right)$$

b The line meets the plane when

$$[(2+\lambda)\mathbf{i}+(3+\lambda)\mathbf{j}+(-2+\lambda)\mathbf{k}]\cdot(\mathbf{i}+\mathbf{j}-2\mathbf{k})=1$$

i.e. $(2+\lambda)+(3+\lambda)-2(-2+\lambda)=1$

$\therefore 2+\lambda+3+\lambda+4-2\lambda=1$

$\therefore 9=1$

This is a contradiction.
There are no values of λ for which the line meets the plane.
The line is parallel to the plane.

c The line meets the plane when

$$[\mathbf{i}+(1+2\lambda)\mathbf{j}+(1-2\lambda)\mathbf{k}]\cdot(3\mathbf{i}-\mathbf{j}-6\mathbf{k})=1$$

i.e. $3-(1+2\lambda)-6(1-2\lambda)=1$

i.e. $3-1-2\lambda-6+12\lambda=1$

$\therefore 10\lambda-4=1$

$\therefore \lambda=\frac{1}{2}$

Substitute into the equation of the line

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Vectors
Exercise F, Question 3

Question:

Find the equation of the line of intersection of the planes Π_1 and Π_2 where

a Π_1 has equation $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 5$ and Π_2 has equation $\mathbf{r} \cdot (4\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 5$

b Π_1 has equation $\mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 16$ and Π_2 has equation $\mathbf{r} \cdot (16\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}) = 53$

c Π_1 has equation $\mathbf{r} \cdot (\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 10$ and Π_2 has equation $\mathbf{r} \cdot (4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 1$.

Solution:

a The planes have equations

$3x-2y-z=5$ and ①

$4x-y-2z=5$ ②

Express the equations of the planes in Cartesian form then eliminate one of the variables (x, y or z) from the equations.

Multiply ① by 2 then subtract ②

$\therefore 2x-3y=5$

$\therefore x=\frac{5+3y}{2}$

Substitute this into ①

$\therefore 3\frac{(5+3y)}{2}-2y-z=5$

$\therefore z=3\frac{(5+3y)}{2}-2y-5$

$=\frac{5+5y}{2}$

Let $y=\lambda$.

Then $x=\frac{5+3\lambda}{2}$ and $z=\frac{5+5\lambda}{2}$

i.e. $\lambda=\frac{x-\frac{5}{2}}{\frac{3}{2}}$ and $\lambda=\frac{z-\frac{5}{2}}{\frac{5}{2}}$

$\therefore$ Equation of the line of intersection is

$$\frac{x-\frac{5}{2}}{\frac{3}{2}}=y=\frac{z-\frac{5}{2}}{\frac{5}{2}}=\lambda$$

or $\mathbf{r}=\left(\frac{5}{2}\mathbf{i}+\frac{5}{2}\mathbf{k}\right)+\lambda\left(\frac{3}{2}\mathbf{i}+\mathbf{j}+\frac{5}{2}\mathbf{k}\right)$

b The planes have equations

$5x-y-2z=16$ ①

and $16x-5y-4z=53$ ②

Express the equations of the planes in Cartesian form then eliminate one of the variables (x, y or z) from the equations.

Multiply equation ① by 5 then subtract equation ②

$\therefore 9x-6z=27$

$\therefore x=\frac{27+6z}{9}=\frac{9+2z}{3}$

Substitute into equation ①

Then $5\frac{(9+2z)}{3}-y-2z=16$

$\therefore y=5\frac{(9+2z)}{3}-2z-16$

$=\frac{4z-3}{3}$

Let $z=\lambda$

Then $x=\frac{9+2\lambda}{3}$ and $y=\frac{4\lambda-3}{3}$ and $z=\lambda$

$$\therefore \frac{x-3}{\frac{2}{3}} = \frac{y+1}{\frac{4}{3}} = z = \lambda$$

This is the equation of the line of intersection.
In vector form:

$$\mathbf{r} = (3\mathbf{i} - \mathbf{j}) + \lambda\left(\frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \mathbf{k}\right)$$

c The planes have equations

$$x - 3y + z = 10 \quad ①$$

and $4x - 3y - 2z = 1 \quad ②$

Subtract equation ① from equation ②

Express the equations of the planes in Cartesian form then eliminate one of the variables (x, y or z) from the equations.

$\therefore 3x - 3z = -9$

$\therefore x = z - 3$

Substitute into equation ①

$\therefore z - 3 - 3y + z = 10$

i.e. $3y = 2z - 13$

$$\therefore y = \frac{2z-13}{3}$$

Let $z = \lambda$

Then $x = \lambda - 3$ and $y = \frac{2\lambda - 13}{3}$ and $z = \lambda$

$$\therefore \frac{x+3}{1} = \frac{y + \frac{13}{3}}{\frac{2}{3}} = z = \lambda$$

This is the Cartesian form of the equation of the line of intersection.
The vector form is

$$\mathbf{r} = \left(-3\mathbf{i} - \frac{13}{3}\mathbf{j}\right) + \lambda\left(\mathbf{i} + \frac{2}{3}\mathbf{j} + \mathbf{k}\right)$$

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Vectors
Exercise F, Question 4

Question:

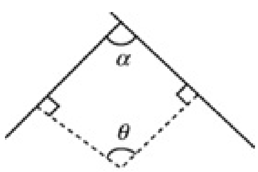
Find the acute angle between the planes with equations $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=1$ and $\mathbf{r}\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})=7$ respectively.

Solution:

The angle θ between the two normal vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ is given by

$$\cos\theta=\frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})}{|\mathbf{i}+2\mathbf{j}-2\mathbf{k}||-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}|}$$
$$=\frac{-4+8-14}{\sqrt{1^2+2^2+(-2)^2}\sqrt{(-4)^2+4^2+7^2}}$$
$$=\frac{-10}{\sqrt{9}\sqrt{81}}$$
$$=-\frac{10}{27}$$

First find the angle between the two normal vectors.



The acute angle, α, between the two planes is such that

$\alpha+\theta=180°$

So $\cos\alpha=-\cos\theta$

$$=\frac{10}{27}$$

$\therefore \alpha=68.3°\quad(3\text{ s.f.})$

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Vectors
Exercise F, Question 5

Question:

Find the acute angle between the planes with equations $\mathbf{r}\cdot(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=9$ and $\mathbf{r}\cdot(5\mathbf{i}-12\mathbf{k})=7$ respectively.

Solution:

The angle θ between the two normal vectors $3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ and $5\mathbf{i}-12\mathbf{k}$ is given by

$$\cos\theta=\frac{(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot(5\mathbf{i}-12\mathbf{k})}{|3\mathbf{i}-4\mathbf{j}+12\mathbf{k}||5\mathbf{i}-12\mathbf{k}|}$$

First find the angle between the two normal vectors.

$$=\frac{15-144}{\sqrt{3^2+(-4)^2+12^2}\sqrt{5^2+(-12)^2}}$$

$$=\frac{-129}{\sqrt{169}\sqrt{169}}$$

$$=\frac{-129}{169}$$

The acute angle α between the planes is such that $\alpha+\theta=180°$

So $\cos\alpha=-\cos\theta=\frac{129}{169}$

$\therefore \alpha=40.2°\quad$ (3 s.f.)

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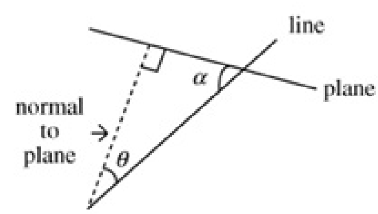
Vectors
Exercise F, Question 6

Question:

Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k})$ and the plane with equation $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 13$.

Solution:



Find the acute angle between the given line and the normal to the plane, then subtract from $90°$.

Let θ be the acute angle between the line and the normal to the plane.

Then $\cos\theta = \left|\dfrac{(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{4^2+4^2+7^2}\sqrt{2^2+1^2+(-2)^2}}\right|$

$= \left|\dfrac{8+4-14}{\sqrt{81}\sqrt{9}}\right|$

$= \left|\dfrac{-2}{27}\right| = \dfrac{2}{27}$

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

So $\sin\alpha = \cos\theta = \dfrac{2}{27}$

$\therefore \alpha = 4.25°$ (3 s.f.)

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Vectors
Exercise F, Question 7

Question:

Find the acute angle between the line with equation $\mathbf{r} = -\mathbf{i} - 7\mathbf{j} + 13\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and the plane with equation $\mathbf{r} \cdot (4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}) = 9$.

Solution:

Let θ be the acute angle between the line and the normal to the plane.

$$\text{Then } \cos\theta = \frac{(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) \cdot (4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k})}{\sqrt{3^2 + 4^2 + (-12)^2}\sqrt{4^2 + (-4)^2 + (-7)^2}}$$

$$= \frac{12 - 16 + 84}{\sqrt{169}\sqrt{81}}$$

Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

$$= \frac{80}{13 \times 9}$$

$$= \frac{80}{117}$$

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90^\circ$

$$\text{So } \sin\alpha = \cos\theta = \frac{80}{117}$$

$$\therefore \alpha = 43.1^\circ \text{ (3 s.f.)}$$

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Vectors
Exercise F, Question 8

Question:

Find the acute angle between the line with equation $(\mathbf{r}-3\mathbf{j})\times(-4\mathbf{i}-7\mathbf{j}+4\mathbf{k})=0$ and the plane with equation $\mathbf{r}=\lambda(4\mathbf{i}-\mathbf{j}-\mathbf{k})+\mu(4\mathbf{i}-5\mathbf{j}+3\mathbf{k})$.

Solution:

First find a normal $\mathbf{n}$ to the plane

$$\mathbf{n}=(4\mathbf{i}-\mathbf{j}-\mathbf{k})\times(4\mathbf{i}-5\mathbf{j}-3\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -1 & -1\\ 4 & -5 & 3\end{vmatrix}$$

$$=-8\mathbf{i}-16\mathbf{j}-16\mathbf{k}$$

So a simple normal to the plane is $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

Let θ be the acute angle between the line and the normal to the plane,

Then $\cos\theta=\left|\dfrac{(-4\mathbf{i}-7\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\sqrt{(-4)^2+(-7)^2+4^2}\sqrt{1^2+2^2+2^2}}\right|=\left|\dfrac{-4-14+8}{9\times 3}\right|$

Let α be the angle between the line and the plane.

Then $\theta+\alpha=90°$, so $\sin\alpha=\cos\theta=\dfrac{10}{27}$.

$\therefore \alpha=21.7°$ (3 s.f.)

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Vectors
Exercise F, Question 9

Question:

The plane Π has equation $\mathbf{r}\cdot(10\mathbf{j}+10\mathbf{j}+23\mathbf{k})=81$.

a Find the perpendicular distance from the origin to plane Π.

b Find the perpendicular distance from the point $(-1,-1,4)$ to the plane Π.

c Find the perpendicular distance from the point (2, 1, 3) to the plane Π.

d Find the perpendicular distance from the point $(6,12,-9)$ to the plane Π.

Solution:

a The length of the normal vector $10\mathbf{i}+10\mathbf{j}+23\mathbf{k}$ is $\sqrt{10^2+10^2+23^2}=\sqrt{729}=27$

$\therefore \frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$ is a unit vector normal to the plane.

The plane has equation

$$\mathbf{r}\cdot(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=81$$

or $\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=\frac{81}{27}=3$

$\therefore$ The perpendicular distance from the origin to the plane is 3.

b A plane parallel to π through the point $(-1,-1,4)$ has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=(-\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$

$$=\frac{-10}{27}-\frac{10}{27}+\frac{92}{27}$$

$$=\frac{72}{27}$$

$$=\frac{8}{3}$$

$\therefore$ The perpendicular distance from the origin to this new plane is $2\frac{2}{3}$.

The distance between the planes is $3-2\frac{2}{3}=\frac{1}{3}$

$\therefore$ The perpendicular distance from the point $(-1,-1,4)$ to the plane π is $\frac{1}{3}$.

c A plane parallel to π through the point (2, 1, 3) has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$

$$=\frac{20}{27}+\frac{10}{27}+\frac{69}{27}$$

$$=\frac{99}{27}$$

$$=\frac{11}{3}$$

$\therefore$ The perpendicular distance from the origin to this new plane is $3\frac{2}{3}$

$\therefore$ The distances between this plane and π is $3\frac{2}{3}-3=\frac{2}{3}$

$\therefore$ The perpendicular distance from (2, 1, 3) to π is $\frac{2}{3}$.

d A plane parallel to π through the point $(6, 12, -9)$ has equation

$$\mathbf{r} \cdot \frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (6\mathbf{i}+12\mathbf{j}-9\mathbf{k}) \cdot \frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$

$$= \frac{60}{27}+\frac{120}{27}-\frac{207}{27}$$

$$= -\frac{27}{27}$$

$$= -1$$

$\therefore$ The perpendicular distance from the origin to this new plane is 1, in the opposite direction.

$\therefore$ The distance between this plane and π is $3-(-1)=4$

$\therefore$ The perpendicular distance from (2, 1, 3) to π is 4.

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Vectors
Exercise F, Question 10

Question:

Find the shortest distance between the parallel planes.

a $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=55$ and $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=22$.

b $\mathbf{r}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{k})+\mu(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$ and

$\mathbf{r}=14\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{j}+\mathbf{k})+\mu(8\mathbf{i}-9\mathbf{j}-\mathbf{k})$

Solution:

a The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=55$ is $\dfrac{55}{\sqrt{6^2+6^2+(-7)^2}}$

$= \dfrac{55}{\sqrt{121}}$

$= \dfrac{55}{11}$

$= 5$

First find the distance from the origin to each plane, then subtract.

The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=22$ is $\dfrac{22}{\sqrt{6^2+6^2+(-7)^2}}$

$= \dfrac{22}{11}$

$= 2$

$\therefore$ The distance between the planes is $5-2=3$

b $\mathbf{r}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{k})+\mu(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$

The normal to the plane is $\mathbf{n}$ where

Express the equations of the planes in the form $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

$$\mathbf{n}=(4\mathbf{i}+\mathbf{k})\times(8\mathbf{i}+3\mathbf{j}+3\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 0 & 1\\ 8 & 3 & 3\end{vmatrix}=-3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$$

$\therefore$ Equation of plane may be written

$$\mathbf{r}\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=(3\mathbf{i}+4\mathbf{j}+\mathbf{k})\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$$

i.e. $\mathbf{r}\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=-13$

The distance from the origin to this plane is $\dfrac{-13}{\sqrt{(-3)^2+(-4)^2+12^2}}=-1$

The second plane

$\mathbf{r}=14\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{j}+\mathbf{k})+\mu(8\mathbf{i}-9\mathbf{j}-\mathbf{k})$ has normal $\mathbf{n}$ where

$$\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 3 & 1\\ 8 & -9 & -1\end{vmatrix}=6\mathbf{i}+8\mathbf{j}-24\mathbf{k}$$

This shows it is parallel to the first plane as the normal vectors are parallel.

$\therefore$ Equation of second plane may be written

$$\mathbf{r}\cdot(6\mathbf{i}+8\mathbf{j}-24\mathbf{k})=(14\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(6\mathbf{i}+8\mathbf{j}-24\mathbf{k})$$

i.e. $\mathbf{r}\cdot(6\mathbf{i}+8\mathbf{j}-24\mathbf{k})=52$ or $\mathbf{r}\cdot(-3\mathbf{i}-4\mathbf{j}+12\mathbf{k})=-26$

The distance from the origin to this plane is $\dfrac{-26}{\sqrt{3^2+4^2+(-12)^2}}=-2$

$\therefore$ The distance between the two planes is $-1-(-2)=1$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise F, Question 11

Question:

Find the shortest distance between the two skew lines with equations $\mathbf{r} = \mathbf{i} + \lambda(-3\mathbf{i} - 12\mathbf{j} + 11\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k})$, where λ and μ are scalars.

Solution:

The shortest distance is found by using the formula $\left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$.

Use the formula for shortest distance between skew lines.

$$\mathbf{a}-\mathbf{c} = \mathbf{i} - (3\mathbf{i} - \mathbf{j} + \mathbf{k}) = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\mathbf{b}\times\mathbf{d} = (-3\mathbf{i} - 12\mathbf{j} + 11\mathbf{k}) \times (2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -12 & 11 \\ 2 & 6 & -5 \end{vmatrix}$$

$$= -6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$$\therefore \text{shortest distance} = \left|\frac{(-2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(-6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})}{\sqrt{(-6)^2 + 7^2 + 6^2}}\right|$$

$$= \left|\frac{12 + 7 - 6}{\sqrt{121}}\right|$$

$$= \frac{13}{11}$$

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Vectors
Exercise F, Question 12

Question:

Find the shortest distance between the parallel lines with equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$, where λ and μ are scalars.

Solution:

Let A be a general point on the first line and B be a general point on the second line,

then $\overrightarrow{AB} = \begin{pmatrix} -2 \\ +2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$, where $t = \mu - \lambda$.

Find the minimum value of the quadratic by using calculus, or completion of the square.

Let the distance $AB = x$ then

$$x^2 = (-2-3t)^2 + (2-4t)^2 + (5t)^2$$
$$= 8 - 4t + 50t^2$$

The minimum value of x^2 occurs when $t = \frac{1}{25}$.

So $x^2 = 8 - \frac{4}{25} + \frac{50}{625}$

$$= \frac{198}{25}$$

$$\therefore x = \frac{\sqrt{198}}{5} \text{ or } 2.81 \quad (3 \text{ s.f.})$$

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Vectors
Exercise F, Question 13

Question:

Determine whether the lines l_1 and l_2 meet. If they do, find their point of intersection. If they do not, find the shortest distance between them. (In each of the following cases λ and μ are scalars.)

a l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})$

b l_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

c l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Solution:

a Assume that l_1 and l_2 meet.

Then $\begin{pmatrix} 1+2\lambda \\ 1-\lambda \\ 5\lambda \end{pmatrix} = \begin{pmatrix} -1+2\mu \\ 1-5\mu \\ 2+\mu \end{pmatrix}$

i.e.

$1+2\lambda = -1+2\mu$ ①

$1-\lambda = 1-5\mu$ ②

$5\lambda = 2+\mu$ ③

Add ① $+2\times$ ②

$\therefore 3 = 1-8\mu$

i.e. $\mu = -\frac{1}{4}$

Substitute into equation ①

$\therefore 1+2\lambda = -1\frac{1}{2}$

$\therefore \lambda = -1\frac{1}{4}$

But for these values of λ and μ equation ③ does not hold true. There is a contradiction.

$\therefore$ The lines do not meet.

They must be skew so the shortest distance between them is calculated from the formula

$$\left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right| \text{ where } \mathbf{a}-\mathbf{c} = 2\mathbf{i}-2\mathbf{k} \text{ and}$$

$$\mathbf{b}\times\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ 2 & -5 & 1 \end{vmatrix}$$

$$= 24\mathbf{i}+8\mathbf{j}-8\mathbf{k}$$

$$\therefore \text{Distance} = \left|\frac{2\mathbf{i}-2\mathbf{k}\cdot(24\mathbf{i}+8\mathbf{j}-8\mathbf{k})}{8\sqrt{3^2+1^2+(-1)^2}}\right| = \frac{32}{8\sqrt{11}} = \frac{4\sqrt{11}}{11} \text{ or } 1.21$$

b Assume that l_1 and l_2 meet:

$\begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ -1-\mu \\ 3+\mu \end{pmatrix}$

i.e. $2+2\lambda = 1+\mu$ ①

$1-2\lambda = -1-\mu$ ②

$-2+2\lambda = 3+\mu$ ③

Adding equations ① and ② gives $3=0$

This is a contradiction.

$\therefore$ Lines do not meet.

The lines are in fact parallel as $2\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ is a multiple of $\mathbf{i}-\mathbf{j}+\mathbf{k}$.

The distance between them is found by considering A on line l_1 and B on line l_2.

Then $\overrightarrow{AB} = -\mathbf{i}-2\mathbf{j}+5\mathbf{k}+t(\mathbf{i}-\mathbf{j}+\mathbf{k})$

$$|\overrightarrow{AB}|^2 = x^2 = (-1+t)^2+(-2-t)^2+(5+t)^2$$
$$= 1-2t+t^2+4+4t+t^2+25+10t+t^2$$
$$= 30+12t+3t^2$$

The minimum value of x^2 occurs when $\frac{\mathrm{d}(x^2)}{\mathrm{d}t}=0$

$$\frac{\mathrm{d}(x)^2}{\mathrm{d}t} = 12+6t$$

When $\frac{\mathrm{d}(x)^2}{\mathrm{d}t}=0, t=-2$

$$\therefore x^2 = 30-24+12$$
$$= 18$$
$$\therefore x = \sqrt{18} = 3\sqrt{2} \text{ or } 4.24 \text{ (3 s.f.)}$$

c Let l_1 meet l_2, then

$$\begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 5-2\lambda \end{pmatrix} = \begin{pmatrix} -1+\mu \\ -1+\mu \\ 2+\mu \end{pmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

Subtract ① − ②

Then $\lambda = 0$

Substitute into equation ①

Then $\mu = 2$

But $\lambda = 0, \mu = 2$ does not satisfy equation ③

So the lines do not meet.

They are skew.

Using the formula distance $= \left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

$$\mathbf{a}-\mathbf{c} = 2\mathbf{i}+2\mathbf{j}+3\mathbf{k}$$
$$\mathbf{b}\times\mathbf{d} = (2\mathbf{i}+\mathbf{j}-2\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$= 3\mathbf{i}-4\mathbf{j}+\mathbf{k}$$

$$\therefore \text{shortest distance} = \left|\frac{(2\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\sqrt{3^2+(-4)^2+1^2}}\right|$$
$$= \frac{6-8+3}{\sqrt{26}}$$
$$= \frac{1}{\sqrt{26}}$$
$$= 0.196 \text{ (3 s.f.)}$$

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Vectors
Exercise F, Question 14

Question:

Find the shortest distance between the point with coordinates $(4,1,-1)$ and the line with equation
$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, where μ is a scalar.

Solution:

Let A be the point $(4,1,-1)$ and B be the point $(3+2t,-1-t,2-t)$ which lies on the line.

Find the distance between $(4,1,-1)$ and $(3+2t,-1-t,2-t)$ at a point on the line.

Then $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

$$= \left[4-(3+2t),1-(-1-t),-1-(2-t)\right]$$

$$= \left[1-2t,2+t,-3+t\right]$$

$$\therefore |\overrightarrow{BA}|^2 = (1-2t)^2 + (2+t)^2 + (-3+t)^2$$

$$= 6t^2 - 6t + 14$$

$|\overrightarrow{BA}|$ is a minimum when $|\overrightarrow{BA}|^2$ is minimum

This minimum value can be found by calculus or completion of the square.

$$|\overrightarrow{BA}|^2 = 6\left(t^2 - t\right) + 14$$

$$= 6\left(t - \frac{1}{2}\right)^2 + 14 - \frac{6}{4}$$

This is a minimum when $t = \frac{1}{2}$ and

$$|\overrightarrow{BA}|^2 = 14 - 1\frac{1}{2} = 12\frac{1}{2}$$

$$\therefore |\overrightarrow{BA}| = \sqrt{12\frac{1}{2}} = 3.54 \text{ (3 s.f.)}$$

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Vectors
Exercise F, Question 15

Question:

The plane Π has equation $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=4$.

a Show that the line with equation $\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ lies in the plane Π.

b Show that the line with equation $\mathbf{r}=-\mathbf{i}+2\mathbf{j}+4\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ is parallel to the plane Π and find the shortest distance from the line to the plane.

Solution:

a The line $\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ passes through the point (2, 3, 1).
The point (2, 3, 1) also lies on the plane $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=4$ as $2\times1+3\times1-1=4$.
So the line and plane have a point in common.
The line is in the direction $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$.
This direction is parallel to the plane as it is perpendicular to the normal $\mathbf{i}+\mathbf{j}-\mathbf{k}$, as $-1\times1+2\times1+1\times-1=0$.
As the line also has a common point with the plane it lies in the plane.

Check that the line is perpendicular to the normal to the plane and check that the line and plane have a common point.

b The line $\mathbf{r}=-\mathbf{i}+2\mathbf{j}+4\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ is also parallel to the plane as its direction is $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$ which is perpendicular to the normal to the plane (see a).
The point $(-1,2,4)$ lies on the line. It does not lie on the plane as

$$(-\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k}) = -1+2-4 = -3 \neq 4$$

Show that there is a point on the line which does not lie on the plane.

$\therefore$ This line is parallel to the plane π but lies on the plane $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=-3$

The distance between the two planes is $\dfrac{4-(-3)}{|\mathbf{i}+\mathbf{j}-\mathbf{k}|}=\dfrac{7}{\sqrt{3}}$

$\therefore$ The shortest distance from the line to the plane is $\dfrac{7\sqrt{3}}{3}=4.04$ (3 s.f.)

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Vectors
Exercise G, Question 1

Question:

Find the shortest distance between the lines with vector equations
$\mathbf{r} = 3\mathbf{i} + s\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
where s, t are scalars. **[E]**

Solution:

Use the formula $\left|\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$ ← Write the first equation in the form $\mathbf{r} = 3\mathbf{i} - \mathbf{k} + s\mathbf{j}$

with $\mathbf{a} = 3\mathbf{i} - \mathbf{k}, \mathbf{b} = \mathbf{j}, \mathbf{c} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Then $\mathbf{a} - \mathbf{c} = -6\mathbf{i} + 2\mathbf{j}$

and $\mathbf{b} \times \mathbf{d} = \mathbf{i} - \mathbf{k}$

$\therefore$ The shortest distance is $\left|\frac{(-6\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}-\mathbf{k})}{\sqrt{1^2+(-1)^2}}\right|$

$= \left|\frac{-6}{\sqrt{2}}\right|$

$= 3\sqrt{2}$ or 4.24

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Vectors
Exercise G, Question 2

Question:

Obtain the shortest distance between the lines with equations
$\mathbf{r} = (3s-3)\mathbf{i} - s\mathbf{j} + (s+1)\mathbf{k}$
and $\mathbf{r} = (3+t)\mathbf{i} + (2t-2)\mathbf{j} + \mathbf{k}$
where s, t are parameters. **[E]**

Solution:

Use the formula $\left|\frac{(\mathbf{a}-\mathbf{c})\cdot\mathbf{b}\times\mathbf{d}}{|\mathbf{b}\times\mathbf{d}|}\right|$
with $\mathbf{a} = -3\mathbf{i}+\mathbf{k}, \mathbf{b} = 3\mathbf{i}-\mathbf{j}+\mathbf{k}, \mathbf{c} = 3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{d} = \mathbf{i}+2\mathbf{j}$

Then $\mathbf{a}-\mathbf{c} = -6\mathbf{i}+2\mathbf{j}$ and $\mathbf{b}\times\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$

$= -2\mathbf{i}+\mathbf{j}+7\mathbf{k}$

So shortest distance $= \left|\frac{(-6\mathbf{i}+2\mathbf{j})\cdot(-2\mathbf{i}+\mathbf{j}+7\mathbf{k})}{\sqrt{(-2)^2+1^2+7^2}}\right|$

$= \frac{12+2}{\sqrt{54}}$

$= \frac{14}{\sqrt{54}}$

$= \frac{14}{3\sqrt{6}}$

$= \frac{14\sqrt{6}}{18}$

$= \frac{7\sqrt{6}}{9}$ or 1.91 (3 s.f.)

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Vectors
Exercise G, Question 3

Question:

The position vectors of the points A, B, C and D relative to a fixed origin O, are $(-\mathbf{j}+2\mathbf{k}), (\mathbf{i}-3\mathbf{j}+5\mathbf{k}), (2\mathbf{i}-2\mathbf{j}+7\mathbf{k})$ and $(\mathbf{j}+2\mathbf{k})$ respectively.

a Find $\mathbf{p} = \overrightarrow{AB} \times \overrightarrow{CD}$.

b Calculate $\overrightarrow{AC} \cdot \mathbf{p}$.

Hence determine the shortest distance between the line containing AB and the line containing CD. **[E]**

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Vectors
Exercise G, Question 4

Question:

Relative to a fixed origin O, the point M has position vector $-4\mathbf{i}+\mathbf{j}-2\mathbf{k}$

The straight line l has equation $\mathbf{r}\times\overrightarrow{OM}=5\mathbf{i}-10\mathbf{k}$.

a Express the equation of the line l in the form $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter.

b Verify that the point N with coordinates $(2,-3,1)$ lies on l and find the area of ΔOMN. **[E]**

Solution:

a $\mathbf{b}=\overrightarrow{OM}=-4\mathbf{i}+\mathbf{j}-2\mathbf{k}$

Let $\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Then as $\mathbf{a}$ represents a point on the line

$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\times(-4\mathbf{i}+\mathbf{j}-2\mathbf{k})=5\mathbf{i}-10\mathbf{k}$

i.e. $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ -4 & 1 & -2\end{vmatrix}=5\mathbf{i}-10\mathbf{k}$

$\therefore(-2y-z)\mathbf{i}+(2x-4z)\mathbf{j}+(x+4y)\mathbf{k}=5\mathbf{i}-10\mathbf{k}$

Compare coefficients

$-2y-z=5$ ①

$2x-4z=0$ ②

$x+4y=-10$ ③

Let $x=2$ say

Then from equation ③ $4y=-12 \quad \therefore y=-3$

Also from equation ② $4-4z=0 \quad \therefore z=1$

$\therefore(2,-3,1)$ is one point on the line.

[Any value that you take for x will give a point on the line.]

So equation of line may be written

$\mathbf{r}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}+\lambda(-4\mathbf{i}+\mathbf{j}-2\mathbf{k})$

b It has already been shown that $(2,-3,1)$ lies on the line.

$$\text{Area }\Delta OMN=\frac{1}{2}|\overrightarrow{OM}\times\overrightarrow{ON}|=\frac{1}{2}|5\mathbf{i}-10\mathbf{k}|$$

$$=\frac{1}{2}\sqrt{5^2+(-10)^2}$$

$$=\frac{5}{2}\sqrt{5}\text{ or }5.59\text{ (3 s.f.)}$$

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Vectors
Exercise G, Question 5

Question:

The line l_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and the line l_2 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

a Find a vector which is perpendicular to both l_1 and l_2.

The point A lies on l_1 and the point B lies on l_2. Given that AB is also perpendicular to l_1 and l_2,

b find the coordinates of A and B. **[E]**

Solution:

a A vector perpendicular to l_1 and l_2 is

$$(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

b $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$= \begin{pmatrix} 2+2\mu \\ 1-\mu \\ 1+\mu \end{pmatrix} - \begin{pmatrix} 1+\lambda \\ -1+2\lambda \\ 3\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1+2\mu-\lambda \\ 2-\mu-2\lambda \\ 1+\mu-3\lambda \end{pmatrix}$$

As this is perpendicular to l_1 and to l_2 it is a multiple of $(\mathbf{i} + \mathbf{j} - \mathbf{k})$

$\therefore 1 + 2\mu - \lambda = 2 - \mu - 2\lambda \Rightarrow 3\mu + \lambda = 1$ ①

and $1 + 2\mu - \lambda = -(1 + \mu - 3\lambda) \Rightarrow 3\mu - 4\lambda = -2$ ②

Subtract ① − ②

Then $5\lambda = 3 \Rightarrow \lambda = \frac{3}{5}$

Substitute into equation ①.

Then $3\mu = 1 - \frac{3}{5}$

$\therefore \mu = \frac{2}{15}$

$\therefore$ A is the point with coordinates $\left(1\frac{3}{5}, \frac{1}{5}, 1\frac{4}{5}\right)$ and B is the point with coordinates $\left(2\frac{4}{15}, \frac{13}{15}, 1\frac{2}{15}\right)$

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Vectors
Exercise G, Question 6

Question:

A plane passes through the three points A, B, C, whose position vectors, referred to an origin O, are $(\mathbf{i}+3\mathbf{j}+3\mathbf{k}), (3\mathbf{i}+\mathbf{j}+4\mathbf{k}), (2\mathbf{i}+4\mathbf{j}+\mathbf{k})$ respectively.

a Find, in the form $(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$, a unit vector normal to this plane.

b Find also a Cartesian equation of the plane.

c Find the perpendicular distance from the origin to this plane. **[E]**

Solution:

a $\overrightarrow{AB} = (3\mathbf{i}+\mathbf{j}+4\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$
$= 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\overrightarrow{AC} = (2\mathbf{i}+4\mathbf{j}+\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$
$= \mathbf{i}+\mathbf{j}-2\mathbf{k}$

A vector normal to this plane ABC is in the direction $\overrightarrow{AB}\times\overrightarrow{AC}$.

i.e. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$

A unit vector normal to the plane is $\dfrac{1}{\sqrt{3^2+5^2+4^2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

$= \dfrac{1}{\sqrt{50}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

b The equation of the plane may be written as

$\mathbf{r}\cdot(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}) = (\mathbf{i}+3\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$
$= 3+15+12$
$= 30$

i.e. $3x+5y+4z=30$

c The perpendicular distance from the origin to the plane is

$\dfrac{30}{\sqrt{3^2+5^2+4^2}} = \dfrac{30}{\sqrt{50}} = \dfrac{30\sqrt{50}}{50} = 3\sqrt{2}$.

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Vectors
Exercise G, Question 7

Question:

a Show that the vector $\mathbf{i}+\mathbf{k}$ is perpendicular to the plane with vector equation $\mathbf{r}=\mathbf{i}+s\mathbf{j}+t(\mathbf{i}-\mathbf{k})$.

b Find the perpendicular distance from the origin to this plane.

c Hence or otherwise obtain a Cartesian equation of the plane. **[E]**

Solution:

The plane with vector equation

$\mathbf{r}=\mathbf{i}+s\mathbf{j}+t(\mathbf{i}-\mathbf{k})$

is perpendicular to $\mathbf{i}+\mathbf{k}$, as $(\mathbf{i}+\mathbf{k})\cdot\mathbf{j}=0$ and $(\mathbf{i}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{k})=1-1=0$

The plane also has equation

$\mathbf{r}\cdot(\mathbf{i}+\mathbf{k})=\mathbf{i}\cdot(\mathbf{i}+\mathbf{k})$, as $\mathbf{i}$ is the position vector of a point on the plane.

i.e. $\mathbf{r}\cdot(\mathbf{i}+\mathbf{k})=1$

The perpendicular distance from the origin to this plane is $\frac{1}{\sqrt{1^2+1^2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$ or

0.707 (3 s.f.)

The Cartesian form of the equation of the plane is $x+z=1$

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Vectors
Exercise G, Question 8

Question:

The points A, B and C have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}, 5\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ respectively, referred to an origin O.

a Find a vector perpendicular to the plane containing the points A, B and C.

b Hence, or otherwise, find an equation for the plane which contains the points A, B and C, in the form $ax+by+cz+d=0$.

The point D has coordinates (1, 5, 6).

c Find the volume of the tetrahedron $ABCD$. **[E]**

Solution:

a $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(5\mathbf{i}-2\mathbf{j}+\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=4\mathbf{i}-3\mathbf{j}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=2\mathbf{i}+\mathbf{j}+5\mathbf{k}$

Perpendicular vector to the plane is in direction

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -3 & 0\\ 2 & 1 & 5\end{vmatrix}=-15\mathbf{i}-20\mathbf{j}+10\mathbf{k}$$

b The equation on of plane containing A, B and C is

$\mathbf{r}\cdot(-15\mathbf{i}-20\mathbf{j}+10\mathbf{k})=(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(-15\mathbf{i}-20\mathbf{j}+10\mathbf{k})$

i.e. $-15x-20y+10z=-25$

or $3x+4y-2z-5\quad=0$

c Volume of tetrahedron $ABCD=\left|\frac{1}{6}\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(\mathbf{i}+5\mathbf{j}+6\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=4\mathbf{j}+5\mathbf{k}$

$$\therefore \text{Volume}=\frac{1}{6}\left|(4\mathbf{j}+5\mathbf{k})\cdot(-15\mathbf{i}-20\mathbf{j}+10\mathbf{k})\right|$$
$$=\frac{1}{6}\left|(-80+50)\right|$$
$$=5$$

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Vectors
Exercise G, Question 9

Question:

The plane Π passes through $A(3,-5,-1), B(-1,5,7)$ and $C(2,-3,0)$.

a Find $\overrightarrow{AC} \times \overrightarrow{BC}$.

b Hence, or otherwise, find the equation, in the form $\mathbf{r} \cdot \mathbf{n} = p$, of the plane Π.

c The perpendicular from the point $(2,3,-2)$ to Π meets the plane at P. Find the coordinates of P. **[E]**

Solution:

a $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = (2\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} - 5\mathbf{j} - \mathbf{k})$

$= -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (2\mathbf{i} - 3\mathbf{j}) - (-\mathbf{i} + 5\mathbf{j} + 7\mathbf{k})$

$= 3\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$

$$\therefore \overrightarrow{AC} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & -8 & -7 \end{vmatrix} = -6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

b Equation of the plane π is

$\mathbf{r} \cdot (-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = (3\mathbf{i} - 5\mathbf{j} - \mathbf{k}) \cdot (-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

$= -18 + 20 - 2$

$= 0$

or $\mathbf{r} \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$

c The perpendicular from $(2,3,-2)$ to π has equation

$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

This meets the plane π when

$((2+3\lambda)\mathbf{i} + (3+2\lambda)\mathbf{j} + (-2-\lambda)\mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$

i.e. $3(2+3\lambda) + 2(3+2\lambda) - 1(-2-\lambda) = 0$

i.e. $14\lambda + 14 = 0$

$\therefore \lambda = -1$

$\therefore$ Substitute into equation of line

$\mathbf{r} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\therefore$ Foot of perpendicular is at $(-1,1,-1)$

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Vectors
Exercise G, Question 10

Question:

Given that P and Q are the points with position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, relative to an origin O, and that

$\mathbf{p} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$\mathbf{q} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$,

a find $\mathbf{p} \times \mathbf{q}$.

b Hence, or otherwise, find an equation of the plane containing O, P and Q in the form $ax + by + cz = d$.

The line with equation $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = 0$ meets the plane with equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ at the point T.

c Find the coordinates of the point T. **[E]**

Solution:

a $\mathbf{p} \times \mathbf{q} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - \mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

b The equation of the plane is

$\mathbf{r} \cdot (-\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 7\mathbf{j} + 5\mathbf{k})$

i.e. $-x + 7y + 5z = 0$

c The line equation may be written in the form

$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

This meets the plane $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ when $(3 + 2\lambda) + (-1 + \lambda) + (2 - \lambda) = 2$

i.e. $2\lambda + 4 = 2$

$\therefore \lambda = -1$

Substitute into the line equation

Then $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

The coordinates of point T are $(1, -2, 3)$

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Vectors
Exercise G, Question 11

Question:

The planes Π_1 and Π_2 are defined by the equations $2x+2y-z=9$ and $x-2y=7$ respectively.

a Find the acute angle between Π_1 and Π_2, giving your answer to the nearest degree.

b Find in the form $\mathbf{r}\times\mathbf{u}=\mathbf{v}$ an equation of the line of intersection of Π_1 and Π_2. **[E]**

Solution:

a The normals to the planes are
$\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j}$
The angle between the normals is θ where

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2\times1-2\times2}{\sqrt{2^2+2^2+(-1)^2}\sqrt{1^2+(-2)^2}}$$

$$= \frac{-2}{\sqrt{9}\sqrt{5}}$$

$$= \frac{-2\sqrt{5}}{15}$$

$\therefore$ The acute angle α between the planes is given by $\cos\alpha = \frac{2\sqrt{5}}{15}$,

i.e. $\alpha = 72.7° = 73°$ (nearest degree)

b The planes have equations $2x + 2y - z = 9$ ①
and $x - 2y = 7$ ②

Add ① + ②
Then $3x - z = 16$
$\therefore x = \frac{z+16}{3}$.
Also from equation ②
$x = \frac{7+2y}{1}$
Let $x = \lambda$

Then $\frac{x-0}{1} = \frac{y+\frac{7}{2}}{\frac{1}{2}} = \frac{z+16}{3} = \lambda$

This may be written

$$\mathbf{r}\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right) = \left(\frac{-7}{2}\mathbf{j}-16\mathbf{k}\right)\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right)$$

$$\text{i.e. } \mathbf{r}\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -\frac{7}{2} & -16\\ 1 & \frac{1}{2} & 3\end{vmatrix}$$

$$= \left(\frac{-21}{2}+8\right)\mathbf{i} - 16\mathbf{j} + \frac{7}{2}\mathbf{k}$$

$$= -\frac{5}{2}\mathbf{i} - 16\mathbf{j} + \frac{7}{2}\mathbf{k}$$

$$\therefore \mathbf{r}\times\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+3\mathbf{k}\right) = \left(-\frac{5}{2}\mathbf{i}-16\mathbf{j}+\frac{7}{2}\mathbf{k}\right)$$

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Vectors
Exercise G, Question 12

Question:

A pyramid has a square base *OPQR* and vertex *S*. Referred to *O*, the points *P*, *Q*, *R* and *S* have position vectors $\overrightarrow{OP} = 2\mathbf{i}, \overrightarrow{OQ} = 2\mathbf{i} + 2\mathbf{j}, \overrightarrow{OR} = 2\mathbf{j}, \overrightarrow{OS} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

a Express *PS* in terms of **i**, **j** and **k**.

b Show that the vector $-4\mathbf{j} + \mathbf{k}$ is perpendicular to *OS* and *PS*.

c Find to the nearest degree the acute angle between the line *SQ* and the plane *OSP*.

[E]

Solution:

a $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$

$= \mathbf{i} + \mathbf{j} + 4\mathbf{k} - 2\mathbf{i}$

$= -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

b $(-4\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = -4 + 4 = 0$

$\therefore -4\mathbf{j} + \mathbf{k}$ is perpendicular to $\overrightarrow{OS}$.

Also $(-4\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = -4 + 4 = 0$

$\therefore -4\mathbf{j} + \mathbf{k}$ is perpendicular to $\overrightarrow{PS}$.

c $-4\mathbf{j} + \mathbf{k}$ is normal to the plane *OSP*.

$\overrightarrow{SQ} = \overrightarrow{OQ} - \overrightarrow{OS}$

$= 2\mathbf{i} + 2\mathbf{j} - (\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= \mathbf{i} + \mathbf{j} - 4\mathbf{k}$

The acute angle θ between $\overrightarrow{SQ}$ and the normal to the plane is given by

$$\cos\theta = \left| \frac{(-4\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 4\mathbf{k})}{\sqrt{(-4)^2 + 1^2}\sqrt{1^2 + 1^2 + (-4)^2}} \right|$$

$$= \left| \frac{-8}{\sqrt{17}\sqrt{18}} \right| = \frac{8}{\sqrt{17}\sqrt{18}}$$

The angle α between the line *SQ* and the plane *OSP* is such that $\alpha + \theta = 90°$ and so $\sin\alpha = \dfrac{8}{\sqrt{17}\sqrt{18}}$ and $\alpha = 27°$ (nearest degree)

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Vectors
Exercise G, Question 13

Question:

The plane Π has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + u\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + v\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \text{ where } u \text{ and } v \text{ are parameters.}$$

The line L has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, where t is a parameter.

a Show that L is parallel to Π.

b Find the shortest distance between L and Π. **[E]**

Solution:

a The normal to the plane Π is in the direction $(4\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$

i.e. $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 1 & 2\\ 3 & 2 & -1\end{vmatrix} = -5\mathbf{i}+10\mathbf{j}+5\mathbf{k}$

The line L is in the direction $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$
As $(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-4\mathbf{k}) = 0$
the line L is perpendicular to the normal to the plane.

Thus L is parallel to the plane Π.

b The line L passes through point $(2, 1, -3)$

The perpendicular to plane π through $(2, 1, -3)$ has equation
$\mathbf{r} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}+\lambda(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k})$

The equation of the plane may be written
$\mathbf{r}\cdot(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k}) = (\mathbf{i}+3\mathbf{j}+4\mathbf{k})\cdot(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k})$
$= 45$

This perpendicular meets plane Π when
$((2-5\lambda)\,\mathbf{i}+(1+10\lambda)\,\mathbf{j}+(-3+5\lambda)\,\mathbf{k})\cdot(-5\mathbf{i}+10\mathbf{j}+5\mathbf{k}) = 45$
i.e. $-10+25\lambda+10+100\lambda-15+25\lambda = 45$
i.e. $150\lambda = 60 \Rightarrow \lambda = \frac{2}{5}$
Substitute $\lambda = \frac{2}{5}$ into the equation of the perpendicular.
Then $\mathbf{r} = 5\mathbf{j}-\mathbf{k}$
i.e. The perpendicular to Π from $(2,1,-3)$ meets the plane at $(0,5,-1)$
$\therefore$ Shortest distance from L to Π is

$\sqrt{(2-0)^2+(1-5)^2+(-3-(-1))^2}$
$= \sqrt{4+16+4}$
$= \sqrt{24} = 2\sqrt{6}$ or 4.90
or

Take point A on Π $\begin{pmatrix}1\\3\\4\end{pmatrix}$ and B on L $\begin{pmatrix}2\\1\\-3\end{pmatrix}$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -7 \end{pmatrix}$$

$$\text{Distance} = |\overrightarrow{AB} \cdot \hat{\mathbf{n}}|$$

$$|\mathbf{n}| = \sqrt{(-5)^2 + 10^2 + 5^2} = \sqrt{150} = 5\sqrt{6}$$

$$\therefore \hat{\mathbf{n}} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore \text{Distance} = \frac{1}{\sqrt{6}} \left| \begin{pmatrix} 1 \\ -2 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right| = \frac{1}{\sqrt{6}} \times 12 = \frac{12}{\sqrt{6}} = 2\sqrt{6} = 4.90$$

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Vectors
Exercise G, Question 14

Question:

Planes Π_1 and Π_2 have equations given by

$\Pi_1 : \mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$,

$\Pi_2 : \mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$.

a Show that the point $A(2, -2, 3)$ lies in Π_2.

b Show that Π_1 is perpendicular to Π_2.

c Find, in vector form, an equation of the straight line through A which is perpendicular to Π_1.

d Determine the coordinates of the point where this line meets Π_1.

e Find the perpendicular distance of A from Π_1.

f Find a vector equation of the plane through A parallel to Π_1. **[E]**

Solution:

a $(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = 2-10+9$

$= 1$

$\therefore (2,-2,3)$ lies on the plane Π_2

b $(2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = 2-5+3$

$= 0$

$\therefore$ the normal to plane Π_1 is perpendicular to the normal to plane Π_2.

$\therefore \Pi_1$ is perpendicular to Π_2.

c $\mathbf{r} = 2\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

d This line meets the plane Π_1 when

$[(2+2\lambda)\mathbf{i}+(-2-\lambda)\mathbf{j}+(3+\lambda)\mathbf{k}]\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=0$

i.e. $4+4\lambda+2+\lambda+3+\lambda=0$

i.e. $6\lambda+9=0$

$\therefore \lambda = -\frac{3}{2}$

Substitute $\lambda=-\frac{3}{2}$ into the equation of the line: then $\mathbf{r}=-\mathbf{i}-\frac{1}{2}\mathbf{j}+\frac{3}{2}\mathbf{k}$

i.e. The line meets Π_1 at the point $\left(-1,-\frac{1}{2},\frac{3}{2}\right)$

e The distance required is

$$\sqrt{(2-(-1))^2+\left(-2-\left(-\frac{1}{2}\right)\right)^2+\left(3-\frac{3}{2}\right)^2} = \sqrt{9+2\frac{1}{4}+2\frac{1}{4}} = \sqrt{13\frac{1}{2}}$$

$$= 3.67 \text{ (3 s.f.)}$$

f $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = (2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

$= 4+2+3$

i.e. $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 9$

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Vectors
Exercise G, Question 15

Question:

The plane Π has equation $2x + y + 3z = 21$ and the origin is O. The line l passes through the point $P(1, 2, 1)$ and is perpendicular to Π.

a Find a vector equation of l.

The line l meets the plane Π at the point M.

b Find the coordinates of M.

c Find $\overrightarrow{OP} \times \overrightarrow{OM}$.

d Hence, or otherwise, find the distance from P to the line OM, giving your answer in surd form.

The point Q is the reflection of P in Π.

e Find the coordinates of Q. **[E]**

Solution:

a $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

b This line meets plane Π when

$(1+2\lambda)\cdot 2 + (2+\lambda)\cdot 1 + (1+3\lambda)\cdot 3 = 21$

i.e. $14\lambda + 7 = 21$

i.e. $\lambda = 1$

Substitute $\lambda = 1$ into the equation of the line l.

Then $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

So M has coordinates (3, 3, 4)

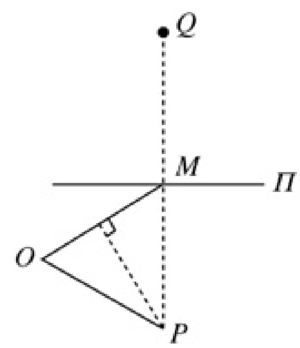
c $\overrightarrow{OP} \times \overrightarrow{OM} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$= 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

d Area of $\Delta\, OPM = \frac{1}{2}|5\mathbf{i} - \mathbf{j} - 3\mathbf{k}|$

$= \frac{1}{2}\sqrt{5^2 + (-1)^2 + (-3)^2}$

$= \frac{1}{2}\sqrt{35}$



$\therefore$ Distance from P to line $OM = \dfrac{\frac{1}{2}\sqrt{35}}{\frac{1}{2}|\mathrm{OM}|}$

$= \dfrac{\frac{1}{2}\sqrt{35}}{\frac{1}{2}\sqrt{3^2 + 3^2 + 4^2}}$

$= \dfrac{\sqrt{35}}{\sqrt{34}}$

e $\overrightarrow{PM} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} - (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\therefore \overrightarrow{MQ} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

And $\overrightarrow{OQ} = \overrightarrow{OM} + \overrightarrow{MQ} = 5\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$

Q has coordinates (5, 4, 7)

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Vectors
Exercise G, Question 16

Question:

With respect to a fixed origin O, the straight lines l_1 and l_2 are given by

$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}),$

$l_2 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(-3\mathbf{i} + 4\mathbf{k}),$

where λ and μ are scalar parameters.

a Show that the lines intersect.

b Find the position vector of their point of intersection.

c Find the cosine of the acute angle contained between the lines.

d Find a vector equation of the plane containing the lines. **[E]**

Solution:

a The lines l_1 and l_2 intersect if

$$\begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ -2\lambda \end{pmatrix} = \begin{pmatrix} 1-3\mu \\ 2 \\ 2+4\mu \end{pmatrix}$$

have consistent solutions.

i.e. $2\lambda = -3\mu$ ①

$\lambda = 3$ ②

and $-2\lambda = 4\mu + 2$ ③

Substitute $\lambda = 3$ from ② into ①, then $\mu = -2$

Check in equation ③ $\lambda = 3$ and $\mu = -2$ satisfy equation ③

$\therefore$ the lines intersect

b Substitute $\lambda = 3$ into equation of l_1

Then $\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$

This is the position vector of the point of intersection.

c Let θ be the acute angle between the lines.

$$\text{Then } \cos\theta = \left| \frac{(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (-3\mathbf{i} + 4\mathbf{k})}{\sqrt{2^2 + 1^2 + (-2)^2}\sqrt{(-3)^2 + 4^2}} \right|$$

$$= \left| \frac{-6-8}{\sqrt{9}\sqrt{25}} \right|$$

$$= \frac{14}{15}$$

d $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \mu(-3\mathbf{i} + 4\mathbf{k})$ is a vector equation for the plane.

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Vectors
Exercise G, Question 17

Question:

Relative to an origin O, the points A and B have position vectors $\mathbf{a}$ metres and $\mathbf{b}$ metres respectively, where
$\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$
The point C moves such that the volume of the tetrahedron $OABC$ is always $5\,\text{m}^3$.
Determine Cartesian equations of the locus of the point C. **[E]**

Solution:

Let C be the point with position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
The volume of the tetrahedron $OABC$ is given by

$$\frac{1}{6}\begin{vmatrix} x & y & z \\ 5 & 2 & 0 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= \frac{1}{6}(-6x + 15y - 9z)$$

As the volume is $5\,\text{m}^3$,

$$\therefore \frac{1}{6}(-6x + 15y - 9z) = 5$$

i.e. $-6x + 15y - 9z = 30$

or $2x - 5y + 3z + 10 = 0$, which is the locus of the point C.

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Vectors
Exercise G, Question 18

Question:

The lines L_1 and L_2 have equations $\mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + t\mathbf{b}_2$ respectively, where

$\mathbf{a}_1 = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b}_1 = \mathbf{j} + 2\mathbf{k}$,

$\mathbf{a}_2 = 8\mathbf{i} + 3\mathbf{j}$, $\mathbf{b}_2 = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

a Verify that the point P with position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ lies on both L_1 and L_2.

b Find $\mathbf{b}_1 \times \mathbf{b}_2$.

c Find a Cartesian equation of the plane containing L_1 and L_2.

The points with position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are A_1 and A_2 respectively.

d By expressing $\overrightarrow{A_1P}$ and $\overrightarrow{A_2P}$ as multiples of $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively, or otherwise, find the area of the triangle PA_1A_2. **[E]**

Solution:

a Equation of l_1 is

$\mathbf{r} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k})$

When $\lambda = 2, \mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ So P lies on l_1.

Equation of l_2 is

$\mathbf{r} = 8\mathbf{i} + 3\mathbf{j} + \mu(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

When $\mu = -1, \mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. So P lies on l_2.

b $$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}$$

c The normal to the plane is in direction of $\mathbf{b}_1 \times \mathbf{b}_2$. So $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is a normal.

$\therefore$ Equation of plane is

$$\mathbf{r} \cdot (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) \cdot (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -6 - 6 + 2$$

$\therefore -2x + 2y - z = -10$

$\therefore +2x - 2y + z = 10$ is a Cartesian equation of the plane.

d $\overrightarrow{A_1P} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 2\mathbf{j} + 4\mathbf{k} = 2\mathbf{b}_1$

$\overrightarrow{A_2P} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (8\mathbf{i} + 3\mathbf{j}) = (-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = -\mathbf{b}_2$

$$\text{Area of } PA_1A_2 = \frac{1}{2}\left|\overrightarrow{A_1P} \times \overrightarrow{A_2P}\right| = \frac{1}{2}\left|2\mathbf{b}_1 \times -\mathbf{b}_2\right|$$
$$= |\mathbf{b}_1 \times \mathbf{b}_2|$$
$$= \sqrt{(-10)^2 + (10)^2 + (-5)^2}$$
$$= \sqrt{225}$$
$$= 15$$

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Vectors
Exercise G, Question 19

Question:

With respect to the origin O the points A, B, C have position vectors $a(5\mathbf{i}-\mathbf{j}-3\mathbf{k}), a(-4\mathbf{i}+4\mathbf{j}-\mathbf{k}), a(5\mathbf{i}-2\mathbf{j}+11\mathbf{k})$ respectively, where a is a non-zero constant.

Find

a a vector equation for the line BC,

b a vector equation for the plane OAB,

c the cosine of the acute angle between the lines OA and OB.

Obtain, in the form $\mathbf{r}\cdot\mathbf{n}=p$, a vector equation for Π, the plane which passes through A and is perpendicular to BC.

Find Cartesian equations for

d the plane Π,

e the line BC.

Solution:

a $\overrightarrow{BC} = a(5\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}) - a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

$= a(9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$

$\therefore$ vector equation for the line BC is

$\mathbf{r} = a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + \lambda a(9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$

b A vector equation for the plane OAB is

$\mathbf{r} = a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + \lambda a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + \mu a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

c Let the acute angle between OA and OB be θ

Then $\cos\theta = \left|\dfrac{a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \cdot a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})}{a\sqrt{25+1+9}\ a\ \sqrt{16+16+1}}\right|$

$= \left|\dfrac{-12}{\sqrt{35}\sqrt{33}}\right|$

$= \dfrac{12}{\sqrt{35}\sqrt{33}} = 0.353$ (3 s.f.)

The plane through A, perpendicular to BC has equation

$\mathbf{r} \cdot (9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \cdot (9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$

i.e. $\mathbf{r} \cdot (9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = 15a$

or $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 5a$

d The Cartesian equation for this plane Π is $3x - 2y + 4z = 5a$

e The Cartesian equation for the line BC comes from

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4a \\ 4a \\ -a \end{pmatrix} + \lambda \begin{pmatrix} 9a \\ -6a \\ 12a \end{pmatrix}$$

$\therefore \dfrac{x+4a}{9} = \dfrac{y-4a}{-6} = \dfrac{z+a}{12} = \lambda a$

or $\dfrac{x+4a}{3} = \dfrac{y-4a}{-2} = \dfrac{z+a}{4} = \lambda'$

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Vectors
Exercise G, Question 20

Question:

In a tetrahedron $ABCD$ the coordinates of the vertices B, C, D are respectively (1, 2, 3), (2, 3, 3), (3, 2, 4). Find

a the equation of the plane BCD.

b the sine of the angle between BC and the plane $x + 2y + 3z = 4$.

If AC and AD are perpendicular to BD and BC respectively and if $AB = \sqrt{26}$, find the coordinates of the two possible positions of A.

Solution:

a $\overrightarrow{BC} = (2\mathbf{i}+3\mathbf{j}+3\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \mathbf{i}+\mathbf{j}$

$\overrightarrow{BD} = (3\mathbf{i}+2\mathbf{j}+4\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 2\mathbf{i}+\mathbf{k}$

$$\therefore \overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

This is normal to the plane BCD.

$\therefore$ The equation of the plane BCD is

$$\mathbf{r} \cdot (\mathbf{i}-\mathbf{j}-2\mathbf{k}) = (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) \cdot (\mathbf{i}-\mathbf{j}-2\mathbf{k})$$

$$= 1-2-6$$

$$= -7$$

This may be written $x - y - 2z + 7 = 0$

b Let the required angle be α. Then $\sin\alpha = \cos\theta$ where θ is the acute angle between BC and the normal vector $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

$$\therefore \cos\theta = \frac{(\mathbf{i}+\mathbf{j}) \cdot (\mathbf{i}+2\mathbf{j}+3\mathbf{k})}{\sqrt{1^2+1^2}\sqrt{1^2+2^2+3^2}}$$

$$= \frac{3}{\sqrt{2}\sqrt{14}} = 0.567 \text{ (3 s.f.)}$$

c Let A have coordinates (x, y, z).

Then $\overrightarrow{AC} = (2-x)\mathbf{i} + (3-y)\mathbf{j} + (3-z)\mathbf{k}$

Also $\overrightarrow{AD} = (3-x)\mathbf{i} + (2-y)\mathbf{j} + (4-z)\mathbf{k}$

As AC is perpendicular to BD, $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$\therefore 2(2-x) + 0(3-y) + 1(3-z) = 0$

$\therefore 2x + z = 7$ ①

As AD is perpendicular to BC, $\overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$\therefore 1(3-x) + 1(2-y) + 0(4-z) = 0$

$\therefore x + y = 5$ ②

Also $AB = \sqrt{26}$.

$\therefore (x-1)^2 + (y-2)^2 + (z-3)^2 = 26$ ③

Substitute $z = 7-2x$ and $y = 5-x$ from equations ① and ② into equation ③

Then $(x-1)^2 + (3-x)^2 + (4-2x)^2 = 26$

$\therefore 6x^2 - 24x + 26 = 26$

$\therefore 6x(x-4) = 0$

$\therefore x = 0$ or 4

When $x = 0, y = 5$ and $z = 7$

When $x = 4, y = 1$ and $z = -1$

$\therefore$ The two possible positions are $(0, 5, 7)$ and $(4, 1, -1)$

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Further matrix algebra
Exercise A, Question 1

Question:

Write down the transposes of the following matrices. In each case give the dimensions of the transposed matrix.

a $\begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 4 \end{pmatrix}$

b $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

c $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Solution:

a $\begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 2 & 4 \end{pmatrix}$ dimension 3×2

b $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ dimension 2×2

c $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$ dimension 3×3

d $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$ dimension 1×3

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Further matrix algebra
Exercise A, Question 2

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 4\\ -3 & 6\end{pmatrix}$.

a Write down $\mathbf{A}^{\mathrm{T}}$.

b Find $\mathbf{AA}^{\mathrm{T}}$.

c Find $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

Solution:

a $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}2 & -3\\ 4 & 6\end{pmatrix}$

b
$$\mathbf{AA}^{\mathrm{T}}=\begin{pmatrix}2 & 4\\ -3 & 6\end{pmatrix}\begin{pmatrix}2 & -3\\ 4 & 6\end{pmatrix}=\begin{pmatrix}4+16 & -6+24\\ -6+24 & 9+36\end{pmatrix}=\begin{pmatrix}20 & 18\\ 18 & 45\end{pmatrix}$$

c
$$\mathbf{A}^{\mathrm{T}}\mathbf{A}=\begin{pmatrix}2 & -3\\ 4 & 6\end{pmatrix}\begin{pmatrix}2 & 4\\ -3 & 6\end{pmatrix}=\begin{pmatrix}4+9 & 8-18\\ 8-18 & 16+36\end{pmatrix}=\begin{pmatrix}13 & -10\\ -10 & 52\end{pmatrix}$$

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Further matrix algebra
Exercise A, Question 3

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix}$.

a Find $\mathbf{BA}$.

b Verify that $\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} = (\mathbf{BA})^{\mathrm{T}}$.

Solution:

a $\mathbf{BA} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3-12 & 2+6 \\ 0+8 & 0-4 \end{pmatrix}$

$= \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$

b From **a**

$(\mathbf{BA})^{\mathrm{T}} = \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$ ($\mathbf{BA}$ is symmetric)

$\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -4 \end{pmatrix}$

$= \begin{pmatrix} 3-12 & 0+8 \\ 2+6 & 0-4 \end{pmatrix}$

$= \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix} = (\mathbf{BA})^{\mathrm{T}}$, as required.

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Further matrix algebra
Exercise A, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -4 & 8 \\ 4 & -7 & -4 \\ 8 & 4 & 1 \end{pmatrix}$.

a Write down $\mathbf{A}^T$.

b Show that $\mathbf{A}\mathbf{A}^T = 81\mathbf{I}$.

Solution:

a $\mathbf{A}^T = \begin{pmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{pmatrix}$

b $$\mathbf{A}\mathbf{A}^T = \begin{pmatrix} 1 & -4 & 8 \\ 4 & -7 & -4 \\ 8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+16+64 & 4+28-32 & 8-16+8 \\ 4+28-32 & 16+49+16 & 32-28-4 \\ 8-16+8 & 32-28-4 & 64+16+1 \end{pmatrix}$$

$$= \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = 81\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 81\mathbf{I}, \text{ as required.}$$

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Further matrix algebra
Exercise A, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}0 & 3 & 5\\ -3 & 0 & -1\\ -5 & 1 & 0\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}-4 & 1 & -1\\ 1 & 5 & 2\\ -3 & 0 & 3\end{pmatrix}$.

Given that $\mathbf{C}=\mathbf{AB}$,

a find $\mathbf{C}$,

b verify that the matrix $\mathbf{C}$ is symmetric.

Solution:

a $\mathbf{C}=\begin{pmatrix}0 & 3 & 5\\ -3 & 0 & -1\\ -5 & 1 & 0\end{pmatrix}\begin{pmatrix}-4 & 1 & -1\\ 1 & 5 & 2\\ -3 & 0 & 3\end{pmatrix}$

$$=\begin{pmatrix}0+3-15 & 0+15+0 & 0+6+15\\ 12+0+3 & -3+0+0 & 3+0+-3\\ 20+1+0 & -5+5+0 & 5+2+0\end{pmatrix}$$

$$=\begin{pmatrix}-12 & 15 & 21\\ 15 & -3 & 0\\ 21 & 0 & 7\end{pmatrix}$$

b $\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-12 & 15 & 21\\ 15 & -3 & 0\\ 21 & 0 & 7\end{pmatrix}=\mathbf{C}$

Hence the matrix $\mathbf{C}$ is symmetric.

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Further matrix algebra
Exercise A, Question 6

Question:

The matrix $\mathbf{A}=\begin{pmatrix}0 & 3 & 5\\ 2 & 0 & -1\\ 1 & 1 & 0\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}$.

a Find $\mathbf{AB}$.

b Verify that $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

Solution:

a $$\mathbf{AB}=\begin{pmatrix}0 & 3 & 5\\ 2 & 0 & -1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & -1\\ 0 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}$$
$$=\begin{pmatrix}0+0-5 & 0+3+0 & 0+0+15\\ 2+0+1 & 2+0+0 & -2+0-3\\ 1+0+0 & 1+1+0 & -1+0+0\end{pmatrix}$$
$$=\begin{pmatrix}-5 & 3 & 15\\ 3 & 2 & -5\\ 1 & 2 & -1\end{pmatrix}$$

b From **a**

$$(\mathbf{AB})^{\mathrm{T}}=\begin{pmatrix}-5 & 3 & 1\\ 3 & 2 & 2\\ 15 & -5 & -1\end{pmatrix}$$
$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1 & 0 & -1\\ 1 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}\begin{pmatrix}0 & 2 & 1\\ 3 & 0 & 1\\ 5 & -1 & 0\end{pmatrix}$$
$$=\begin{pmatrix}0+0-5 & 2+0+1 & 1+0+0\\ 0+3+0 & 2+0+0 & 1+1+0\\ 0+0+15 & -2+0-3 & -1+0+0\end{pmatrix}$$
$$=\begin{pmatrix}-5 & 3 & 1\\ 3 & 2 & 2\\ 15 & -5 & -1\end{pmatrix}=(\mathbf{AB})^{\mathrm{T}}\text{, as required.}$$

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Further matrix algebra
Exercise B, Question 1

Question:

Find the values of the determinants.

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

b $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$

c $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix}$

d $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix}$

Solution:

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$

$= 1(6-0) - 0 + 0 = 6$

b $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 0\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} + 0\begin{vmatrix} 5 & -2 \\ 2 & 1 \end{vmatrix}$

$= 0 - 4(20-6) + 0 = -56$

c $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$

$= 1(8-5) - 0 + 1(10-12)$

$= 3 - 2 = 1$

d $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} + 4\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix}$

$= 2(10-10) + 3(10-10) + 4(10-10) = 0$

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Further matrix algebra
Exercise B, Question 2

Question:

Find the values of the determinants.

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$

b $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$

c $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

Solution:

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix} = 4\begin{vmatrix} -2 & 0 \\ 4 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix}$

$= 4(4-0) - 3(-4-0) - 1(8-0)$

$= 16 + 12 - 8 = 20$

b $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix} = 3\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -3 \\ 7 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix}$

$= 3(-4+6) + 2(-16+21) + 1(8-7)$

$= 6 + 10 + 1 = 17$

c $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix} = 5\begin{vmatrix} 4 & 2 \\ -4 & -3 \end{vmatrix} - (-2)\begin{vmatrix} 6 & 2 \\ -2 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 6 & 4 \\ -2 & -4 \end{vmatrix}$

$= 5(-12+8) + 2(-18+4) - 3(-24+8)$

$= 5 \times (-4) + 2 \times (-14) - 3 \times (-16)$

$= -20 - 28 + 48 = 0$

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Further matrix algebra
Exercise B, Question 3

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{pmatrix}$.

Given that $\mathbf{A}$ is singular, find the value of the constant k.

Solution:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & k \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2k+1 & k \\ 1 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2k+1 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 2(3-0) - 1(2k+1-k) - 4(0-3)$$

$$= 6 - k - 1 + 12 = 17 - k$$

As $\mathbf{A}$ is singular,

$$\det(\mathbf{A}) = 0$$

$$17 - k = 0$$

$$k = 17$$

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Further matrix algebra
Exercise B, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant.

Given that the determinant of $\mathbf{A}$ is 8, find the possible values of k.

Solution:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 1 & k+3 \end{vmatrix} - (-1)\begin{vmatrix} k & 4 \\ -2 & k+3 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 2(2k+6-4) + 1(k^2+3k+8) + 3(k+4)$$

$$= 4k+4+k^2+3k+8+3k+12$$

$$= k^2+10k+24$$

As $\det(\mathbf{A}) = 8$

$$k^2+10k+24 = 8$$

$$k^2+10k+16 = 0$$

$$(k+8)(k+2) = 0$$

$$k = -8, -2$$

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Further matrix algebra
Exercise B, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 5 & 3\\-2 & 0 & 4\\3 & 10 & 8\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}1 & 1 & 0\\1 & 2 & 2\\0 & -2 & -1\end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.
b Find $\mathbf{AB}$.
c Show that $\mathbf{AB}$ is also singular.

Solution:

a $\det(\mathbf{A})=\begin{vmatrix}2 & 5 & 3\\-2 & 0 & 4\\3 & 10 & 8\end{vmatrix}$

$$=2\begin{vmatrix}0 & 4\\10 & 8\end{vmatrix}-5\begin{vmatrix}-2 & 4\\3 & 8\end{vmatrix}+3\begin{vmatrix}-2 & 0\\3 & 10\end{vmatrix}$$

$$=2(0-40)-5(-16-12)+3(-20-0)$$

$$=-80+140-60=0$$

Hence $\mathbf{A}$ is singular.

b $\mathbf{AB}=\begin{pmatrix}2 & 5 & 3\\-2 & 0 & 4\\3 & 10 & 8\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\1 & 2 & 2\\0 & -2 & -1\end{pmatrix}$

$$=\begin{pmatrix}2+5+0 & 2+10-6 & 0+10-3\\-2+0+0 & -2+0-8 & 0+0-4\\3+10+0 & 3+20-16 & 0+20-8\end{pmatrix}$$

$$=\begin{pmatrix}7 & 6 & 7\\-2 & -10 & -4\\13 & 7 & 12\end{pmatrix}$$

c $\det(\mathbf{AB})=\begin{vmatrix}7 & 6 & 7\\-2 & -10 & -4\\13 & 7 & 12\end{vmatrix}$

$$=7\begin{vmatrix}-10 & -4\\7 & 12\end{vmatrix}-6\begin{vmatrix}-2 & -4\\13 & 12\end{vmatrix}+7\begin{vmatrix}-2 & -10\\13 & 7\end{vmatrix}$$

$$=7(-120+28)-6(-24+52)+7(-14+130)$$

$$=7\times(-92)-6\times 28+7\times 116$$

$$=-644-168+812=0$$

Hence $\mathbf{AB}$ is also singular.

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Further matrix algebra
Exercise B, Question 6

Question:

The matrix $A = \begin{pmatrix} 4 & 5 & -2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{pmatrix}$.

a Find det (A).

b Write down A^T.

c Verify that $\det(A^T) = \det(A)$.

Solution:

a $\det(A) = \begin{vmatrix} 4 & 5 & -2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{vmatrix}$

$$= 4\begin{vmatrix} -3 & 2 \\ -4 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & -3 \\ 2 & -4 \end{vmatrix}$$

$$= 4(-9+8) - 5(6-4) - 2(-8+6)$$

$$= -4 - 10 + 4 = -10$$

b $A^T = \begin{pmatrix} 4 & 2 & 2 \\ 5 & -3 & -4 \\ -2 & 2 & 3 \end{pmatrix}$

c $\det(A^T) = \begin{vmatrix} 4 & 2 & 2 \\ 5 & -3 & -4 \\ -2 & 2 & 3 \end{vmatrix}$

$$= 4\begin{vmatrix} -3 & -4 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & -4 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & -3 \\ -2 & 2 \end{vmatrix}$$

$$= 4(-9+8) - 2(15-8) + 2(10-6)$$

$$= -4 - 14 + 8 = -10$$

$$= \det(A), \text{ as required.}$$

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Further matrix algebra
Exercise B, Question 7

Question:

a Show that, for all values of a, b and c, the matrix $\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$ is singular.

b Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non-singular.

Solution:

a $$\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} - a\begin{vmatrix} -a & c \\ b & 0 \end{vmatrix} + (-b)\begin{vmatrix} -a & 0 \\ b & -c \end{vmatrix}$$
$$= 0 - a(0 - cb) - b(ac - 0)$$
$$= abc - abc = 0$$

Hence the matrix is singular for all a, b and c.

b $$\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix} - (-2)\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$$
$$= 2(x^2 + 6) + 2(3x - 2) + 4(9 + x)$$
$$= 2x^2 + 12 + 6x - 4 + 36 + 4x$$
$$= 2x^2 + 10x + 44$$
$$= 2(x^2 + 5x) + 44$$
$$= 2\left(x^2 + 5x + \left(\frac{5}{2}\right)^2\right) + 44 - 2\times\left(\frac{5}{2}\right)^2$$
$$= 2\left(x + \frac{5}{2}\right)^2 + 31\frac{1}{2} \geq 31\frac{1}{2}, \text{ for all real } x.$$

Hence the determinant cannot be zero and the matrix is non-singular for all real x.

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Further matrix algebra
Exercise B, Question 8

Question:

Find all the values of x for which the matrix $\begin{pmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{pmatrix}$ is singular.

Solution:

$$\begin{vmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{vmatrix} = (x-3)\begin{vmatrix} x & -2 \\ -1 & x+1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -2 \\ -2 & x+1 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ -2 & -1 \end{vmatrix}$$

$$= (x-3)(x^2+x-2)+2(x+1-4)+0$$

$$= x^3+x^2-2x-3x^2-3x+6+2x-6$$

$$= x^3-2x^2-3x$$

For the matrix to be singular, the determinant must be zero.

$$x^3-2x^2-3x = x(x^2-2x-3) = x(x-3)(x+1) = 0$$

$$x = -1, 0, 3$$

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Further matrix algebra
Exercise C, Question 1

Question:

Find the inverses of these matrices.

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

Solution:

a Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1(4-1) - 0 + 0 = 3$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

b By inspection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

c Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} - 0\begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} + 0\begin{vmatrix} 0 & \frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix}$$

$$= 1\left(\frac{9}{25} + \frac{6}{25}\right) - 0 + 0 = 1$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & \frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{3}{5} & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 2

Question:

Find the inverses of these matrices.

a $\begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$

b $\begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

c $\begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

Solution:

a Let $\mathbf{A}=\begin{pmatrix}1 & -3 & 2\\ 0 & -2 & 1\\ 3 & 0 & 2\end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix}-2 & 1\\ 0 & 2\end{vmatrix} - (-3)\begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix} + 2\begin{vmatrix}0 & -2\\ 3 & 0\end{vmatrix}$$
$$= 1(-4-0)+3(0-3)+2(0+6)$$
$$= -4-9+12 = -1$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix}\begin{vmatrix}-2 & 1\\ 0 & 2\end{vmatrix} & \begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}0 & -2\\ 3 & 0\end{vmatrix}\\ \begin{vmatrix}-3 & 2\\ 0 & 2\end{vmatrix} & \begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix} & \begin{vmatrix}1 & -3\\ 3 & 0\end{vmatrix}\\ \begin{vmatrix}-3 & 2\\ -2 & 1\end{vmatrix} & \begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix} & \begin{vmatrix}1 & -3\\ 0 & -2\end{vmatrix}\end{pmatrix}$$
$$= \begin{pmatrix}-4 & -3 & 6\\ -6 & -4 & 9\\ 1 & 1 & -2\end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix}-4 & 3 & 6\\ 6 & -4 & -9\\ 1 & -1 & -2\end{pmatrix}$$
$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix}-4 & 6 & 1\\ 3 & -4 & -1\\ 6 & -9 & -2\end{pmatrix}$$
$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix}-4 & 6 & 1\\ 3 & -4 & -1\\ 6 & -9 & -2\end{pmatrix} = \begin{pmatrix}4 & -6 & -1\\ -3 & 4 & 1\\ -6 & 9 & 2\end{pmatrix}$$

b Let $\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

$$\det(\mathbf{A}) = 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}$$
$$= 2(-2-1) - 3(3-2) + 2(3+4)$$
$$= -6 - 3 + 14 = 5$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & 7 \\ 1 & -2 & -4 \\ 7 & -4 & -13 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{5}\begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{7}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{4}{5} & -\frac{13}{5} \end{pmatrix}$$

c Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

$$\det(\mathbf{A}) = 3\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + (-7)\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}$$

$$= 3(6-2) - 2(-2-0) - 7(2-0)$$

$$= 12 + 4 - 14 = 2$$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 2 \\ -10 & -6 & -6 \\ -19 & -10 & -11 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix} = \begin{pmatrix} 2 & -5 & -\frac{19}{2} \\ 1 & -3 & -5 \\ 1 & -3 & -\frac{11}{2} \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 3

Question:

The matrix $\mathbf{A}=\begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\2 & 0 & 1\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}2 & 1 & -1\\1 & 0 & 1\\1 & 2 & 1\end{pmatrix}$.

a Find $\mathbf{A}^{-1}$.

b Find $\mathbf{B}^{-1}$.

Given that $(\mathbf{AB})^{-1}=\begin{pmatrix}-\frac{2}{3} & \frac{1}{2} & \frac{1}{2}\\1 & -\frac{1}{2} & -\frac{1}{2}\\\frac{2}{3} & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}$,

c verify that $\mathbf{B}^{-1}\mathbf{A}^{-1}=(\mathbf{AB})^{-1}$.

Solution:

a $\det(\mathbf{A}) = 1\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} - 0\begin{vmatrix}0 & 0\\2 & 1\end{vmatrix} + 1\begin{vmatrix}0 & 1\\2 & 0\end{vmatrix}$

$= 1 - 0 - 2 = -1$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} & \begin{vmatrix}0 & 0\\2 & 1\end{vmatrix} & \begin{vmatrix}0 & 1\\2 & 0\end{vmatrix} \\ \begin{vmatrix}0 & 1\\0 & 1\end{vmatrix} & \begin{vmatrix}1 & 1\\2 & 1\end{vmatrix} & \begin{vmatrix}1 & 0\\2 & 0\end{vmatrix} \\ \begin{vmatrix}0 & 1\\1 & 0\end{vmatrix} & \begin{vmatrix}1 & 1\\0 & 0\end{vmatrix} & \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

b $\det(\mathbf{B}) = 2\begin{vmatrix}0 & 1\\2 & 1\end{vmatrix} - 1\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} + (-1)\begin{vmatrix}1 & 0\\1 & 2\end{vmatrix}$

$= -4 - 0 - 2 = -6$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}0 & 1\\2 & 1\end{vmatrix} & \begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} & \begin{vmatrix}1 & 0\\1 & 2\end{vmatrix} \\ \begin{vmatrix}1 & -1\\2 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\1 & 1\end{vmatrix} & \begin{vmatrix}2 & 1\\1 & 2\end{vmatrix} \\ \begin{vmatrix}1 & -1\\0 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\1 & 1\end{vmatrix} & \begin{vmatrix}2 & 1\\1 & 0\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 2 \\ 3 & 3 & 3 \\ 1 & 3 & -1 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -2 & 0 & 2 \\ -3 & 3 & -3 \\ 1 & -3 & -1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-6}\begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

c $$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}+0-\frac{1}{3} & 0+\frac{1}{2}+0 & \frac{1}{3}+0+\frac{1}{6} \\ 0+0+1 & 0-\frac{1}{2}+0 & 0+0-\frac{1}{2} \\ \frac{1}{3}+0+\frac{1}{3} & 0+\frac{1}{2}+0 & -\frac{1}{3}+0-\frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = (\mathbf{AB})^{-1}\text{, as required.}$$

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Further matrix algebra
Exercise C, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

a Show that $\det(\mathbf{A}) = 3(k+1)$

b Given that $k \neq -1$, find $\mathbf{A}^{-1}$.

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Further matrix algebra
Exercise C, Question 5

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$

Given that $\mathbf{A} = \mathbf{A}^{-1}$, find the values of the constants a, b and c.

Solution:

$\mathbf{A} = \mathbf{A}^{-1}$
Multiplying throughout by $\mathbf{A}$

$\mathbf{AA} = \mathbf{AA}^{-1}$
$\mathbf{A}^2 = \mathbf{I}$

$$\mathbf{A}^2 = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}\begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$$

$$= \begin{pmatrix} ab+33 & -2a-8 & 8a+4c+20 \\ 16-2b & ab+33 & 4b+8c-56 \\ -2b+2c+10 & 2a-2c+14 & c^2-8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equating the second elements in the first row
$-2a-8=0 \Rightarrow a=-4$
Equating the first elements in the second row
$16-2b=0 \Rightarrow b=8$
Equating the first elements in the third row and using $b=8$
$-2b+2c+10=0 \Rightarrow -16+2c+10=0$
$2c=6 \Rightarrow c=3$
$a=-4, b=8, c=3$

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Further matrix algebra
Exercise C, Question 6

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = \mathbf{I}$.

b Hence find $\mathbf{A}^{-1}$.

Solution:

a $$\mathbf{A}^2 = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4-4-3 & -2+3+3 & 2+0+1 \\ 8-12+0 & -4+9+0 & 4+0+0 \\ -6+12-3 & 3-9+3 & -3+0+1 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -6+16-9 & 3-12+9 & -3+0+3 \\ -8+20-12 & 4-15+12 & -4+0+4 \\ 6-12+6 & -3+9-6 & 3+0-2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

b $\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \mathbf{I}$

Comparing with the definition of an inverse

$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

$$\mathbf{A}^{-1} = \mathbf{A}^2 = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 7

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$.

b Deduce that $15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$.

c Hence find $\mathbf{A}^{-1}$.

Solution:

a $\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1+3+0 & 1-3+0 & 0+1+0 \\ 3-9+0 & 3+9+3 & 0-3+2 \\ 0+9+0 & 0-9+6 & 0+3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4-6+0 & 4+6+3 & 0-2+2 \\ -6+45+0 & -6-45-3 & 0+15-2 \\ 9-9+0 & 9+9+21 & 0-3+14 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 13 & 0 \\ 39 & -54 & 13 \\ 0 & 39 & 11 \end{pmatrix}$$

$$13\mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 13 & 13 & 0 \\ 39 & -39 & 13 \\ 0 & 39 & 26 \end{pmatrix} - \begin{pmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 13 & 0 \\ 39 & -54 & 13 \\ 0 & 39 & 11 \end{pmatrix} = \mathbf{A}^3$$

Hence

$\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$, as required.

b Multiply the result of part **a** throughout by $\mathbf{A}^{-1}$

$$\mathbf{A}^3\mathbf{A}^{-1} = 13\mathbf{A}\mathbf{A}^{-1} - 15\mathbf{I}\mathbf{A}^{-1}$$
$$\mathbf{A}^2 = 13\mathbf{I} - 15\mathbf{A}^{-1}$$

Rearranging

$15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$, as required.

c Using the result of part **b**

$$15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2 = \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$$

Hence

$$\mathbf{A}^{-1} = \frac{1}{15}\begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$$

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Further matrix algebra
Exercise C, Question 8

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 0 & 1\\ 4 & 3 & -2\\ 0 & 3 & -4\end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

The matrix $\mathbf{C}$ is the matrix of the cofactors of $\mathbf{A}$.

b Find $\mathbf{C}$.

c Show that $\mathbf{AC}^{\mathrm{T}}=\mathbf{0}$.

Solution:

a $\det(\mathbf{A}) = 2\begin{vmatrix}3 & -2\\3 & -4\end{vmatrix} - 0\begin{vmatrix}4 & -2\\0 & -4\end{vmatrix} + 1\begin{vmatrix}4 & 3\\0 & 3\end{vmatrix}$

$$= 2(-12+6) - 0 + 1(12-0)$$

$$= -12 + 12 = 0$$

Hence $\mathbf{A}$ is singular.

b The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}3 & -2\\3 & -4\end{vmatrix} & \begin{vmatrix}4 & -2\\0 & -4\end{vmatrix} & \begin{vmatrix}4 & 3\\0 & 3\end{vmatrix} \\ \begin{vmatrix}0 & 1\\3 & -4\end{vmatrix} & \begin{vmatrix}2 & 1\\0 & -4\end{vmatrix} & \begin{vmatrix}2 & 0\\0 & 3\end{vmatrix} \\ \begin{vmatrix}0 & 1\\3 & -2\end{vmatrix} & \begin{vmatrix}2 & 1\\4 & -2\end{vmatrix} & \begin{vmatrix}2 & 0\\4 & 3\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -16 & 12 \\ -3 & -8 & 6 \\ -3 & -8 & 6 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -6 & 16 & 12 \\ 3 & -8 & -6 \\ -3 & 8 & 6 \end{pmatrix}$$

c $\mathbf{AC}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}\begin{pmatrix} -6 & 3 & -3 \\ 16 & -8 & 8 \\ 12 & -6 & 6 \end{pmatrix}$

$$= \begin{pmatrix} -12+0+12 & 6+0-6 & -6+0+6 \\ -24+48-24 & 12-24+12 & -12+24-12 \\ 0+48-48 & 0-24+24 & 0+24-24 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0\text{, as required.}$$

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Further matrix algebra
Exercise D, Question 1

Question:

Given that $T:\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ y+z \\ 2x-3z \end{pmatrix}$ and $U:\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x-3y-z \\ 2y+3z \\ 5z \end{pmatrix}$, find matrices representing

a T

b U

c TU.

Solution:

a $\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1-0\\0+0\\2-0\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0-1\\1+0\\0-0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0-0\\0+1\\0-3\end{pmatrix} = \begin{pmatrix}0\\1\\-3\end{pmatrix}$

The matrix representing T is $\begin{pmatrix}1 & -1 & 0\\0 & 1 & 1\\2 & 0 & -3\end{pmatrix}$

b $\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}2-0-0\\0+0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0-3-0\\2+0\\0\end{pmatrix} = \begin{pmatrix}-3\\2\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0-0-1\\0+3\\5\end{pmatrix} = \begin{pmatrix}-1\\3\\5\end{pmatrix}$

The matrix representing U is $\begin{pmatrix}2 & -3 & -1\\0 & 2 & 3\\0 & 0 & 5\end{pmatrix}$

c The matrix representing TU is given by

$$\begin{pmatrix}1 & -1 & 0\\0 & 1 & 1\\2 & 0 & -3\end{pmatrix}\begin{pmatrix}2 & -3 & -1\\0 & 2 & 3\\0 & 0 & 5\end{pmatrix} = \begin{pmatrix}2+0+0 & -3-2+0 & -1-3+0\\0+0+0 & 0+2+0 & 0+3+5\\4+0+0 & -6+0+0 & -2+0-15\end{pmatrix} = \begin{pmatrix}2 & -5 & -4\\0 & 2 & 8\\4 & -6 & -17\end{pmatrix}$$

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Further matrix algebra
Exercise D, Question 2

Question:

The point with position vector $\begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix}$ is transformed by the linear transformation represented by the matrix $\begin{pmatrix} 4 & -1 & 0 \\ -2 & 2 & 3 \\ 5 & -2 & 1 \end{pmatrix}$ to the point with position vector $\begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$.

Find the values of the constants a, b and c.

Solution:

$$\begin{pmatrix} 4 & -1 & 0 \\ -2 & 2 & 3 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix} = \begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 4+3a \\ -1+a \end{pmatrix} = \begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$$

Equating the top elements

$b = 1$

Equating the middle elements

$4+3a = -5 \Rightarrow a = -3$

Equating the lowest elements and using $a = -3$

$-1+a = -1-3 = c \Rightarrow c = -4$

$a = -3, b = 1, c = -4$

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Further matrix algebra
Exercise D, Question 3

Question:

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$.

The vector $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$.

The vector $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$.

The vector $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

Find $\mathbf{T}$.

Solution:

Let $\mathbf{T} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ 2d \\ 2g \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

Equating the elements

$2a = 6 \Rightarrow a = 3$

$2d = 2 \Rightarrow d = 1$

$2g = 4 \Rightarrow g = 2$

$$\begin{pmatrix} 3 & b & c \\ 1 & e & f \\ 2 & h & i \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 9-c \\ 3-f \\ 6-i \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$$

Equating the elements

$9 - c = -2 \Rightarrow c = 11$

$3 - f = 3 \Rightarrow f = 0$

$6 - i = 5 \Rightarrow i = 1$

$$\begin{pmatrix} 3 & b & 11 \\ 1 & e & 0 \\ 2 & h & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} b-11 \\ e \\ h-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

Equating the elements

$b - 11 = 2 \Rightarrow b = 13$

$e = -1$

$h - 1 = -2 \Rightarrow h = -1$

$$\mathbf{T} = \begin{pmatrix} 3 & 13 & 11 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

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Further matrix algebra
Exercise D, Question 4

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T}=\begin{pmatrix}0 & -1 & 2\\ 2 & 5 & -4\\ 3 & 2 & 1\end{pmatrix}.$$

The line l_1 is transformed by T to the line l_2. The line l_1 has vector equation

$$\mathbf{r}=\begin{pmatrix}2\\4\\1\end{pmatrix}+t\begin{pmatrix}-1\\-2\\3\end{pmatrix},$$ where t is a real parameter.

Find a vector equation of l_2.

Solution:

$$\mathbf{r}=\begin{pmatrix}2\\4\\1\end{pmatrix}+t\begin{pmatrix}-1\\-2\\3\end{pmatrix}=\begin{pmatrix}2-t\\4-2t\\1+3t\end{pmatrix}$$

$$\begin{pmatrix}0 & -1 & 2\\ 2 & 5 & -4\\ 3 & 2 & 1\end{pmatrix}\begin{pmatrix}2-t\\4-2t\\1+3t\end{pmatrix}=\begin{pmatrix}0(2-t)-1(4-2t)+2(1+3t)\\2(2-t)+5(4-2t)-4(1+3t)\\3(2-t)+2(4-2t)+1(1+3t)\end{pmatrix}$$

$$=\begin{pmatrix}-2+8t\\20-24t\\15-4t\end{pmatrix}=\begin{pmatrix}-2\\20\\15\end{pmatrix}+t\begin{pmatrix}8\\-24\\-4\end{pmatrix}$$

An equation of l_2 is $\mathbf{r}=\begin{pmatrix}-2\\20\\15\end{pmatrix}+t\begin{pmatrix}8\\-24\\-4\end{pmatrix}$

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Further matrix algebra
Exercise D, Question 5

Question:

The points A and B have position vectors $\begin{pmatrix}2\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\3\\4\end{pmatrix}$ respectively. The points A and B are transformed by the linear transformation T to the points A' and B' respectively.

The transformation T is represented by the matrix $\mathbf{T}$, where $\mathbf{T}=\begin{pmatrix}1&-3&4\\2&3&-2\\0&2&5\end{pmatrix}$.

a Find the position vectors of A' and B'.
b Hence find a vector equation of the line $A'B'$.

Solution:

a $\begin{pmatrix}1&-3&4\\2&3&-2\\0&2&5\end{pmatrix}\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}2-3+0\\4+3+0\\0+2+0\end{pmatrix}=\begin{pmatrix}-1\\7\\2\end{pmatrix}$

$\begin{pmatrix}1&-3&4\\2&3&-2\\0&2&5\end{pmatrix}\begin{pmatrix}-2\\3\\4\end{pmatrix}=\begin{pmatrix}-2-9+16\\-4+9-8\\0+6+20\end{pmatrix}=\begin{pmatrix}5\\-3\\26\end{pmatrix}$

The position vector of A' is $\begin{pmatrix}-1\\7\\2\end{pmatrix}$ and the position vector of B' is $\begin{pmatrix}5\\-3\\26\end{pmatrix}$.

b $\mathbf{r}=\mathbf{a}'+t(\mathbf{b}'-\mathbf{a}')$

$=\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}5-(-1)\\-3-7\\26-2\end{pmatrix}=\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}6\\-10\\24\end{pmatrix}$

A vector equation of $A'B'$ is $\mathbf{r}=\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}6\\-10\\24\end{pmatrix}$.

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Further matrix algebra
Exercise D, Question 6

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T}=\begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}.$$

The plane Π_1 is transformed by T to the plane Π_2. The plane Π_1 has Cartesian equation $x-2y+z=0$.

Find a Cartesian equation of Π_2.

Solution:

Let $y=s$ and $z=t$, then $x=2y-z=2s-t$

A parametric form of the general point on Π_1 is $\begin{pmatrix} 2s-t \\ s \\ t \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2s-t \\ s \\ t \end{pmatrix}=\begin{pmatrix} 6s-3t-2s-2t \\ -4s+2t-8s+4t \\ -4s+2t+4s \end{pmatrix}=\begin{pmatrix} 4s-5t \\ -12s+6t \\ 2t \end{pmatrix}$$

Parametric equations of Π_2 are

$x=4s-5t$ ①

$y=-12s+6t$ ②

$z=2t$ ③

From ③ $t=\frac{z}{2}$

Substituting for t in ① and ②

$x=4s-\frac{5z}{2}$ ④

$y=-12s+3z$ ⑤

$3\times$④ + ⑤ $\quad 3x+y=-\frac{9z}{2} \Rightarrow 6x+2y+9z=0$

A Cartesian equation of Π_2 is $6x+2y+9z=0$.

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Further matrix algebra
Exercise D, Question 7

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 4 & 5 & -3 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The plane Π_1 is transformed by T to the plane Π_2. The plane Π_1 has vector equation

$$r = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$, where s and t are real parameters.

Find an equation of Π_2 in the form $\mathbf{r} \cdot \mathbf{n} = p$.

Solution:

$$\mathbf{r}=\begin{pmatrix}0\\1\\1\end{pmatrix}+s\begin{pmatrix}1\\-1\\2\end{pmatrix}+t\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}s+3t\\1-s\\1+2s+4t\end{pmatrix}$$

$$\begin{pmatrix}4&5&-3\\-1&2&1\\1&0&1\end{pmatrix}\begin{pmatrix}s+3t\\1-s\\1+2s+4t\end{pmatrix}=\begin{pmatrix}4s+12t+5-5s-3-6s-12t\\-s-3t+2-2s+1+2s+4t\\s+3t+1+2s+4t\end{pmatrix}$$

$$=\begin{pmatrix}2-7s\\3-s+t\\1+3s+7t\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix}+s\begin{pmatrix}-7\\-1\\3\end{pmatrix}+t\begin{pmatrix}0\\1\\7\end{pmatrix}$$

A vector equation of Π_2 is $\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+s\begin{pmatrix}-7\\-1\\3\end{pmatrix}+t\begin{pmatrix}0\\1\\7\end{pmatrix}$ *

To find a vector perpendicular to both $\begin{pmatrix}-7\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}0\\1\\7\end{pmatrix}$

$$\begin{pmatrix}-7\\-1\\3\end{pmatrix}\times\begin{pmatrix}0\\1\\7\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-7&-1&3\\0&1&7\end{vmatrix}=\mathbf{i}\begin{vmatrix}-1&3\\1&7\end{vmatrix}-\mathbf{j}\begin{vmatrix}-7&3\\0&7\end{vmatrix}+\mathbf{k}\begin{vmatrix}-7&-1\\0&1\end{vmatrix}$$

$$=-10\mathbf{i}+49\mathbf{j}-7\mathbf{k}=\begin{pmatrix}-10\\49\\-7\end{pmatrix}$$

Taking the scalar product of equation * throughout with $\begin{pmatrix}-10\\49\\-7\end{pmatrix}$ and using the property that the scalar product of perpendicular vectors is 0

$$\mathbf{r}\cdot\begin{pmatrix}-10\\49\\-7\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}-10\\49\\-7\end{pmatrix}=-20+147-7=120$$

A vector equation of Π_2 is $\mathbf{r}\cdot\begin{pmatrix}-10\\49\\-7\end{pmatrix}=120$.

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Further matrix algebra
Exercise D, Question 8

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 4 & 1 & -2 \\ -2 & 3 & 4 \\ -1 & 0 & 2 \end{pmatrix}.$$

There is a line through the origin for which every point on the line is mapped onto itself under T.
Find a vector equation of this line.

Solution:

Let a point which is unchanged by T have coordinates $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$$\begin{pmatrix} 4 & 1 & -2 \\ -2 & 3 & 4 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 4a+b-2c \\ -2a+3b+4c \\ -a+2c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Equating the lowest elements

$-a+2c=c \Rightarrow c=a$

Equating the top elements and substituting $c=a$

$4a+b-2a=a \Rightarrow b=-a$

(Equating the middle elements also gives $b=-a$)

The general form of a point which is unchanged is $\begin{pmatrix} a \\ -a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

A vector equation of the line is $\mathbf{r} = t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 1

Question:

A transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T}^{-1} = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}.$$

The point with position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the point with position vector $\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$.

a Find the values of the constants a, b and c.

A line l_1 which passes through the origin is transformed by T to the line l_2.

A vector equation of l_2 is $\mathbf{r} = t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b Find a vector equation of l_1.

Solution:

a $$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} -24-21+24 \\ 12-28+40 \\ -24-7+8 \end{pmatrix} = \begin{pmatrix} -21 \\ 24 \\ -23 \end{pmatrix}$$

$a = -21, b = 24, c = -23$

b $$\begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4-6+3 \\ -2-8+5 \\ 4-2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$$

A vector equation of l_1 is $\mathbf{r} = t\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 2

Question:

The transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 2 \\ -3 & 2 & 8 \end{pmatrix}.$$

a Find $\mathbf{T}^{-1}$.

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix}$.

b Find the values of the constants a, b and c.

Solution:

a $\det(\mathbf{T}) = 2\begin{vmatrix}1 & 2\\2 & 8\end{vmatrix} - 0\begin{vmatrix}0 & 2\\-3 & 8\end{vmatrix} + (-3)\begin{vmatrix}0 & 1\\-3 & 2\end{vmatrix}$

$= 2(8-4) - 0 - 3(0+3)$

$= 8 - 9 = -1$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}1 & 2\\2 & 8\end{vmatrix} & \begin{vmatrix}0 & 2\\-3 & 8\end{vmatrix} & \begin{vmatrix}0 & 1\\-3 & 2\end{vmatrix} \\ \begin{vmatrix}0 & -3\\2 & 8\end{vmatrix} & \begin{vmatrix}2 & -3\\-3 & 8\end{vmatrix} & \begin{vmatrix}2 & 0\\-3 & 2\end{vmatrix} \\ \begin{vmatrix}0 & -3\\1 & 2\end{vmatrix} & \begin{vmatrix}2 & -3\\0 & 2\end{vmatrix} & \begin{vmatrix}2 & 0\\0 & 1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix}4 & 6 & 3\\6 & 7 & 4\\3 & 4 & 2\end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix}4 & -6 & 3\\-6 & 7 & -4\\3 & -4 & 2\end{pmatrix}$$

As $\mathbf{C}$ is symmetric $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix}4 & -6 & 3\\-6 & 7 & -4\\3 & -4 & 2\end{pmatrix} = \begin{pmatrix}-4 & 6 & -3\\6 & -7 & 4\\-3 & 4 & -2\end{pmatrix}$$

b $$\begin{pmatrix}a\\b\\c\end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix}-5\\5\\16\end{pmatrix} = \begin{pmatrix}-4 & 6 & -3\\6 & -7 & 4\\-3 & 4 & -2\end{pmatrix}\begin{pmatrix}-5\\5\\16\end{pmatrix} = \begin{pmatrix}20+30-48\\-30-35+64\\15+20-32\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$$

$a = 2, b = -1, c = 3$

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Further matrix algebra
Exercise E, Question 3

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -3 & 0 & -4 \end{pmatrix}.$$

a Find $\mathbf{T}^{-1}$.

The line l_1 is transformed by T to the line l_2. The line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$ where t is a real parameter.

b Find a vector equation of l_1.

Solution:

a $\det(\mathbf{T}) = 1\begin{vmatrix}2 & 2\\0 & -4\end{vmatrix} - 1\begin{vmatrix}0 & 2\\-3 & -4\end{vmatrix} + 2\begin{vmatrix}0 & 2\\-3 & 0\end{vmatrix}$

$= 1(-8-0) - 1(0+6) + 2(0+6)$

$= -8-6+12 = -2$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}2 & 2\\0 & -4\end{vmatrix} & \begin{vmatrix}0 & 2\\-3 & -4\end{vmatrix} & \begin{vmatrix}0 & 2\\-3 & 0\end{vmatrix} \\ \begin{vmatrix}1 & 2\\0 & -4\end{vmatrix} & \begin{vmatrix}1 & 2\\-3 & -4\end{vmatrix} & \begin{vmatrix}1 & 1\\-3 & 0\end{vmatrix} \\ \begin{vmatrix}1 & 2\\2 & 2\end{vmatrix} & \begin{vmatrix}1 & 2\\0 & 2\end{vmatrix} & \begin{vmatrix}1 & 1\\0 & 2\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 6 & 6 \\ -4 & 2 & 3 \\ -2 & 2 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -8 & -6 & 6 \\ 4 & 2 & -3 \\ -2 & -2 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -8 & 4 & -2 \\ -6 & 2 & -2 \\ 6 & -3 & 2 \end{pmatrix}$$

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-2}\begin{pmatrix} -8 & 4 & -2 \\ -6 & 2 & -2 \\ 6 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 3 & -1 & 1 \\ -3 & \frac{3}{2} & -1 \end{pmatrix}$$

b A general point on l_2 has coordinates $\begin{pmatrix}2\\4\\1\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}2-t\\4\\1+t\end{pmatrix}$

$$\mathbf{T}^{-1}\begin{pmatrix}2-t\\4\\1+t\end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 3 & -1 & 1 \\ -3 & \frac{3}{2} & -1 \end{pmatrix}\begin{pmatrix}2-t\\4\\1+t\end{pmatrix}$$

$$= \begin{pmatrix} 8-4t-8+1+t \\ 6-3t-4+1+t \\ -6+3t+6-1-t \end{pmatrix} = \begin{pmatrix} 1-3t \\ 3-2t \\ -1+2t \end{pmatrix} = \begin{pmatrix}1\\3\\-1\end{pmatrix} + t\begin{pmatrix}-3\\-2\\2\end{pmatrix}$$

A vector equation of l_1 is $\mathbf{r} = \begin{pmatrix}1\\3\\-1\end{pmatrix} + t\begin{pmatrix}-3\\-2\\2\end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 4

Question:

The matrix $\mathbf{T} = \begin{pmatrix} a & 1 & 2 \\ 4 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, where a is a constant.

a Find $\mathbf{T}^{-1}$, in terms of a.

The transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$. The point with position vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is transformed by T to the point with position vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

b Find p, q and r.

Solution:

a $\det(\mathbf{T}) = a\begin{vmatrix}0 & 0\\ 0 & -1\end{vmatrix} - 1\begin{vmatrix}4 & 0\\ 0 & -1\end{vmatrix} + 2\begin{vmatrix}4 & 0\\ 0 & 0\end{vmatrix}$

$= 0 + 4 + 0 = 4$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}0 & 0\\ 0 & -1\end{vmatrix} & \begin{vmatrix}4 & 0\\ 0 & -1\end{vmatrix} & \begin{vmatrix}4 & 0\\ 0 & 0\end{vmatrix} \\ \begin{vmatrix}1 & 2\\ 0 & -1\end{vmatrix} & \begin{vmatrix}a & 2\\ 0 & -1\end{vmatrix} & \begin{vmatrix}a & 1\\ 0 & 0\end{vmatrix} \\ \begin{vmatrix}1 & 2\\ 0 & 0\end{vmatrix} & \begin{vmatrix}a & 2\\ 4 & 0\end{vmatrix} & \begin{vmatrix}a & 1\\ 4 & 0\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 & 0 \\ -1 & -a & 0 \\ 0 & -8 & -4 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 0 & 4 & 0 \\ 1 & -a & 0 \\ 0 & 8 & -4 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 0 \\ 4 & -a & 8 \\ 0 & 0 & -4 \end{pmatrix}$$

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}} = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 4 & -a & 8 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -\frac{a}{4} & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

b $$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -\frac{a}{4} & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 + \frac{3}{4} + 0 \\ 2 - \frac{3a}{4} - 2 \\ 0 + 0 + 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3a}{4} \\ 1 \end{pmatrix}$$

$p = \frac{3}{4}, q = -\frac{3a}{4}, r = 1$

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Further matrix algebra
Exercise E, Question 5

Question:

The matrix $\mathbf{S}=\begin{pmatrix}1 & -\sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}$.

a Show that $\mathbf{SS}^{\mathrm{T}}=k\mathbf{I}$, stating the value of k.

The transformation $S:\mathbb{R}^3\to\mathbb{R}^3$ is represented by the matrix $\mathbf{S}$.

The vector $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$ is transformed by S to the vector $\begin{pmatrix}2\sqrt{2}\\ \sqrt{2}\\ -2\sqrt{2}\end{pmatrix}$.

b Find the values of the constants a, b and c.

Solution:

a $$\mathbf{SS}^{\mathrm{T}}=\begin{pmatrix}1 & -\sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}=\begin{pmatrix}1+2+1 & \sqrt{2}+0-\sqrt{2} & 1-2+1\\ \sqrt{2}+0-\sqrt{2} & 2+0+2 & \sqrt{2}+0-\sqrt{2}\\ 1-2+1 & \sqrt{2}+0-\sqrt{2} & 1+2+1\end{pmatrix}$$

$$=\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}=4\mathbf{I}$$

$$k=4$$

b $\mathbf{SS}^{\mathrm{T}}=4\mathbf{I}\Rightarrow\mathbf{S}^{-1}=\frac{1}{4}\mathbf{S}^{\mathrm{T}}$

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\frac{1}{4}\mathbf{S}^{\mathrm{T}}\begin{pmatrix}2\sqrt{2}\\ \sqrt{2}\\ -2\sqrt{2}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}2\sqrt{2}\\ \sqrt{2}\\ -2\sqrt{2}\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}2\sqrt{2}+2-2\sqrt{2}\\ -4+0-4\\ 2\sqrt{2}-2-2\sqrt{2}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}2\\ -8\\ -2\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ -2\\ -\frac{1}{2}\end{pmatrix}$$

$$a=\frac{1}{2},b=-2,c=-\frac{1}{2}$$

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Further matrix algebra
Exercise E, Question 6

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 3 & a & -3 \\ b & -1 & -2 \\ -18 & 11 & c \end{pmatrix}$. Given that $\mathbf{AB} = \mathbf{I}$,

a find the values of the constants a, b and c.

The transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{A}$.

The plane Π_1 is transformed by A to the plane Π_2. The plane Π_2 has vector equation

$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, where s and t are real parameters.

b Find a vector equation of the plane Π_1.

Solution:

a $\mathbf{AB} = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & -3 \\ b & -1 & -2 \\ -18 & 11 & c \end{pmatrix}$

$$= \begin{pmatrix} 9+5b-18 & 3a-5+11 & -9-10+c \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equating the elements in the first row

$9+5b-18 = 1 \Rightarrow 5b = 10 \Rightarrow b = 2$

$3a-5+11 = 0 \Rightarrow 3a = -6 \Rightarrow a = -2$

$-9-10+c = 0 \Rightarrow c = 19$

$a = -2, b = 2, c = 19$

b $\mathbf{AB} = \mathbf{I} \Rightarrow \mathbf{A}^{-1} = \mathbf{B} = \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}$

The general point on Π_2 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-s \\ 1+t \\ 2s+t \end{pmatrix}$

$$\mathbf{A}^{-1}\begin{pmatrix} 1-s \\ 1+t \\ 2s+t \end{pmatrix} = \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}\begin{pmatrix} 1-s \\ 1+t \\ 2s+t \end{pmatrix} = \begin{pmatrix} 3-3s-2-2t-6s-3t \\ 2-2s-1-t-4s-2t \\ -18+18s+11+11t+38s+19t \end{pmatrix}$$

$$= \begin{pmatrix} 1-9s-5t \\ 1-6s-3t \\ -7+56s+30t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} + s\begin{pmatrix} -9 \\ -6 \\ 56 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 30 \end{pmatrix}$$

A vector equation of Π_2 is $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} + s\begin{pmatrix} -9 \\ -6 \\ 56 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 30 \end{pmatrix}$.

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Further matrix algebra
Exercise E, Question 7

Question:

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} -1 & 3 & 6 \\ 1 & 4 & 2 \\ 2 & -5 & 1 \end{pmatrix}.$$

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the vector $\begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix}$.

Find the values of the constants a, b and c.

Solution:

$$\begin{pmatrix} -1 & 3 & 6 \\ 1 & 4 & 2 \\ 2 & -5 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -a+3b+6c \\ a+4b+2c \\ 2a-5b+c \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix}$$

Equating the elements

$-a+3b+6c=-8$ ①

$a+4b+2c=0$ ②

$2a-5b+c=3$ ③

① + ②

$7b+8c=-8$ ④

2× ① + ③

$b+13c=-13$ ⑤

7× ⑤

$7b+91c=-91$ ⑥

⑥ − ④

$83c=-83 \Rightarrow c=-1$

Substituting $c=-1$ into ⑤

$b-13=-13 \Rightarrow b=0$

Substituting $b=0$ and $c=-1$ into ②

$a+0-2=0 \Rightarrow a=2$

$a=2, b=0, c=-1$

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Further matrix algebra
Exercise E, Question 8

Question:

The matrix $\mathbf{S}=\begin{pmatrix}2 & -1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 1\end{pmatrix}$ and the matrix $\mathbf{T}=\begin{pmatrix}3 & 4 & 4\\ -6 & -7 & -6\\ 4 & 4 & 3\end{pmatrix}$.

a Find $\mathbf{S}^{-1}$.

b Show that $\mathbf{T}^2=\mathbf{I}$.

The transformation $S:\mathbb{R}^3\to\mathbb{R}^3$ is represented by the matrix $\mathbf{S}$ and the transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ is represented by the matrix $\mathbf{T}$.

The transformation U is the transformation T followed by the transformation S.

The point $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$ is transformed by U to the point $\begin{pmatrix}6\\ -3\\ 2\end{pmatrix}$.

c Find the values of the constants a, b and c.

Solution:

a $\det(\mathbf{S}) = 2\begin{vmatrix}2 & 1\\0 & 1\end{vmatrix} - (-1)\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} + 2\begin{vmatrix}0 & 2\\1 & 0\end{vmatrix}$

$= 2(2-0)+1(0-1)+2(0-2)$

$= 4-1-4=-1$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}2 & 1\\0 & 1\end{vmatrix} & \begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} & \begin{vmatrix}0 & 2\\1 & 0\end{vmatrix} \\ \begin{vmatrix}-1 & 2\\0 & 1\end{vmatrix} & \begin{vmatrix}2 & 2\\1 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\1 & 0\end{vmatrix} \\ \begin{vmatrix}-1 & 2\\2 & 1\end{vmatrix} & \begin{vmatrix}2 & 2\\0 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\0 & 2\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -2 \\ -1 & 0 & 1 \\ -5 & 2 & 4 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ -5 & -2 & 4 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ -2 & -1 & 4 \end{pmatrix}$$

$$\mathbf{S}^{-1} = \frac{1}{\det(\mathbf{S})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ -2 & -1 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 5 \\ -1 & 0 & 2 \\ 2 & 1 & -4 \end{pmatrix}$$

b $$\mathbf{T}^2 = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 9-24+16 & 12-28+16 & 12-24+12 \\ -18+42-24 & -24+49-24 & -24+42-18 \\ 12-24+12 & 16-28+12 & 16-24+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

c From part **b**, $\mathbf{T}^2 = \mathbf{I} \Rightarrow \mathbf{T}^{-1} = \mathbf{T}$

$$\mathbf{ST}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$(\mathbf{ST})^{-1}\mathbf{ST}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\mathbf{ST})^{1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\mathbf{S}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = \mathbf{TS}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 0 & 2 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} -12 +3 +10 \\ -6 +0 +4 \\ 12 -3 -8 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 -8 +4 \\ -6 +14 -6 \\ 4 -8 +3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$a = -1, b = 2, c = -1$

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Further matrix algebra
Exercise F, Question 1

Question:

Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

c $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$.

Solution:

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2 & 4\\ 1 & 5\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}2-\lambda & 4\\ 1 & 5-\lambda\end{pmatrix}$$

a $\begin{vmatrix}2-\lambda & 4\\ 1 & 5-\lambda\end{vmatrix}=(2-\lambda)(5-\lambda)-4$

$$=10-7\lambda+\lambda^2-4=\lambda^2-7\lambda+6=(\lambda-1)(\lambda-6)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(\lambda-1)(\lambda-6)=0\Rightarrow\lambda=1,6$

The eigenvalues are 1 and 6.

For $\lambda=1$

$\begin{pmatrix}2 & 4\\ 1 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=1\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}2x+4y\\ x+5y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$

Equating the upper elements

$2x+4y=x\Rightarrow x=-4y$

Let $y=1$, then $x=-4$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix}-4\\ 1\end{pmatrix}$.

For $\lambda=6$

$\begin{pmatrix}2 & 4\\ 1 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=6\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}2x+4y\\ x+5y\end{pmatrix}=\begin{pmatrix}6x\\ 6y\end{pmatrix}$

Equating the upper elements

$2x+4y=6x\Rightarrow y=x$

Let $x=1$, then $y=1$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix}1\\ 1\end{pmatrix}$.

b $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}4-\lambda & -1\\ -1 & 4-\lambda\end{pmatrix}$

$\begin{vmatrix}4-\lambda & -1\\ -1 & 4-\lambda\end{vmatrix}=(4-\lambda)^2-1$

$$=16-8\lambda+\lambda^2-1=\lambda^2-8\lambda+15=(\lambda-3)(\lambda-5)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(\lambda-3)(\lambda-5)=0\Rightarrow\lambda=3,5$

The eigenvalues are 3 and 5.

For $\lambda=3$

$\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=3\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}4x-y\\ -x+4y\end{pmatrix}=\begin{pmatrix}3x\\ 3y\end{pmatrix}$

Equating the upper elements

$4x-y=3x\Rightarrow y=x$

Let $x=1$, then $y=1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\1\end{pmatrix}$.

For $\lambda = 5$

$$\begin{pmatrix}4 & -1\\-1 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 5\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}4x-y\\-x+4y\end{pmatrix} = \begin{pmatrix}5x\\5y\end{pmatrix}$$

Equating the upper elements

$4x - y = 5x \Rightarrow y = -x$

Let $x = 1$, then $y = -1$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix}1\\-1\end{pmatrix}$.

c $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix}3 & -2\\0 & 4\end{pmatrix} - \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix} = \begin{pmatrix}3-\lambda & -2\\0 & 4-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda & -2\\0 & 4-\lambda\end{vmatrix} = (3-\lambda)(4-\lambda)$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (3-\lambda)(4-\lambda) = 0 \Rightarrow \lambda = 3, 4$

The eigenvalues are 3 and 4.

For $\lambda = 3$

$$\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 3\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}3x-2y\\4y\end{pmatrix} = \begin{pmatrix}3x\\3y\end{pmatrix}$$

Equating the lower elements

$4y = 3y \Rightarrow y = 0$

As x can take any non-zero value, let $x = 1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\0\end{pmatrix}$.

For $\lambda = 4$

$$\begin{pmatrix}3 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 4\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}3x-2y\\4y\end{pmatrix} = \begin{pmatrix}4x\\4y\end{pmatrix}$$

Equating the upper elements

$3x - 2y = 4x \Rightarrow x = -2y$

Let $y = 1$, then $x = -2$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}-2\\1\end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 2

Question:

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}$

a Find the eigenvalues of $\mathbf{A}$.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

Solution:

a $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & 4 \\ -2 & 9-\lambda \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & 4 \\ -2 & 9-\lambda \end{vmatrix} = (3-\lambda)(9-\lambda)+8$$

$$= 27-12\lambda+\lambda^2+8 = \lambda^2-12\lambda+35 = (\lambda-5)(\lambda-7)$$

$$\det(\mathbf{A}-\lambda\mathbf{I}) = 0 \Rightarrow (\lambda-5)(\lambda-7) = 0 \Rightarrow \lambda = 5, 7$$

The eigenvalues of $\mathbf{A}$ are 5 and 7.

b For $\lambda = 5$

$$\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+4y \\ -2x+9y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$$3x+4y = 5x \Rightarrow 4y = 2x \Rightarrow y = \frac{1}{2}x$$

For $\lambda = 7$

$$\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+4y \\ -2x+9y \end{pmatrix} = \begin{pmatrix} 7x \\ 7y \end{pmatrix}$$

Equating the upper elements

$$3x+4y = 7x \Rightarrow 4y = 4x \Rightarrow y = x$$

Cartesian equations of the invariant lines are $y = \frac{1}{2}x$ and $y = x$.

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Further matrix algebra
Exercise F, Question 3

Question:

Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}3-\lambda&0&0\\2&4-\lambda&2\\-2&0&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda&0&0\\2&4-\lambda&2\\-2&0&1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}4-\lambda&2\\0&1-\lambda\end{vmatrix}-0\begin{vmatrix}2&2\\-2&1-\lambda\end{vmatrix}+0\begin{vmatrix}2&4-\lambda\\-2&0\end{vmatrix}$$

$$=(3-\lambda)(4-\lambda)(1-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(4-\lambda)(1-\lambda)=0\Rightarrow\lambda=3,4,1$$

The eigenvalues are 1, 3 and 4

For $\lambda=1$

$$\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3x\\2x+4y+2z\\-2x+z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating the top elements

$3x=x\Rightarrow x=0$

Equating the middle elements and substituting $x=0$

$0+4y+2z=y\Rightarrow 3y=-2z$

Let $z=3$, then $y=-2$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix}0\\-2\\3\end{pmatrix}$.

For $\lambda=3$

$$\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3x\\2x+4y+2z\\-2x+z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}$$

Equating the lowest elements

$-2x+z=3z\Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements and substituting $x=1$ and $z=-1$

$2+4y-2=3y\Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x+4y+2z \\ -2x+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the lowest elements

$-2x+z=3z \Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements and substituting $x=1$ and $z=-1$

$2+4y-2=3y \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x+4y+2z \\ -2x+z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$3x=4x \Rightarrow x=0$

Equating the lowest elements and substituting $x=0$

$0+z=4z \Rightarrow z=0$

As y can take any non-zero value, let $y=1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b $$\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & -2 & -4 \\ 2 & 3-\lambda & 0 \\ 2 & -5 & -4-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 4-\lambda & -2 & -4 \\ 2 & 3-\lambda & 0 \\ 2 & -5 & -4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ -5 & -4-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 2 & 0 \\ 2 & -4-\lambda \end{vmatrix} + (-4)\begin{vmatrix} 2 & 3-\lambda \\ 2 & -5 \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda)(-4-\lambda)+2(-8-2\lambda)-4-(-10-6+2\lambda)$$

$$= (\lambda^2-16)(3-\lambda)-16-4\lambda+64-8\lambda$$

$$= 3\lambda^2-\lambda^3-48+16\lambda-12\lambda+48$$

$$= -\lambda^3+3\lambda^2+4\lambda = -\lambda(\lambda^2-3\lambda-4) = -\lambda(\lambda-4)(\lambda+1)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0 \Rightarrow -\lambda(\lambda-4)(\lambda+1)=0 \Rightarrow \lambda=0,4,-1$

The eigenvalues are $-1, 0$ and 4

For $\lambda=-1$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

Equating the middle elements

$2x+3y=-y \Rightarrow x=-2y$

Let $y=1$, then $x=-2$

Equating the top elements and substituting $y=1$ and $x=-2$

$-8-2-4z=2 \Rightarrow z=-3$

An eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$.

For $\lambda=0$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the middle elements

$2x+3y=0 \Rightarrow 3y=-2x$

Let $x=3$, then $y=-2$

Equating the top elements and substituting $x=3$ and $y=-2$

$12+4-4z=0 \Rightarrow z=4$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the middle elements

$2x+3y=4y \Rightarrow y=2x$

Let $x=1$, then $y=2$

Equating the top elements and substituting $x=1$ and $y=2$

$4-4-4z=4 \Rightarrow z=-1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ -3 & 2 & 0 \\ 1 & 4 & -3 \end{pmatrix}$.

a Show that -1 is the only real eigenvalue of $\mathbf{A}$.

b Find an eigenvector corresponding to the eigenvalue -1.

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Further matrix algebra
Exercise F, Question 5

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & 4\\ 0 & 2 & 0\end{pmatrix}$.

a Show that 4 is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues of $\mathbf{A}$.

b Find an eigenvector corresponding to the eigenvalue 4.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & 4\\ 0 & 2 & 0\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}=\begin{pmatrix}2-\lambda & -1 & 3\\ 0 & 2-\lambda & 4\\ 0 & 2 & -\lambda\end{pmatrix}$

$$\begin{vmatrix}2-\lambda & -1 & 3\\ 0 & 2-\lambda & 4\\ 0 & 2 & -\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda & 4\\ 2 & -\lambda\end{vmatrix}-(-1)\begin{vmatrix}0 & 4\\ 0 & -\lambda\end{vmatrix}+3\begin{vmatrix}0 & 2-\lambda\\ 0 & 2\end{vmatrix}$$

$$=(2-\lambda)(-2\lambda+\lambda^2-8)+0+0$$

$$=(2-\lambda)(\lambda^2-2\lambda-8)=(2-\lambda)(\lambda-4)(\lambda+2)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(2-\lambda)(\lambda-4)(\lambda+2)=0\Rightarrow\lambda=2,4,-2$

The eigenvalues of $\mathbf{A}$ are 4, as required, 2 and -2.

b For $\lambda=4$

$$\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & 4\\ 0 & 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\mathbf{4}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x-y+3z\\ 2y+4z\\ 2y\end{pmatrix}=\begin{pmatrix}4x\\ 4y\\ 4z\end{pmatrix}$$

Equating the lowest elements

$2y=4z\Rightarrow y=2z$

Let $z=1$, then $y=2$

Equating the top elements and substituting $y=2$ and $z=1$

$2x-2+3=4x\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}\frac{1}{2}\\ 2\\ 1\end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 6

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}$.

Given that 3 is an eigenvalue of $\mathbf{A}$,

a find the other two eigenvalues of $\mathbf{A}$,

b find eigenvectors corresponding to each of the eigenvalues of $\mathbf{A}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{pmatrix}$

$$\begin{vmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}4-\lambda&-1\\4&3-\lambda\end{vmatrix}-1\begin{vmatrix}2&-1\\4&3-\lambda\end{vmatrix}+3\begin{vmatrix}2&4-\lambda\\4&4\end{vmatrix}$$

$$=(1-\lambda)\left((4-\lambda)(3-\lambda)+4\right)-(6-2\lambda+4)+3(8-16+4\lambda)$$

$$=(1-\lambda)\left(\lambda^2-7\lambda+16\right)+14\lambda-34$$

$$=-\lambda^3+8\lambda^2-23\lambda+16+14\lambda-34$$

$$=-\lambda^3+8\lambda^2-9\lambda-18$$

Let $\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2+k\lambda-6)$

Equating the coefficients of λ^2

$-8=-3+k\Rightarrow k=-5$

Hence $\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2-5\lambda-6)=(\lambda-3)(\lambda-6)(\lambda+1)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-3)(\lambda-6)(\lambda+1)=0\Rightarrow\lambda=3,6,-1$

The other eigenvalues of $\mathbf{A}$ are -1 and 6.

b For $\lambda=-1$

$$\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+y+3z\\2x+4y-z\\4x+4y+3z\end{pmatrix}=\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$$

Equating the top elements

$x+y+3z=-x$

$2x+y+3z=0$ ①

Equating the middle elements

$2x+4y-z=-y$

$2x+5y-z=0$ ②

② − ①

$4y-4z=0\Rightarrow y=z$

Let $z=1$, then $y=1$

Substituting $y=1$ and $z=1$ into ①

$2x+1+3=0\Rightarrow x=-2$

An eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix}-2\\1\\1\end{pmatrix}$.

For $\lambda = 3$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the lowest elements

$4x+4y+3z = 3z \Rightarrow y = -x$

Let $x=1$, then $y=-1$

Equating the top elements and substituting $x=1$ and $y=-1$

$1-1+3z = 3 \Rightarrow z = 1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda = 6$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$x+y+3z = 6x$

$-5x+y+3z = 0$ ①

Equating the lowest elements

$4x+4y+3z = 6z$

$4x+4y-3z = 0$ ②

① + ②

$-x+5y = 0 \Rightarrow x = 5y$

Let $y=1$, then $x=5$

Substituting $x=5$ and $y=1$ into ①

$-25+1+3z = 0 \Rightarrow z = 8$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 7

Question:

The matrix $\mathbf{A}=\begin{pmatrix}2 & 2 & 1\\ -2 & 4 & 0\\ 4 & 2 & 5\end{pmatrix}$.

a Show that 2 is an eigenvalue of $\mathbf{A}$.
b Find the other two eigenvalues of $\mathbf{A}$.
c Find a normalised eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&2&1\\-2&4&0\\4&2&5\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}2-\lambda&2&1\\-2&4-\lambda&0\\4&2&5-\lambda\end{pmatrix}$

When $\lambda=2$

$\mathbf{A}-2\mathbf{I}=\begin{pmatrix}0&2&1\\-2&2&0\\4&2&3\end{pmatrix}$

$\det(\mathbf{A}-2\mathbf{I})=\begin{vmatrix}0&2&1\\-2&2&0\\4&2&3\end{vmatrix}=0\begin{vmatrix}2&0\\2&3\end{vmatrix}-2\begin{vmatrix}-2&0\\4&3\end{vmatrix}+1\begin{vmatrix}-2&2\\4&2\end{vmatrix}$

$=0-2\times(-6)+1(-4-8)=12-12=0$

Hence 2 is an eigenvector of **A**.

b $\begin{vmatrix}2-\lambda&2&1\\-2&4-\lambda&0\\4&2&5-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}4-\lambda&0\\2&5-\lambda\end{vmatrix}-2\begin{vmatrix}-2&0\\4&5-\lambda\end{vmatrix}+1\begin{vmatrix}-2&4-\lambda\\4&2\end{vmatrix}$

$=(2-\lambda)(4-\lambda)(5-\lambda)+20-4\lambda+(-4-16+4\lambda)$

$=(2-\lambda)(4-\lambda)(5-\lambda)$

$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)(4-\lambda)(5-\lambda)=0\Rightarrow\lambda=2,4,5$

The other eigenvalues of **A** are 4 and 5.

c For $\lambda=2$

$\begin{pmatrix}2&2&1\\-2&4&0\\4&2&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\begin{pmatrix}2x+2y+z\\-2x+4y\\4x+2y+5z\end{pmatrix}=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}$

Equating the middle elements

$-2x+4y=2y\Rightarrow y=x$

Let $x=1$, then $y=1$

Equating the top elements and substituting $x=1$ and $y=1$

$2+2+z=2\Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}1\\1\\-2\end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2+1^2+(-2)^2\right)}=\sqrt{6}$

A normalised eigenvector corresponding to the eigenvalue 2 is

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

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Further matrix algebra
Exercise F, Question 8

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}$.

a Show that -2 is an eigenvalue of $\mathbf{A}$ and that there is only one other distinct eigenvalue.

b Find an eigenvector corresponding to each of the eigenvalues.

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Further matrix algebra
Exercise F, Question 9

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

Given that 2 is an eigenvalue of $\mathbf{A}$,

a find the other two eigenvalues of $\mathbf{A}$,

b find eigenvectors corresponding to each of the eigenvalues of $\mathbf{A}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda&-1&0\\-1&-\lambda&1\\1&2&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}1-\lambda&-1&0\\-1&-\lambda&1\\1&2&1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}-\lambda&1\\2&1-\lambda\end{vmatrix}-(-1)\begin{vmatrix}-1&1\\1&1-\lambda\end{vmatrix}+0\begin{vmatrix}-1&-\lambda\\1&2\end{vmatrix}$$

$$=(1-\lambda)(-\lambda+\lambda^2-2)+1(-1+\lambda-1)+0$$

$$=(1-\lambda)(\lambda-2)(\lambda+1)+1(\lambda-2)$$

$$=(\lambda-2)((1-\lambda)(1+\lambda)+1)=(\lambda-2)(2-\lambda^2)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(2-\lambda)(2-\lambda^2)=0\Rightarrow\lambda=2,\pm\sqrt{2}$

The other eigenvalues of $\mathbf{A}$ are $\pm\sqrt{2}$.

b For $\lambda=\sqrt{2}$

$$\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\sqrt{2}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x-y\\-x+z\\x+2y+z\end{pmatrix}=\begin{pmatrix}\sqrt{2}x\\\sqrt{2}y\\\sqrt{2}z\end{pmatrix}$$

Equating the top elements

$x-y=\sqrt{2}x\Rightarrow y=(1-\sqrt{2})x$

Let $x=1$, then $y=1-\sqrt{2}$

Equating the middle elements and substituting $x=1$ and $y=1-\sqrt{2}$

$-1+z=\sqrt{2}(1-\sqrt{2})=\sqrt{2}-2\Rightarrow z=\sqrt{2}-1$

An eigenvector corresponding to the eigenvalue $\sqrt{2}$ is $\begin{pmatrix}1\\1-\sqrt{2}\\\sqrt{2}-1\end{pmatrix}$.

For $\lambda=-\sqrt{2}$

$$\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-\sqrt{2}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x-y\\-x+z\\x+2y+z\end{pmatrix}=\begin{pmatrix}-\sqrt{2}x\\-\sqrt{2}y\\-\sqrt{2}z\end{pmatrix}$$

Equating the top elements

$x-y=-\sqrt{2}x\Rightarrow y=(\sqrt{2}+1)x$

Equating the middle elements and substituting $x=1$ and $y=1+\sqrt{2}$

$-1+z=-\sqrt{2}(1+\sqrt{2})=-\sqrt{2}-2 \Rightarrow z=-1-\sqrt{2}$

An eigenvector corresponding to the eigenvalue $-\sqrt{2}$ is $\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ -1-\sqrt{2} \end{pmatrix}$.

For $\lambda=2$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix}=\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements

$x-y=2x \Rightarrow y=-x$

Let $x=1$, then $y=-1$

Equating the middle elements and substituting $x=1$ and $y=-1$

$-1+z=-2 \Rightarrow z=-1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

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Further matrix algebra
Exercise F, Question 10

Question:

Given that $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}$,

a find the eigenvalue of $\mathbf{A}$ corresponding to $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$,

b find the value of a and the value of b,

c show that $\mathbf{A}$ has only one real eigenvalue.

Solution:

a $\begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 8+2-2 \\ 2+2a \\ -2+2-b \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 2\lambda \\ -1\lambda \end{pmatrix}$

Equating the top elements

$8 = 2\lambda \Rightarrow \lambda = 4$

The eigenvalue is 4.

b Equating the middle elements and substituting $\lambda = 4$

$2 + 2a = 8 \Rightarrow a = 3$

Equating the lowest elements and substituting $\lambda = 4$

$-b = -\lambda = -4 \Rightarrow b = 4$

$a = 3$ and $b = 4$

c $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 1 & 2 \\ 1 & 3-\lambda & 0 \\ -1 & 1 & 4-\lambda \end{pmatrix}$

$$\begin{vmatrix} 4-\lambda & 1 & 2 \\ 1 & 3-\lambda & 0 \\ -1 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 3-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (4-\lambda)^2(3-\lambda) - 1(4-\lambda) + 2(1+3-\lambda)$$

$$= (4-\lambda)^2(3-\lambda) + 1(4-\lambda) = (4-\lambda)\big((4-\lambda)(3-\lambda)+1\big)$$

$$= (4-\lambda)(\lambda^2 - 7\lambda + 13)$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (4-\lambda)(\lambda^2 - 7\lambda + 13) = 0 \Rightarrow \lambda = 4$ or $\lambda^2 - 7\lambda + 13 = 0$

The discriminant of $\lambda^2 - 7\lambda + 13 = 0$ is given by

$b^2 - 4ac = 49 - 52 = -3 < 0$

There are no real solutions of $\lambda^2 - 7\lambda + 13 = 0$

4 is the only real eigenvalue of $\mathbf{A}$.

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Further matrix algebra
Exercise G, Question 1

Question:

Reduce the following matrices to diagonal matrices.

a $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

Solution:

a Using $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 1 - 2\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2) = 0$$

$$\lambda = -2, 4$$

For $\lambda = -2$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

Equating the upper elements

$x+3y=-2x \Rightarrow y=-x$

Let $x=1$, then $y=-1$

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is $\sqrt{\left(1^2+(-1)^2\right)} = \sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

For $\lambda = 4$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$$

Equating the upper elements

$x+3y=4x \Rightarrow y=x$

Let $x=1$, then $y=1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(1^2+1^2\right)} = \sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{P}^{\mathrm{T}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -1-1 & 2-2 \\ -1+1 & 2+2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$$

b Using $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = 4 - 5\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5$$

For $\lambda = 5$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$x - 2y = 5x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2 + (-2)^2\right)} = \sqrt{5}$.

A normalised eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$.

For $\lambda = 0$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Equating the upper elements
$x-2y=0 \Rightarrow x=2y$

Let $y=1$, then $x=2$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}2\\1\end{pmatrix}$.

The magnitude of $\begin{pmatrix}2\\1\end{pmatrix}$ is $\sqrt{(2^2+1^2)}=\sqrt{5}$.

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{pmatrix}$.

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\\ -\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{pmatrix},\quad \mathbf{P}^{\mathrm{T}}\begin{pmatrix}\frac{1}{\sqrt5} & -\frac{2}{\sqrt5}\\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP}=\begin{pmatrix}\frac{1}{\sqrt5} & -\frac{2}{\sqrt5}\\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{pmatrix}\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\\ -\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt5} & -\frac{2}{\sqrt5}\\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt5}+\frac{4}{\sqrt5} & \frac{2}{\sqrt5}-\frac{2}{\sqrt5}\\ -\frac{2}{\sqrt5}-\frac{8}{\sqrt5} & -\frac{4}{\sqrt5}+\frac{4}{\sqrt5}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt5} & -\frac{2}{\sqrt5}\\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{pmatrix}\begin{pmatrix}\frac{5}{\sqrt5} & 0\\ -\frac{10}{\sqrt5} & 0\end{pmatrix}$$

$$=\begin{pmatrix}1+4 & 0\\ 2-2 & 0\end{pmatrix}=\begin{pmatrix}5 & 0\\ 0 & 0\end{pmatrix}$$

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Further matrix algebra
Exercise G, Question 2

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$.

b Find normalised eigenvectors of $\mathbf{A}$ corresponding to each of the two eigenvalues of $\mathbf{A}$.

c Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$.

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Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 3

Question:

The matrix $\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0\end{pmatrix}$.

a Show that $\mathbf{P}$ is an orthogonal matrix.

The matrix $\mathbf{A}=\begin{pmatrix}\frac{3}{2} & -\frac{3}{2} & 1\\ -\frac{3}{2} & \frac{3}{2} & 1\\ 1 & 1 & 1\end{pmatrix}$.

b Show that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

Solution:

a $\mathbf{PP}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{2}{6}-\frac{1}{3} \\ \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{2}{6}-\frac{1}{3} \\ \frac{2}{6}-\frac{1}{3} & \frac{2}{6}-\frac{1}{3} & \frac{4}{6}+\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Hence $\mathbf{P}$ is an orthogonal matrix.

b $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & \frac{3}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2\sqrt{6}} - \frac{3}{2\sqrt{6}} + \frac{2}{\sqrt{6}} & -\frac{3}{2\sqrt{3}} + \frac{3}{2\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \\ -\frac{3}{2\sqrt{6}} + \frac{3}{2\sqrt{6}} + \frac{2}{\sqrt{6}} & \frac{3}{2\sqrt{3}} - \frac{3}{2\sqrt{3}} + \frac{1}{\sqrt{3}} & -\frac{3}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} \\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{2}} \\ \frac{4}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{6} + \frac{2}{6} + \frac{8}{6} & \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} & \frac{3}{\sqrt{12}} - \frac{3}{\sqrt{12}} \\ -\frac{2}{18} - \frac{2}{\sqrt{18}} + \frac{4}{\sqrt{18}} & -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} & -\frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} \\ \frac{2}{\sqrt{12}} - \frac{2}{\sqrt{12}} & \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{3}{2} + \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$ a diagonal matrix.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 4

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$. Reduce $\mathbf{A}$ to a diagonal matrix.

Solution:

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & 2-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}-0\begin{vmatrix}0 & 0\\ 2 & 2-\lambda\end{vmatrix}+2\begin{vmatrix}0 & 2-\lambda\\ 2 & 0\end{vmatrix}$$

$$=(2-\lambda)^3-4(2-\lambda)=(2-\lambda)\left((2-\lambda)^2-4\right)=(2-\lambda)(-\lambda)(4-\lambda)$$

$$=-\lambda(2-\lambda)(4-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow -\lambda(\lambda-2)(\lambda-4)=0\Rightarrow \lambda=0,2,4$$

For $\lambda=0$

$$\begin{pmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x+2z\\ 2y\\ 2x+2z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Equating the top elements

$2x+2z=0\Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements

$2y=0\Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$.

The magnitude of $\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$ is $\sqrt{\left(1^2+0^2+(-1)^2\right)}=\sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$.

For $\lambda=2$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2z \\ 2y \\ 2x+2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements

$2x+2z=2x \Rightarrow z=0$

Equating the lowest elements

$2x+2z=2z \Rightarrow x=0$

y can take any value

Let $y=1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The magnitude of this vector is 1, so it is already normalised.

For $\lambda=4$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2z \\ 2y \\ 2x+2z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$2x+2z=4x \Rightarrow z=x$

Let $x=1$, then $z=1$

Equating the middle elements

$2y=4y \Rightarrow 2y=0 \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $\sqrt{\left(1^2+0^2+1^2\right)}=\sqrt{2}$

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$$\text{Let } \mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{Then } \mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ 0 & 2 & 0 \\ \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{4}{\sqrt{2}} \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2-2 \\ 0 & 2 & 0 \\ 0 & 0 & 2+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 5

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

The eigenvalues of $\mathbf{A}$ are 0, -1 and 8.

a Find a normalised eigenvector corresponding to the eigenvalue 0.

Given that $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1 and that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 8,

b find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

Solution:

a For $\lambda = 0$

$$\begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x+3y+3z \\ 3x+y+z \\ 3x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the top elements

$5x+3y+3z=0$ ①

Equating the middle elements

$3x+y+z=0$ ②

$3\times$② $-$ ①

$x=0$

Substituting $x=0$ into ②

$y+z=0 \Rightarrow z=-y$

Let $y=1$, then $z=-1$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is $\sqrt{\left(0^2+1^2+(-1)^2\right)}=\sqrt{2}$

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

b The magnitude of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is $\sqrt{\left((-1)^2+1^2+1^2\right)}=\sqrt{3}$

A normalised eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

The magnitude of $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is $\sqrt{(2^2+1^2+1^2)}=\sqrt{6}$

A normalised eigenvector corresponding to the eigenvalue 8 is $\begin{pmatrix}\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$.

$$\mathbf{P}=\begin{pmatrix}0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}0 & 0 & 0\\0 & -1 & 0\\0 & 0 & 8\end{pmatrix}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 6

Question:

The matrix $\mathbf{A}=\begin{pmatrix}7 & 0 & 2\\ 0 & 5 & -2\\ -2 & -2 & 6\end{pmatrix}$.

a Given that 9 is an eigenvalue of **A**, find the other two eigenvalues of **A**.

b Find eigenvectors of **A** corresponding to each of the three eigenvalues of **A**.

c Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$.

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{vmatrix}$$

$$=(7-\lambda)\begin{vmatrix}5-\lambda & -2\\ -2 & 6-\lambda\end{vmatrix}-0\begin{vmatrix}0 & -2\\ -2 & 6-\lambda\end{vmatrix}+(-2)\begin{vmatrix}0 & 5-\lambda\\ -2 & -2\end{vmatrix}$$

$$=(7-\lambda)((5-\lambda)(6-\lambda)-4)-2(10-2\lambda)$$

$$=(7-\lambda)(26-11\lambda+\lambda^2)-20+4\lambda$$

$$=182-103\lambda+18\lambda^2-\lambda^3-20+4\lambda=-(\lambda^3-18\lambda^2+99\lambda-162)$$

Let $\lambda^3-18\lambda^2+99\lambda-162=(\lambda-9)(\lambda^2+k\lambda+18)$

Equating coefficients of λ^2

$-18=-9+k\Rightarrow k=-9$

Hence

$\lambda^3-18\lambda^2+99\lambda-162=(\lambda-9)(\lambda^2-9\lambda+18)=(\lambda-9)(\lambda-6)(\lambda-3)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-3)(\lambda-6)(\lambda-9)=0\Rightarrow\lambda=3,6,9$

The other two eigenvalues of $\mathbf{A}$ are 3 and 6.

b For $\lambda=3$

$$\begin{pmatrix}7 & 0 & -2\\ 0 & 5 & -2\\ -2 & -2 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}7x-2z\\ 5y-2z\\ -2x-2y+6z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the top elements

$7x-2z=3x\Rightarrow z=2x$

Let $x=1$, then $z=2$

Equating the middle elements and substituting $z=2$

$5y-4=3y\Rightarrow y=2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$.

For $\lambda=6$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$7x-2z=6x \Rightarrow x=2z$

Let $z=1$, then $x=2$

Equating the middle elements and substituting $z=1$

$5y-2=6y \Rightarrow y=-2$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

For $\lambda=9$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$$

Equating the top elements

$7x-2z=9x \Rightarrow z=-x$

Let $x=2$, then $z=-2$

Equating the middle elements and substituting $z=-2$

$5y+4=9y \Rightarrow y=1$

An eigenvector corresponding to the eigenvalue 9 is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

c The magnitudes of the vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ are all

$\sqrt{\left(1^2+2^2+2^2\right)}=\sqrt{9}=3$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 7

Question:

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}$.

a Show that 4 is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues of $\mathbf{A}$.
b Find a normalised eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 4.

Given that $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$,

c find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Solution:

a $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & \sqrt{5} \\ 0 & \sqrt{5} & 1-\lambda \end{vmatrix}$

Substituting $\lambda = 4$,

$$\begin{vmatrix} 1-4 & 2 & 0 \\ 2 & 1-4 & \sqrt{5} \\ 0 & \sqrt{5} & 1-4 \end{vmatrix} = \begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & \sqrt{5} \\ 0 & \sqrt{5} & -3 \end{vmatrix} = (-3)\begin{vmatrix} -3 & \sqrt{5} \\ \sqrt{5} & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & \sqrt{5} \\ 0 & -3 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & \sqrt{5} \end{vmatrix}$$

$$= (-3)(9-5) - 2(-6-0) = -12+12 = 0$$

Hence, by the factor theorem, 4 is an eigenvalue of $\mathbf{A}$.

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & \sqrt{5} \\ 0 & \sqrt{5} & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & \sqrt{5} \\ \sqrt{5} & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & \sqrt{5} \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 1-\lambda \\ 0 & \sqrt{5} \end{vmatrix}$$

$$= (1-\lambda)\left((1-\lambda)^2 - 5\right) - 4 + 4\lambda$$

$$= (1-\lambda)(\lambda^2 - 2\lambda - 4) - 4 + 4\lambda = -\lambda^3 + 3\lambda^2 + 6\lambda - 8$$

$$= -\lambda^3 + 4\lambda^2 - \lambda^2 + 4\lambda + 2\lambda - 8 = -\lambda^2(\lambda-4) - \lambda(\lambda-4) + 2(\lambda-4)$$

$$= -(\lambda-4)(\lambda^2+\lambda-2) = -(\lambda-4)(\lambda+2)(\lambda-1)$$

$\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-4)(\lambda+2)(\lambda-1) = 0 \Rightarrow \lambda = 4, -2, 1$

The other two eigenvalues of $\mathbf{A}$ are -2 and 1.

b For $\lambda = 4$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+2y \\ 2x+y+\sqrt{5}z \\ \sqrt{5}y+z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$x+2y = 4x \Rightarrow 2y = 3x$

Let $x = 2$, then $y = 3$

Equating the lowest elements and substituting $y = 3$

$3\sqrt{5} + z = 4z \Rightarrow z = \sqrt{5}$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 2 \\ 3 \\ \sqrt{5} \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ 3 \\ \sqrt{5} \end{pmatrix}$ is $\sqrt{\left(2^2 + 3^2 + \left(\sqrt{5}\right)^2\right)} = \sqrt{18}$

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{2}{\sqrt{18}} \\ \frac{3}{\sqrt{18}} \\ \frac{\sqrt{5}}{\sqrt{18}} \end{pmatrix}$.

c $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2+6 \\ -4+3-5 \\ 3\sqrt{5}-\sqrt{5} \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2\sqrt{5} \end{pmatrix} = (-2)\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$

The magnitude of $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ is $\sqrt{\left((-2)^2+3^2+(-\sqrt{5})^2\right)} = \sqrt{18}$

A normalised eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} -\frac{2}{\sqrt{18}} \\ \frac{3}{\sqrt{18}} \\ -\frac{5}{\sqrt{18}} \end{pmatrix}$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ is $\sqrt{\left((\sqrt{5})^2+0^2+2^2\right)} = \sqrt{9} = 3$

A normalised eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \frac{\sqrt{5}}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}$.

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{18}} & -\frac{2}{\sqrt{18}} & \frac{\sqrt{5}}{3} \\ \frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} & 0 \\ \frac{\sqrt{5}}{\sqrt{18}} & -\frac{\sqrt{5}}{\sqrt{18}} & -\frac{2}{3} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Further matrix algebra
Exercise G, Question 8

Question:

The eigenvalue of the matrix $\mathbf{A}=\begin{pmatrix}2 & 2 & -3\\ 2 & 2 & 3\\ -3 & 3 & 3\end{pmatrix}$ are $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 > \lambda_2 > \lambda_3$.

a Show that $\lambda_1 = 6$ and find the other two eigenvalues λ_2 and λ_3.

b Verify that $\det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3$.

c Find an eigenvector corresponding to the value $\lambda_1 = 6$.

Given that $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ are eigenvectors corresponding to λ_2 and λ_3,

d write down a matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2-\lambda & 2 & -3\\ 2 & 2-\lambda & 3\\ -3 & 3 & 3-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 2 & -3\\ 2 & 2-\lambda & 3\\ -3 & 3 & 3-\lambda\end{vmatrix}$

$=(2-\lambda)\begin{vmatrix}2-\lambda & 3\\ 3 & 3-\lambda\end{vmatrix}-2\begin{vmatrix}2 & 3\\ -3 & 3-\lambda\end{vmatrix}+(-3)\begin{vmatrix}2 & 2-\lambda\\ -3 & 3\end{vmatrix}$

$=(2-\lambda)\left((2-\lambda)(3-\lambda)-9\right)-2(6-2\lambda+9)-3(6+6-3\lambda)$

$=(2-\lambda)(\lambda^2-5\lambda-3)-30+4\lambda-36+9\lambda$

$=-\lambda^3+7\lambda^2-7\lambda-6-66+13\lambda=-\lambda^3+7\lambda^2+6\lambda-72$

$=-\lambda^3+6\lambda^2+\lambda^2-6\lambda+12\lambda-72$

$=-\lambda^2(\lambda-6)+\lambda(\lambda-6)+12(\lambda-6)=-(\lambda-6)(\lambda^2-\lambda-12)$

$=-(\lambda-6)(\lambda-4)(\lambda+3)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-6)(\lambda-4)(\lambda+3)=0\Rightarrow\lambda=6,4,-3$

As $\lambda_1>\lambda_2>\lambda_3, \lambda_1=6$, as required, $\lambda_2=4$ and $\lambda_3=-3$.

b $\det(\mathbf{A})=\begin{vmatrix}2 & 2 & -3\\ 2 & 2 & 3\\ -3 & 3 & 3\end{vmatrix}=2\begin{vmatrix}2 & 3\\ 3 & 3\end{vmatrix}-2\begin{vmatrix}2 & 3\\ -3 & 3\end{vmatrix}+(-3)\begin{vmatrix}2 & 2\\ -3 & 3\end{vmatrix}$

$=2(6-9)-2(6+9)-3(6+6)=-6-30-36$

$=-72=6\times4\times(-3)=\lambda_1\lambda_2\lambda_3$, as required.

c For $\lambda_1=6$

$\begin{pmatrix}2 & 2 & -3\\ 2 & 2 & 3\\ -3 & 3 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=6\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

$\begin{pmatrix}2x+2y-3z\\ 2x+2y+3z\\ -3x+3y+3z\end{pmatrix}=\begin{pmatrix}6x\\ 6y\\ 6z\end{pmatrix}$

Equating the top elements

$2x+2y-3z=6x\Rightarrow-4x+2y-3z=0$ ①

Equating the middle elements

$2x+2y+3z=6y\Rightarrow2x-4y+3z=0$ ②

① + ②

$-2x-2y=0\Rightarrow y=-x$

Let $x=1$, then $y=-1$

Substitute $x=1$ and $y=-1$ into ①

$-4-2-3z=0\Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

d The magnitude of $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2 + (-1)^2 + (-2)^2\right)} = \sqrt{6}$

The magnitude of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is $\sqrt{\left(1^2 + 1^2 + 0^2\right)} = \sqrt{2}$

The magnitude of $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(1^2 + (-1)^2 + 1^2\right)} = \sqrt{3}$

Hence $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$

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Further matrix algebra
Exercise H, Question 1

Question:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

Given that $\mathbf{A}$ is singular, find the value of t. **[E]**

Solution:

$$\begin{vmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} t & 1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} t & 3 \\ -2 & -1 \end{vmatrix}$$

$$= 1(3+1) + 2(-t+6) = 16 - 2t$$

As $\mathbf{A}$ is singular

$\det(\mathbf{A}) = 16 - 2t = 0 \Rightarrow t = 8$

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Further matrix algebra
Exercise H, Question 2

Question:

$$\mathbf{M}=\begin{pmatrix}1&0&0\\x&2&0\\3&1&1\end{pmatrix}$$

Find $\mathbf{M}^{-1}$ in terms of x. **[E]**

Solution:

$$\det(\mathbf{M})=\begin{vmatrix}1&0&0\\x&2&0\\3&1&1\end{vmatrix}=1\begin{vmatrix}2&0\\1&1\end{vmatrix}-0\begin{vmatrix}x&0\\3&1\end{vmatrix}+0\begin{vmatrix}x&2\\3&1\end{vmatrix}$$

$$=2-0+0=2$$

The matrix of minors is

$$\begin{pmatrix}\begin{vmatrix}2&0\\1&1\end{vmatrix}&\begin{vmatrix}x&0\\3&1\end{vmatrix}&\begin{vmatrix}x&2\\3&1\end{vmatrix}\\\begin{vmatrix}0&0\\1&1\end{vmatrix}&\begin{vmatrix}1&0\\3&1\end{vmatrix}&\begin{vmatrix}1&0\\3&1\end{vmatrix}\\\begin{vmatrix}0&0\\2&0\end{vmatrix}&\begin{vmatrix}1&0\\x&0\end{vmatrix}&\begin{vmatrix}1&0\\x&2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}2&x&x-6\\0&1&1\\0&0&2\end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C}=\begin{pmatrix}2&-x&x-6\\0&1&-1\\0&0&2\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}2&0&0\\-x&1&0\\x-6&-1&2\end{pmatrix}$$

$$\mathbf{M}^{-1}=\frac{1}{\det(\mathbf{M})}\mathbf{C}^{\mathrm{T}}=\frac{1}{2}\begin{pmatrix}2&0&0\\-x&1&0\\x-6&-1&2\end{pmatrix}$$

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Further matrix algebra
Exercise H, Question 3

Question:

The matrix $\mathbf{M}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -15$ and $\mathbf{M} = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}$.

a For each eigenvalue, find a corresponding eigenvector.

b Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{pmatrix} 5 & 0 \\ 0 & -15 \end{pmatrix}$. **[E]**

Solution:

a For $\lambda_1 = 5$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$x + 8y = 5x \Rightarrow x = 2y$

Let $y = 1$, then $x = 2$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -15$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -15\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} -15x \\ -15y \end{pmatrix}$$

Equating the upper elements

$x + 8y = -15x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

An eigenvector corresponding to the eigenvalue -15 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

b The magnitude of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\sqrt{\left(2^2 + 1^2\right)} = \sqrt{5}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2 + (-2)^2\right)} = \sqrt{5}$

Hence $\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$

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Further matrix algebra
Exercise H, Question 4

Question:

The matrix $\mathbf{A}=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$ and the matrix $\mathbf{B}=\begin{pmatrix}2 & -1\\ -4 & 2\end{pmatrix}$.

a Find $\mathbf{AB}$.

b Verify that $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=(\mathbf{AB})^{\mathrm{T}}$.

Solution:

a $\mathbf{AB}=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -4 & 2\end{pmatrix}=\begin{pmatrix}10-8 & -5+4\\ 4-4 & -2+2\end{pmatrix}=\begin{pmatrix}2 & -1\\ 0 & 0\end{pmatrix}$

b $(\mathbf{AB})^{\mathrm{T}}=\begin{pmatrix}2 & 0\\ -1 & 0\end{pmatrix}$

$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}=\begin{pmatrix}10-8 & 4-4\\ -5+4 & -2+2\end{pmatrix}=\begin{pmatrix}2 & 0\\ -1 & 0\end{pmatrix}$

$=(\mathbf{AB})^{\mathrm{T}}$, as required.

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Further matrix algebra
Exercise H, Question 5

Question:

A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A} = \begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

Solution:

a $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -5-\lambda & 8 \\ 3 & -7-\lambda \end{pmatrix}$

$$\begin{vmatrix} -5-\lambda & 8 \\ 3 & -7-\lambda \end{vmatrix} = (5+\lambda)(7+\lambda) - 24 = \lambda^2 + 12\lambda + 11 = (\lambda+1)(\lambda+11)$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (\lambda+1)(\lambda+11) = 0 \Rightarrow \lambda = -1, -11$$

The eigenvalues of $\mathbf{A}$ are -1 and -11.

b For $\lambda = -1$

$$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Equating the upper elements

$$-5x + 8y = -x \Rightarrow y = \frac{1}{2}x$$

For $\lambda = -11$

$$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -11\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -11x \\ -11y \end{pmatrix}$$

Equating the upper elements

$$-5x + 8y = -11x \Rightarrow y = -\frac{3}{4}x$$

Cartesian equations of the lines through the origin which are invariant under T are

$y = \frac{1}{2}x$ and $y = -\frac{3}{4}x$.

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Further matrix algebra
Exercise H, Question 6

Question:

Given that 1 is an eigenvalue of the matrix $\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$,

a find a corresponding eigenvector,

b find the other eigenvalues of the matrix. **[E]**

Solution:

a For $\lambda = 1$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x+y \\ 2x+4y \\ x+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements

$3x + y = x \Rightarrow 2x + y = 0$ ①

Equating the middle elements

$2x + 4y = y \Rightarrow 2x + 3y = 0$ ②

② − ①

$2y = 0 \Rightarrow y = 0$

Substituting $y = 0$ into ①

$2x = 0 \Rightarrow x = 0$

z can take any non-zero value

Let $z = 1$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

b Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 2 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & 1 & 0 \\ 2 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 1 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 4-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(1-\lambda) - 2(1-\lambda)$$

$$= (1-\lambda)\big((3-\lambda)(4-\lambda) - 2\big) = (1-\lambda)\left(\lambda^2 - 7\lambda + 10\right)$$

$$= (1-\lambda)(\lambda-2)(\lambda-5)$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (1-\lambda)(\lambda-2)(\lambda-5) = 0 \Rightarrow \lambda = 1, 2, 5$$

The other eigenvalues are 2 and 5.

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Further matrix algebra
Exercise H, Question 7

Question:

The transformation $T:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix $\mathbf{T}$ where

$$\mathbf{T}=\begin{pmatrix}4 & 3 & 0\\ 0 & -2 & 1\\ 3 & 1 & -2\end{pmatrix}.$$

The line l_1 is transformed by T to the line l_2. The line l_1 has vector equation

$$r=\begin{pmatrix}1\\0\\2\end{pmatrix}+t\begin{pmatrix}2\\-3\\0\end{pmatrix}$$, where t is a real parameter.

Find Cartesian equations of l_2.

Solution:

$$\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+t\begin{pmatrix}2\\-3\\0\end{pmatrix}=\begin{pmatrix}1+2t\\-3t\\2\end{pmatrix}$$

$$\mathbf{Tr}=\begin{pmatrix}4 & 3 & 0\\ 0 & -2 & 1\\ 3 & 1 & -2\end{pmatrix}\begin{pmatrix}1+2t\\-3t\\2\end{pmatrix}=\begin{pmatrix}4+8t-9t\\6t+2\\3+6t-3t-4\end{pmatrix}=\begin{pmatrix}4-t\\2+6t\\-1+3t\end{pmatrix}$$

Equations of l_2 are given by

$$\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4-t\\2+6t\\-1+3t\end{pmatrix}$$

Equating elements

$x=4-t, y=2+6t, z=-1+3t$

$$\frac{x-4}{-1}=\frac{y-2}{6}=\frac{z+1}{3}=\quad t$$

Cartesian equations of l_2 are

$$\frac{x-4}{-1}=\frac{y-2}{6}=\frac{z+1}{3}$$

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Further matrix algebra
Exercise H, Question 8

Question:

$$\mathbf{A}=\begin{pmatrix}3 & 4 & -4\\ 4 & 5 & 0\\ -4 & 0 & 1\end{pmatrix}$$

a Show that 3 is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b Find an eigenvector corresponding to the eigenvalue 3.

Given that the vectors $\begin{pmatrix}2\\ 2\\ -1\end{pmatrix}$ and $\begin{pmatrix}2\\ -1\\ 2\end{pmatrix}$ are eigenvectors corresponding to the other two eigenvalues,

c find a matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix. **[E]**

Solution:

a $$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 4 & -4\\ 4 & 5-\lambda & 0\\ -4 & 0 & 1-\lambda\end{pmatrix}$$

$$\begin{vmatrix}3-\lambda & 4 & -4\\ 4 & 5-\lambda & 0\\ -4 & 0 & 1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}5-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix}-4\begin{vmatrix}4 & 0\\ -4 & 1-\lambda\end{vmatrix}+(-4)\begin{vmatrix}4 & 5-\lambda\\ -4 & 0\end{vmatrix}$$

$$=(3-\lambda)(5-\lambda)(1-\lambda)-16+16\lambda-80+16\lambda$$

$$=(3-\lambda)(5-\lambda)(1-\lambda)-96+32\lambda$$

$$=(3-\lambda)(5-\lambda)(1-\lambda)-32(3-\lambda)$$

$$=(3-\lambda)((5-\lambda)(1-\lambda)-32)=(3-\lambda)(\lambda^2-6\lambda-27)$$

$$=(3-\lambda)(\lambda+3)(\lambda-9)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(\lambda+3)(\lambda-9)=0\Rightarrow\lambda=3,-3,9$

3 is an eigenvalue of $\mathbf{A}$ and the other eigenvalues are -3 and 9.

b $$\begin{pmatrix}3 & 4 & -4\\ 4 & 5 & 0\\ -4 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x+4y-4z\\ 4x+5y\\ -4x+z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the middle elements

$4x+5y=3y\Rightarrow y=-2x$

Let $x=1$, then $y=-2$

Equating the lowest elements and substituting $x=1$

$-4+z=3z\Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ -2\\ -2\end{pmatrix}$.

c The magnitudes of $\begin{pmatrix}1\\ -2\\ -2\end{pmatrix}$, $\begin{pmatrix}2\\ 2\\ -1\end{pmatrix}$ and $\begin{pmatrix}2\\ -1\\ 2\end{pmatrix}$ are all

$\sqrt{(1^2+2^2+2^2)}=\sqrt{9}=3$

Hence

$$\mathbf{P}=\begin{pmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

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Further matrix algebra
Exercise H, Question 9

Question:

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

a Show that $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$, giving their corresponding eigenvalues.

b Given that 6 is the third eigenvalue of $\mathbf{A}$, find a corresponding eigenvector.

c Hence write down a matrix such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. **[E]**

Solution:

a $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-6+0 \\ -4+3-2 \\ 0+6-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is an eigenvalue of $\mathbf{A}$ corresponding to the eigenvalue -1.

$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+2+0 \\ -4-1+2 \\ 0-2+5 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, is an eigenvalue of $\mathbf{A}$ corresponding to the eigenvalue 3.

b For $\lambda = 6$

$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 2x-2y \\ -2x+y+2z \\ 2y+5z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$

Equating the top elements

$2x-2y = 6x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

Equating the lowest elements and substituting $y = -2$

$-4+5z = 6z \Rightarrow z = -4$

$\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ is an eigenvalue of $\mathbf{A}$ corresponding to the eigenvalue 6.

c The magnitude of $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is $\sqrt{\left(2^2+3^2+(-1)^2\right)} = \sqrt{14}$

The magnitude of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(2^2+(-1)^2+1^2\right)} = \sqrt{6}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ is $\sqrt{\left(1^2+(-2)^2+(-4)^2\right)} = \sqrt{21}$

Hence

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \\ -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{21}} \end{pmatrix}$$

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Further matrix algebra
Exercise H, Question 10

Question:

a Calculate the inverse of the matrix $\mathbf{A}(x)=\begin{pmatrix}1 & x & -1\\ 3 & 0 & 2\\ 1 & 1 & 0\end{pmatrix}, x\neq\frac{5}{2}$.

The image of the vector $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$ when it is transformed by the matrix $\begin{pmatrix}1 & 3 & -1\\ 3 & 0 & 2\\ 1 & 1 & 0\end{pmatrix}$ is the vector $\begin{pmatrix}4\\ 3\\ 5\end{pmatrix}$.

b Find the values of a, b and c. **[E]**

Solution:

a $\det\left(\mathbf{A}(x)\right)=\begin{vmatrix}1 & x & -1\\ 3 & 0 & 2\\ 1 & 1 & 0\end{vmatrix}=1\begin{vmatrix}0 & 2\\ 1 & 0\end{vmatrix}-x\begin{vmatrix}3 & 2\\ 1 & 0\end{vmatrix}+(-1)\begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}$

$=-2+2x-3=2x-5$

The matrix of the minors is given by

$$\mathbf{M}=\begin{vmatrix}\begin{vmatrix}0 & 2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}3 & 2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}\\ \begin{vmatrix}x & -1\\ 1 & 0\end{vmatrix} & \begin{vmatrix}1 & -1\\ 1 & 0\end{vmatrix} & \begin{vmatrix}1 & x\\ 1 & 1\end{vmatrix}\\ \begin{vmatrix}x & -1\\ 0 & 2\end{vmatrix} & \begin{vmatrix}1 & -1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}1 & x\\ 3 & 0\end{vmatrix}\end{vmatrix}=\begin{pmatrix}-2 & -2 & 3\\ 1 & 1 & 1-x\\ 2x & 5 & -3x\end{pmatrix}$$

The matrix of the cofactors is given by

$$\mathbf{C}=\begin{pmatrix}-2 & 2 & 3\\ -1 & 1 & x-1\\ 2x & -5 & -3x\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-2 & -1 & 2x\\ 2 & 1 & -5\\ 3 & x-1 & -3x\end{pmatrix}$$

$$\left(\mathbf{A}(x)\right)^{-1}=\frac{1}{\det\left(\mathbf{A}(x)\right)}\mathbf{C}^{\mathrm{T}}=\frac{1}{2x-5}\begin{pmatrix}-2 & -1 & 2x\\ 2 & 1 & -5\\ 3 & x-1 & -3x\end{pmatrix}$$

b Substituting $x=3$

$$\left(\mathbf{A}(3)\right)^{-1}=\begin{pmatrix}-2 & -1 & 6\\ 2 & 1 & -5\\ 3 & 2 & -9\end{pmatrix}$$

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}-2 & -1 & 6\\ 2 & 1 & -5\\ 3 & 2 & -9\end{pmatrix}\begin{pmatrix}4\\ 3\\ 5\end{pmatrix}=\begin{pmatrix}-8-3+30\\ 8+3-25\\ 12+6-45\end{pmatrix}=\begin{pmatrix}19\\ -14\\ -27\end{pmatrix}$$

Equating elements

$a=19, b=-14, c=-27$

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Further matrix algebra
Exercise H, Question 11

Question:

a Show that for all values of the constant α, an eigenvalue of the matrix $\mathbf{A}$ is 1,

where $\mathbf{A} = \begin{pmatrix} \alpha & 0 & 2 \\ 4 & 3 & 0 \\ -2 & -1 & 1 \end{pmatrix}$.

An eigenvector of the matrix $\mathbf{A}$ is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and the corresponding eigenvalue is $\beta (\beta \neq 1)$.

b Find the value of α and the value of β.

c For your value of α, find the third eigenvalue of $\mathbf{A}$. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}\alpha-\lambda & 0 & 2\\ 4 & 3-\lambda & 0\\ -2 & -1 & 1-\lambda\end{pmatrix}$

$$\begin{vmatrix}\alpha-\lambda & 0 & 2\\ 4 & 3-\lambda & 0\\ -2 & -1 & 1-\lambda\end{vmatrix}=(\alpha-\lambda)\begin{vmatrix}3-\lambda & 0\\ -1 & 1-\lambda\end{vmatrix}-0\begin{vmatrix}4 & 0\\ -2 & 1-\lambda\end{vmatrix}+2\begin{vmatrix}4 & 3-\lambda\\ -2 & -1\end{vmatrix}$$

$$=(\alpha-\lambda)(3-\lambda)(1-\lambda)+2(-4+6-2\lambda)$$

$$=(\alpha-\lambda)(3-\lambda)(1-\lambda)+4(1-\lambda)$$

$$=(1-\lambda)\left((\alpha-\lambda)(3-\lambda)+4\right) \quad *$$

Hence, for all α, $\lambda=1$ is a solution of $\det(\mathbf{A}-\lambda\mathbf{I})=0$, and, for all α, an eigenvalue of $\mathbf{A}$ is 1.

b

$$\begin{pmatrix}\alpha & 0 & 2\\ 4 & 3 & 0\\ -2 & -1 & 1\end{pmatrix}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}=\beta\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}$$

$$\begin{pmatrix}2\alpha+2\\ 8-6\\ -4+2+1\end{pmatrix}=\begin{pmatrix}2\beta\\ -2\beta\\ \beta\end{pmatrix}=\begin{pmatrix}2\alpha+2\\ 2\\ -1\end{pmatrix}$$

Equating the lowest elements

$\beta=-1$

Equating the top elements and substituting $\beta=-1$

$2\alpha+2=-2\Rightarrow\alpha=-2$

$$\alpha=-2,\ \beta=-1$$

c Substituting $\alpha=-2$ into $*$ in part **a** and equating to 0

$(1-\lambda)\left((-2-\lambda)(3-\lambda)+4\right)=0$

$$(1-\lambda)\left(\lambda^2-\lambda-2\right)=(1-\lambda)(\lambda-2)(\lambda+1)$$

$$\lambda=1,2,-1$$

The third eigenvalue is 2.

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Further matrix algebra
Exercise H, Question 12

Question:

The matrix $\mathbf{A}$ is defined by $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & u \\ 0 & 1 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^{-1}$ in terms of u, stating the condition for which $\mathbf{A}$ is non-singular.

The image vector of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by the matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix}$.

b Find the values of a, b and c. **[E]**

Solution:

a

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & u \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & u \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & u \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1 - u + 2 + 6 = 9 - u$$

$\mathbf{A}$ is singular if $\det(\mathbf{A}) = 0 \Rightarrow 9 - u = 0 \Rightarrow u = 9$

The condition for which $\mathbf{A}$ is non-singular is $u \neq 9$.

The matrix of the minors is given by

$$\mathbf{M} = \begin{Vmatrix} \begin{vmatrix} 1 & u \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & u \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & u \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & u \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{Vmatrix} = \begin{pmatrix} 1-u & 2 & 2 \\ -4 & 1 & 1 \\ -u-3 & u-6 & 3 \end{pmatrix}$$

The matrix of the cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1-u & -2 & 2 \\ 4 & 1 & -1 \\ -u-3 & 6-u & 3 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1-u & 4 & -3-u \\ -2 & 1 & 6-u \\ 2 & -1 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{9-u}\begin{pmatrix} 1-u & 4 & -3-u \\ -2 & 1 & 6-u \\ 2 & -1 & 3 \end{pmatrix}$$

b Substituting $u = 4$

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 8.4+21.2-16.1 \\ 5.6+5.3+4.6 \\ -5.6-5.3+6.9 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 13.5 \\ 15.5 \\ -4 \end{pmatrix} = \begin{pmatrix} 2.7 \\ 3.1 \\ -0.8 \end{pmatrix}$$

Equating elements

$a = 2.7, b = 3.1, c = -0.8$

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Further matrix algebra
Exercise H, Question 13

Question:

$$\mathbf{M} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & -1 & 3 \end{pmatrix}$$

a Show that the matrix $\mathbf{M}$ has only two distinct eigenvalues.

b Find an eigenvector corresponding to each of these eigenvalues. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 0 & 0\\ 1 & 1-\lambda & 1\\ 4 & -1 & 3-\lambda\end{pmatrix}$

$$=\begin{vmatrix}3-\lambda & 0 & 0\\ 1 & 1-\lambda & 1\\ 4 & -1 & 3-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}1-\lambda & 1\\ -1 & 3-\lambda\end{vmatrix}-0\begin{vmatrix}1 & 1\\ 4 & 3-\lambda\end{vmatrix}+0\begin{vmatrix}1 & 1-\lambda\\ 4 & -1\end{vmatrix}$$

$$=(3-\lambda)\left((1-\lambda)(3-\lambda)+1\right)=(3-\lambda)\left(\lambda^2-4\lambda+4\right)$$

$$=(3-\lambda)(\lambda-2)^2$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(\lambda-2)^2=0\Rightarrow\lambda=3,2$ repeated.

There are only two distinct eigenvalues of **A**, 2 and 3.

b For $\lambda=2$

$$\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x\\ x+y+z\\ 4x-y+3z\end{pmatrix}=\begin{pmatrix}2x\\ 2y\\ 2z\end{pmatrix}$$

Equating the top elements

$3x=2x\Rightarrow x=0$

Equating the middle elements and substituting $x=0$

$0+y+z=2y\Rightarrow y=z$

Let $z=1$, then $y=1$

An eigenvalue corresponding to the eigenvalue 2 is $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$.

For $\lambda=3$

$$\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x\\ x+y+z\\ 4x-y+3z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the lower elements

$4x-y+3z=3z\Rightarrow y=4x$

Let $x=1$, then $y=4$

Equating the middle elements and substituting $x=1$ and $y=4$

$1+4+z=12\Rightarrow z=7$

An eigenvalue corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ 4\\ 7\end{pmatrix}$.

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Further matrix algebra
Exercise H, Question 14

Question:

The matrix $\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$.

a Show that the matrix **P** is orthogonal.

The transformation $P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix **P**.
The plane Π_1 is transformed by A to the plane Π_2. The plane Π_2 has Cartesian equation $x + y - \sqrt{2}z = 0$.

b Find a Cartesian equation of the plane Π_1.

Solution:

a $$\mathbf{PP}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}+\frac{1}{4}+\frac{1}{2} & \frac{1}{4}+\frac{1}{4}-\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 \\ \frac{1}{4}+\frac{1}{4}-\frac{1}{2} & \frac{1}{4}+\frac{1}{4}+\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 \\ \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 & \frac{1}{2}+\frac{1}{2}+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Hence **P** is orthogonal.

b As **P** is orthogonal, $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$

$x + y - \sqrt{2}z = 0$

Let $x = s$ and $y = t$, then $z = \frac{1}{\sqrt{2}}(s+t)$

A parametric form of the general point on Π_2 is $\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix}$

A parametric form for the general point of Π_1 is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix} = \mathbf{P}^{\mathrm{T}}\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} s \\ t \\ \frac{1}{\sqrt{2}}(s+t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}s+\frac{1}{2}t+\frac{1}{2}(s+t) \\ -\frac{1}{2}s-\frac{1}{2}t+\frac{1}{2}(s+t) \\ \frac{1}{\sqrt{2}}s-\frac{1}{\sqrt{2}}t+0 \end{pmatrix} = \begin{pmatrix} s+t \\ 0 \\ \frac{1}{\sqrt{2}}(s-t) \end{pmatrix}$$

Equating elements

$x = s+t, y = 0, z = \frac{1}{\sqrt{2}}(s-t)$

x and z can take any values

A Cartesian equation of Π_1 is $y = 0$.

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Further matrix algebra
Exercise H, Question 15

Question:

a Determine the eigenvalues of the matrix $\mathbf{A}=\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}$

b Show that $\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{A}$.

$\mathbf{B}=\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}$

c Show that $\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{B}$ and write down the corresponding eigenvalue.

d Hence, or otherwise, write down an eigenvector of the matrix $\mathbf{AB}$, and state the corresponding eigenvalue. **[E]**

Solution:

a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}2-\lambda & -8\\ 0 & -2-\lambda\end{vmatrix}-(-3)\begin{vmatrix}0 & -8\\ 0 & -2-\lambda\end{vmatrix}+6\begin{vmatrix}0 & 2-\lambda\\ 0 & 0\end{vmatrix}$$

$$=(3-\lambda)(2-\lambda)(-2-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(2-\lambda)(-2-\lambda)=0\Rightarrow\lambda=-2,2,3$$

The eigenvalues are $-2, 2$ and 3.

b $\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=\begin{pmatrix}9-3\\2\\0\end{pmatrix}=\begin{pmatrix}6\\2\\0\end{pmatrix}=2\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

c $\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=\begin{pmatrix}21-6\\3+2\\3-3\end{pmatrix}=\begin{pmatrix}15\\5\\0\end{pmatrix}=5\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue 5.

d $\mathbf{AB}\begin{pmatrix}3\\1\\0\end{pmatrix}=\mathbf{A}.\left[\mathbf{B}\begin{pmatrix}3\\1\\0\end{pmatrix}\right]=\mathbf{A}.5\begin{pmatrix}3\\1\\0\end{pmatrix}=5\mathbf{A}\begin{pmatrix}3\\1\\0\end{pmatrix}=5\times2\begin{pmatrix}3\\1\\0\end{pmatrix}=10\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{AB}$ corresponding to the eigenvalue 10.

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Further matrix algebra
Exercise H, Question 16

Question:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 4 & 2 & 7 \end{pmatrix}$$

a Showing your working, find A^{-1}.

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix A.

b Find Cartesian equations of the line which is mapped by T onto the line $x = \frac{y}{4} = \frac{z}{3}$.

[E]

Solution:

a $\det(\mathrm{A}) = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 4 & 2 & 7 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}$

$= 7 - 2 + 6 - 4 = 7$

The matrix of the minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 5 & 17 & 2 \\ -2 & 3 & 2 \\ -1 & -2 & 1 \end{pmatrix}$$

The matrix of the cofactors is given by

$$\mathrm{C} = \begin{pmatrix} 5 & -17 & 2 \\ 2 & 3 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\mathrm{C}^{\mathrm{T}} = \begin{pmatrix} 5 & 2 & -1 \\ -17 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\mathrm{A}^{-1} = \frac{1}{\det(\mathrm{A})}\mathrm{C}^{\mathrm{T}} = \frac{1}{7}\begin{pmatrix} 5 & 2 & -1 \\ -17 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

b Let $x = \frac{y}{4} = \frac{z}{3} = t$, then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 4t \\ 3t \end{pmatrix}$

Equations of the original line are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathrm{A}^{-1}\begin{pmatrix} t \\ 4t \\ 3t \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5 & 2 & -1 \\ -17 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} t \\ 4t \\ 3t \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 5t + 8t - 3t \\ -17t + 12t + 6t \\ 2t - 8t + 3t \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 10t \\ t \\ -3t \end{pmatrix}$$

Equating elements

$$x = \frac{10t}{7}, y = \frac{t}{7}, z = -\frac{3t}{7}$$

Hence

$$\frac{x}{10} = \frac{y}{1} = \frac{z}{-3} = \frac{t}{7}$$

Cartesian equations of the line are

$$\frac{x}{10} = \frac{y}{1} = \frac{z}{-3}$$

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Review Exercise 1
Exercise A, Question 1

Question:

Find the value of x for which

$2\tanh x - 1 = 0$,

giving your answer in terms of a natural logarithm. **[E]**

Solution:

$$2\tanh x - 1 = 0$$

$$2\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1$$

$$2e^x - 2e^{-x} = e^x + e^{-x}$$

$$e^x = 3e^{-x}$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

You multiply both sides of this equation by e^x.

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

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Review Exercise 1
Exercise A, Question 2

Question:

Starting from the definition of cosh x in terms of exponentials, find, in terms of natural logarithms, the values of x for which $5 = 3\cosh x$. **[E]**

Solution:

$$5 = 3\cosh x$$

$$5 = 3\left(\frac{e^x + e^{-x}}{2}\right)$$

The wording of the question requires you to use the definition $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$10 = 3e^x + 3e^{-x}$$

$$3e^x - 10 + 3e^{-x} = 0$$

You multiply this equation throughout by e^x.

$$3e^{2x} - 10e^x + 3 = 0$$

$$(3e^x - 1)(e^x - 3) = 0$$

You may find it helpful to substitute $y = e^x$ and then, factorise, $3y^2 - 10y + 3 = (3y-1)(y-3) = 0$. This gives $y = \frac{1}{3}$ and $y = 3$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 3$. With practice, the substitution can be omitted.

$$e^x = \frac{1}{3}, e^x = 3$$

$$x = \ln\left(\frac{1}{3}\right), \ln 3$$

The answer $x = -\ln 3$ would also be acceptable.

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Review Exercise 1
Exercise A, Question 3

Question:

The curves with equations $y = 5\sinh x$ and $y = 4\cosh x$ meet at the point $A(\ln p, q)$.
Find the exact values of p and q. **[E]**

Solution:

The curves intersect when

$$5\sinh x = 4\cosh x$$

$$5\left(\frac{e^x - e^{-x}}{2}\right) = 4\left(\frac{e^x + e^{-x}}{2}\right)$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$5e^x - 5e^{-x} = 4e^x + 4e^{-x}$$

$$e^x = 9e^{-x}$$

$$e^{2x} = 9$$

$$2x = \ln 9$$

$$x = \frac{1}{2}\ln 9 = \ln\sqrt{9} = \ln 3$$

Using the law of logarithms $n\ln a = \ln a^n$ with $n = \frac{1}{2}$ and $a = 9$.

$$y = 5\sinh(\ln 3) = 5\left(\frac{e^{\ln 3} - e^{-\ln 3}}{2}\right) = \frac{5}{2}\times\left(3 - \frac{1}{3}\right)$$

$e^{\ln 3} = 3$ and $e^{-\ln 3} = e^{\ln 1 - \ln 3} = e^{\ln\frac{1}{3}} = \frac{1}{3}$, using $\ln 1 = 0$ and the law of logarithms $\ln a - \ln b = \ln\frac{a}{b}$.

$$= \frac{5}{2}\times\frac{8}{3} = \frac{20}{3}$$

$$p = 3, q = \frac{20}{3}$$

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Review Exercise 1
Exercise A, Question 4

Question:

Find the values of x for which
$5\cosh x - 2\sinh x = 11$,
giving your answers as natural logarithms. **[E]**

Solution:

$$5\cosh x - 2\sinh x = 11$$
$$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$$
$$5e^x + 5e^{-x} - 2e^x + 2e^{-x} = 22$$
$$3e^x - 22 + 7e^{-x} = 0$$
$$3e^{2x} - 22e^x + 7 = 0$$
$$(3e^x - 1)(e^x - 7) = 0$$
$$e^x = \frac{1}{3}, 7$$
$$x = \ln\frac{1}{3}, \ln 7$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

You multiply this equation throughout by e^x.

You may find it helpful to substitute $y = e^x$ and then, factorising $3y^2 - 22y + 7 = (3y-1)(y-7) = 0$. This gives $y = \frac{1}{3}$ and $y = 7$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 7$. With practice, the substitution can be omitted.

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Review Exercise 1
Exercise A, Question 5

Question:

By expressing $\sinh 2x$ and $\cosh 2x$ in terms of exponentials, find the exact values of x for which

$6\sinh 2x + 9\cosh 2x = 7$,

giving each answer in the form $\frac{1}{2}\ln p$, where p is a rational number. **[E]**

Solution:

$$6\sinh 2x + 9\cosh 2x = 7$$

$$6\left(\frac{e^{2x}-e^{-2x}}{2}\right)+9\left(\frac{e^{2x}+e^{-2x}}{2}\right) = 7$$

You use the definitions $\sinh x = \frac{e^{x}-e^{-x}}{2}$ and $\cosh x = \frac{e^{x}+e^{-x}}{2}$ replacing x by $2x$.

$$6e^{2x} - 6e^{-2x} + 9e^{2x} + 9e^{-2x} = 14$$

$$15e^{2x} - 14 + 3e^{-2x} = 0$$

You multiply this equation throughout by e^{2x}.

$$15e^{4x} - 14e^{2x} + 3 = 0$$

$$(3e^{2x}-1)(5e^{2x}-3) = 0$$

$$e^{2x} = \frac{1}{3}, \frac{3}{5}$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

$$2x = \ln\frac{1}{3}, \ln\frac{3}{5}$$

$$x = \frac{1}{2}\ln\frac{1}{3}, \frac{1}{2}\ln\frac{3}{5}$$

$$p = \frac{1}{3}, \frac{3}{5}$$

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Review Exercise 1
Exercise A, Question 6

Question:

Given that
$\sinh x + 2\cosh x = k$,
where k is a positive constant,

a find the set of values of k for which at least one real solution of this equation exists,
b solve the equation when $k = 2$. **[E]**

Solution:

a $\sinh x + 2\cosh x = k$

$$\frac{e^x - e^{-x}}{2} + 2\left(\frac{e^x + e^{-x}}{2}\right) = k$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$e^x - e^{-x} + 2e^x + 2e^{-x} = 2k$$
$$3e^x - 2k + e^{-x} = 0$$
$$3e^{2x} - 2ke^x + 1 = 0$$

Let $y = e^x$

$3y^2 - 2ky + 1 = 0$

$$y = \frac{2k \pm \sqrt{(4k^2 - 12)}}{6}$$

Using the quadratic formula $y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

$$= \frac{k \pm \sqrt{(k^2 - 3)}}{3} \quad *$$

For real y

$k^2 - 3 \geq 0 \Rightarrow k \geq \sqrt{3}, k \leq -\sqrt{3}$

If $x \leq -\sqrt{3}$, then both $\frac{k + \sqrt{(k^2 - 3)}}{3}$ and $\frac{k - \sqrt{(k^2 - 3)}}{3}$ are negative.

As $y = e^x > 0$ for all real x, $k \leq -\sqrt{3}$ is rejected.

$k \geq \sqrt{3}$.

b Using * above with $k = 2$

You could solve the equation in part **b** without using part **a** but it is efficient to use the work you have already done.

$$y = e^x = \frac{2 \pm \sqrt{(4-3)}}{3} = \frac{2 \pm 1}{3}$$

$$e^x = 1, \frac{1}{3} \Rightarrow x = \ln 1, \ln\frac{1}{3} = 0, -\ln 3$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 7

Question:

Using the definitions of cosh x and sinh x in terms of exponentials,

a prove that $\cosh^2 x - \sinh^2 x = 1$,

b solve the equation $\operatorname{cosech} x - 2\coth x = 2$, giving your answer in the form $k \ln a$, where k and a are integers. **[E]**

Solution:

a

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x} - \left(e^{2x} - 2 + e^{-2x}\right)}{4}$$

$$= \frac{4}{4} = 1, \text{ as required.}$$

$$\left(e^x + e^{-x}\right)^2 = \left(e^x\right)^2 + 2e^x \cdot e^{-x} + \left(e^{-x}\right)^2$$
$$= e^{2x} + 2 + e^{-2x}$$

b

$$\operatorname{cosech} x - 2\coth x = 2$$

$$\frac{1}{\sinh x} - \frac{2\cosh x}{\sinh x} = 2$$

You use $\operatorname{cosech} x = \frac{1}{\sinh x}$ and $\coth x = \frac{\cosh x}{\sinh x}$.

$\times \sinh x$

$$1 - 2\cosh x = 2\sinh x$$

$$2\sinh x + 2\cosh x = 1$$

$$2\left(\frac{e^x - e^{-x}}{2}\right) + 2\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

You use the definitions sinh x and cosh x in terms of exponentials to obtain an equation in exponentials which you solve using logarithms.

$$e^x - e^{-x} + e^x + e^{-x} = 1$$

$$2e^x = 1 \Rightarrow e^x = \frac{1}{2}$$

$$x = \ln\frac{1}{2} = -\ln 2$$

$\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, as $\ln 1 = 0$.

$$k = -1, a = 2$$

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Review Exercise 1
Exercise A, Question 8

Question:

a From the definition of $\cosh x$ in terms of exponentials, show that $\cosh 2x = 2\cosh^2 x - 1$.

b Solve the equation $\cosh 2x - 5\cosh x = 2$, giving the answers in terms of natural logarithms. **[E]**

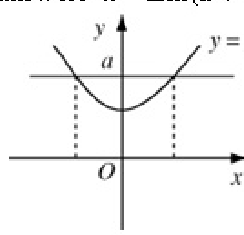
Solution:

a
$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$
$$= 2 \times \frac{e^{2x} + 2 + e^{-2x}}{4} - 1$$
$$= \frac{2e^{2x}}{4} + \frac{4}{4} + \frac{2e^{-2x}}{4} - 1$$
$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x, \text{ as required}$$

$$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$$
$$= e^{2x} + 2 + e^{-2x}$$

b Using the result in part **a**
$$\cosh 2x - 5\cosh x = 2$$
$$2\cosh^2 x - 1 - 5\cosh x = 2$$
$$2\cosh^2 x - 5\cosh x - 3 = 0$$
$$(2\cosh x + 1)(\cosh x - 3) = 0$$
$$\cosh x = -\frac{1}{2}, \cosh x = 3$$
$\cosh x = -\frac{1}{2}$ is impossible
$$x = \pm \operatorname{arcosh} x = \pm \ln\left(3 + \sqrt{8}\right)$$

If $\operatorname{arcosh} x > 0$ then $\operatorname{arcosh} x = \ln(x + \sqrt{(x^2 - 1)})$.
However if $\cosh x = a$, where $a > 0$, then there are two answers $x = \pm\ln(a + \sqrt{(a^2 - 1)})$



These answers can also be written as $x = \ln\left(3 \pm \sqrt{8}\right)$.

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Review Exercise 1
Exercise A, Question 9

Question:

a Using the definition of $\cosh x$ in terms of exponentials, prove that
$4\cosh^3 x - 3\cosh x = \cosh 3x$.

b Hence, or otherwise, solve the equation $\cosh 3x = 5\cosh x$, giving your answer as natural logarithms. **[E]**

Solution:

a

$$4\cosh^3 x - 3\cosh x = 4\left(\frac{e^x + e^{-x}}{2}\right)^3 - 3\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{3x} + 3e^x + 3e^{-x} + e^{-3x}}{2} - \frac{3e^x + 3e^{-x}}{2}$$

$$= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x, \text{ as required.}$$

Using the binomial expansion
$(e^x + e^{-x})^3$
$= (e^x)^3 + 3(e^x)^2 \cdot e^{-x}$
$+ 3e^x(e^{-x})^2 + (e^{-x})^3$
$= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}$.

b $\cosh 3x = 5\cosh x$

Using the result in part **a**

$$4\cosh^3 x - 3\cosh x = 5\cosh x$$

$$4\cosh^3 x - 8\cosh x = 0$$

$$4\cosh x(\cosh^2 x - 2) = 0$$

As for all x, $\cosh x \geq 1$,

$$\cosh x = \sqrt{2}$$

There are 3 possible answers to this cubic, $\cosh x = 0$, $\cosh x = -\sqrt{2}$ and $\cosh x = \sqrt{2}$. As for all real x, $\cosh x \geq 1$ only the last of the three gives real values of x.

$$x = \pm\ln(\sqrt{2} + 1)$$

Using $\operatorname{arcosh} x = \ln\left(x + \sqrt{(x^2 - 1)}\right)$ with

$$-\ln(\sqrt{2}+1) = \ln\left(\frac{1}{\sqrt{2}+1}\right) = \ln\left(\frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$$= \ln\left(\frac{\sqrt{2}-1}{1}\right) = \ln(\sqrt{2}-1)$$

The solutions of $\cosh 3x = 5\cosh x$, as natural logarithms, are $x = \ln(\sqrt{2} \pm 1)$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 10

Question:

a Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that $\cosh(A-B)=\cosh A\cosh B-\sinh A\sinh B$.

b Hence, or otherwise, given that $\cosh(x-1)=\sinh x$, show that

$$\tanh x=\frac{e^2+1}{e^2+2e-1}.$$ **[E]**

Solution:

a $\cosh A\cosh B-\sinh A\sinh B$

$$=\left(\frac{e^A+e^{-A}}{2}\right)\left(\frac{e^B+e^{-B}}{2}\right)-\left(\frac{e^A-e^{-A}}{2}\right)\left(\frac{e^B-e^{-B}}{2}\right)$$

When multiplying out the brackets you must be careful to obtain all eight terms with the correct signs.

$$=\frac{1}{4}\left(e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}-e^{A+B}+e^{-A+B}+e^{A-B}-e^{-A-B}\right)$$

$$=\frac{1}{4}\left(2e^{-A+B}+2e^{A-B}\right)=\frac{e^{A-B}+e^{-(A-B)}}{2}$$

You use the definition $\cosh x=\dfrac{e^x+e^{-x}}{2}$ with $x=A-B$.

$=\cosh(A-B)$, as required.

b $\cosh x\cosh 1-\sinh x\sinh 1=\sinh x$

You expand $\cosh(x-1)$ using the result of part **a**.

$$\cosh x\cosh 1=\sinh x(1+\sinh 1)$$

$$\tanh x=\frac{\cosh 1}{1+\sinh 1}$$

Divide both sides of this equation by $\cosh x(1+\sinh 1)$ and use $\tanh x=\dfrac{\sinh x}{\cosh x}$.

$$\tanh x=\frac{\frac{e+e^{-1}}{2}}{1+\frac{e-e^{-1}}{2}}=\frac{e+e^{-1}}{2+e-e^{-1}}=\frac{e^2+1}{e^2+2e-1}\text{, as required.}$$

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Review Exercise 1
Exercise A, Question 11

Question:

a Starting from the definition $\sinh y = \frac{e^y - e^{-y}}{2}$, prove that, for all real values of x,

$\text{arsinh}x = \ln[x + \sqrt{(1+x^2)}]$.

b Hence, or otherwise, prove that, for $0 < \theta < \pi$,

$\text{arsinh}(\cot\theta) = \ln\left(\cot\frac{\theta}{2}\right)$. **[E]**

Solution:

a Let $y = \text{arsinh}x$

then $x = \sinh y = \frac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x + \sqrt{(4x^2+4)}}{2}$

$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

b $\text{arsinh}(\cot\theta) = \ln\left[\cot\theta + \sqrt{(1+\cot^2\theta)}\right]$

Using the result of part **a** with $x = \cot\theta$.

$= \ln(\cot\theta + \text{cosec}\theta)$

Using $\text{cosec}^2\theta = 1 + \cot^2\theta$.

$= \ln\left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right) = \ln\left(\frac{\cos\theta + 1}{\sin\theta}\right)$

$= \ln\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

You use both double angle formulae $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x\cos x$ with $2x = \theta$.

$= \ln\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) = \ln\left(\cot\frac{\theta}{2}\right)$, as required.

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Review Exercise 1
Exercise A, Question 12

Question:

Given that $n \in \mathbb{Z}^+, x \in \mathbb{R}$ and $\mathbf{M} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$, prove that $\mathbf{M}^n = \mathbf{M}$. **[E]**

Solution:

Let $n = 1$

The result $\mathbf{M}^n = \mathbf{M}$ becomes $\mathbf{M}^1 = \mathbf{M}$, which is true.

Assume the result is true for $n = k$.

That is

$$\mathbf{M}^k = \mathbf{M} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M}$$

$$= \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix} \cdot \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^4 x - \cosh^2 x \sinh^2 & x\cosh^4 x - \cosh^2 x \sinh^2 x \\ -\sinh^2 x \cosh^2 x + \sinh^4 x - \sinh^2 x \cosh^2 + \sinh^4 x \end{pmatrix}$$

$$\cosh^4 x - \cosh^2 x \sinh^2 x = \cosh^2 x(\cosh^2 x - \sinh^2 x)$$
$$= \cosh^2 x$$

$$-\sinh^2 x\cosh^2 x + \sinh^4 x = \sinh^2 x(-\cosh^2 x + \sinh^2 x)$$
$$= -\sinh^2 x$$

You can prove this result using mathematical induction, a method of proof you learnt in the FP1 module. The prerequisites in the FP3 specification state that a knowledge of FP1 is assumed and may be tested.

You use the identity $\cosh^2 x - \sinh^2 x = 1$ to simplify the terms in the matrix.

Hence $\mathbf{M}^{k+1} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$

and this is the result for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 13

Question:

Solve for real x and y, the simultaneous equations

$\cosh x = 3 \sinh y$

$2 \sinh x = 5 - 6 \cosh y$,

expressing your answers in terms of natural logarithms. **[E]**

Solution:

$\cosh x = 3\sinh y$

Multiply by 2

$2\cosh x = 6\sinh y$

Squaring both sides

$4\cosh^2 x = 36\sinh^2 y$ ①

$2\sinh x = 5 - 6\cosh y$

Squaring both sides

$4\sinh^2 x = (5 - 6\cosh y)^2$ ②

① − ②

$4\cosh^2 x - 4\sinh^2 x = 36\sinh^2 y - (5 - 6\cosh y)^2$

$4 = 36\sinh^2 y - 25 + 60\cosh y - 36\cosh^2 y$

$4 = 60\cosh y - 25 - 36(\cosh^2 y - \sinh^2 y)$

$4 = 60\cosh y - 25 - 36$

$60\cosh y = 65 \Rightarrow \cosh y = \frac{13}{12}$

$y = \pm\ln\left(\frac{13}{12} + \sqrt{\left(\frac{169}{144} - 1\right)}\right) = \pm\ln\left(\frac{13}{12} + \sqrt{\left(\frac{25}{144}\right)}\right)$

$= \pm\ln\left(\frac{13}{12} + \frac{5}{12}\right) = \pm\ln\frac{3}{2}$

If $y = -\ln\frac{3}{2}$, then

$\sinh\left(-\ln\frac{3}{2}\right) = \frac{e^{-\ln\frac{3}{2}} - e^{\ln\frac{3}{2}}}{2} = \frac{\frac{2}{3} - \frac{3}{2}}{2} = -\frac{5}{12}$

As $\cosh x = 3\sinh y$, this gives

$\cosh x = 3 \times -\frac{5}{12} = -\frac{5}{4}$

As $\cosh x \geq 1$ for all real x, this is impossible and the solution $y = -\ln\frac{3}{2}$ is rejected.

$2\sinh x = 5 - 6\cosh y = 5 - 6 \times \frac{13}{12} = -\frac{3}{2}$

$\sinh x = -\frac{3}{4}$

$x = \ln\left(-\frac{3}{4} + \sqrt{\left(\frac{9}{16} + 1\right)}\right) = \ln\left(-\frac{3}{4} + \sqrt{\left(\frac{25}{16}\right)}\right)$

$= \ln\left(-\frac{3}{4} + \frac{5}{4}\right) = \ln\frac{1}{2} = -\ln 2$

$x = -\ln 2, y = \ln\frac{3}{2}$

> When you square an equation, you may introduce false solutions. In this case equation ① will contain any solutions of $2\cosh x = -6\sinh y$ as well as $2\cosh x = 6\sinh y$, so you will need to check any solutions you obtain.

> The identity $\cosh^2\theta - \sinh^2\theta = 1$ is used twice.

> If $\cosh x = a$, then there are two possible values of x, $x = \pm\ln\left(a + \sqrt{(a^2 - 1)}\right)$. You need to check that both answers are possible.

> If $y = \ln\frac{3}{2}$, then $\sinh y = \frac{5}{12}$ and $\cosh x = \frac{5}{4}$ and this is the correct solution.

> If $\sinh x = a$, then there is just one possible value of x, $x = \ln\left(a + \sqrt{(a^2 + 1)}\right)$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 14

Question:

a Starting from the definition of tanh x in terms of e^x, show that

$$\text{artanh } x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

and sketch the graph of $y = \text{artanh } x$.

b Solve the equation $x = \tanh[\ln\sqrt{(6x)}]$ for $0 < x < 1$. **[E]**

Solution:

a Let $y = \text{artanh}\, x$

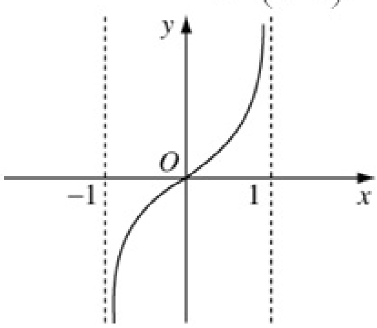
$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$
$$= \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y}$$
$$= \frac{e^{2y} - 1}{e^{2y} + 1}$$

You have been asked to prove a standard result in this question. You should learn the proof of this and other similar results as part of your preparation for the examination.

$$xe^{2y} + x = e^{2y} - 1$$
$$e^{2y}(1 - x) = 1 + x$$
$$e^{2y} = \frac{1+x}{1-x}$$
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$

Make e^{2y} the subject of the formulas and then take logarithms.

$$y = \text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$ as required.



You need to be able to sketch the graphs of the hyperbolic and inverse hyperbolic functions. When you sketch a graph you should show any important features of the curve. In this case, you should show the asymptotes $x = -1$ and $x = 1$ of the curve.

Graph of $y = \text{artanh}\, x$

b $$x = \tanh\left[\ln\sqrt{(6x)}\right]$$

You use the result in part **a**.

$$\ln\sqrt{(6x)} = \text{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\ln\sqrt{(6x)} = \ln\sqrt{\left(\frac{1+x}{1-x}\right)}$$
$$\sqrt{(6x)} = \sqrt{\left(\frac{1+x}{1-x}\right)}$$

As you have squared this equation, you might have introduced an incorrect solution. It would be sensible to check on your calculator that $x = \frac{1}{2}, \frac{1}{3}$ are solutions of $x = \tanh\left[\ln\sqrt{(6x)}\right]$. In this case, both are correct.

Squaring

$$6x = \frac{1+x}{1-x}$$
$$6x - 6x^2 = 1 + x$$
$$6x^2 - 5x + 1 = (3x-1)(2x-1) = 0$$
$$x = \frac{1}{2}, \frac{1}{3}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 15

Question:

a Show that, for $0 < x \le 1$,

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right) = -\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right).$$

b Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, show that, for $0 < x \le 1$,

$$\text{arsech}\, x = \ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right).$$

c Solve the equation

$$3\tanh^2 x - 4\,\text{sech}\, x + 1 = 0,$$

giving exact answers in terms of natural logarithms. **[E]**

Solution:

a $\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)+\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$

$=\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$

There are a number of different ways of starting this question. The method used here begins by using the log rule $\log a+\log b=\log ab$.

$=\ln\frac{1-(1-x^2)}{x^2}=\ln\frac{x^2}{x^2}=\ln 1=0$

This is the difference of two squares $a^2-b^2=(a-b)(a+b)$ with $a=1$ and $b=\sqrt{(1-x^2)}$.

Hence

$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)=-\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$, as required.

b Let $y=\text{arsech}\, x$

$\text{sech}\, y = x$

$\frac{2}{e^y+e^{-y}}=x$

$2 = xe^y+xe^{-y}$

Multiply throughout by e^y and treat the result as a quadratic in e^y.

$xe^{2y}-2e^y+x=0$

$e^y=\frac{2\pm\sqrt{(4-4x^2)}}{2x}=\frac{1\pm\sqrt{(1-x^2)}}{x}$

$y=\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right), \ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)$

$=\pm\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$, using the result of **a**

Either of the two answers is possible but it is conventional to take $0<\text{arsech}\, x\le 1$.

$y=\text{arsech}\, x=\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$, as required

c Using $\text{sech}^2 x=1-\tanh^2 x$

$3\tanh^2 x-4\text{sech}\, x+1 = 0,$

$3-3\text{sech}^2 x-4\text{sech}\, x+1=0$

$3\text{sech}^2 x+4\text{sech}\, x-4 = 0$

$(3\text{sech}x-2)(\text{sech}\, x+2) = 0$

$\text{sech}\, x=-2$ is impossible with real values of x.

$\text{sech}\, x = \frac{2}{3}$

$x = \pm\ln\left(\frac{1+\sqrt{\left(1-\frac{4}{9}\right)}}{\frac{2}{3}}\right)=\pm\ln\left(\frac{3+\sqrt{5}}{2}\right)$

$x=\ln\left(\frac{3\pm\sqrt{5}}{2}\right)$ is another correct form of the answer.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 16

Question:

a Express $\cosh 3\theta$ and $\cosh 5\theta$ in terms of $\cosh\theta$.
b Hence determine the real roots of the equation
$2\cosh 5x + 10\cosh 3x + 20\cosh x = 243$,
giving your answers to 2 decimal places. **[E]**

Solution:

a $\cosh 3\theta = \cosh(2\theta + \theta)$
$= \cosh 2\theta\cosh\theta + \sinh 2\theta\sinh\theta$
$\cosh\theta = c$ and $\sinh\theta = s$
$\cosh 3\theta = (2c^2 - 1)c + 2sc \times s$
$= 2c^3 - c + 2s^2c$
$= 2c^3 - c + 2(c^2 - 1)c$
$= 2c^3 - c + 2c^3 - 2c$
$= 4\cosh^3\theta - 3\cosh\theta$
$\cosh 5\theta = \cosh(3\theta + 2\theta) = \cosh 3\theta\cosh 2\theta + \sinh 3\theta\sinh 2\theta$
$\cosh 3\theta\cosh 2\theta = (4c^3 - 3c)(2c^2 - 1)$
$= 8c^5 - 10c^3 + 3c$
$\sinh 3\theta\sinh 2\theta = \sinh(2\theta + \theta)\sinh 2\theta$
$= (\sinh 2\theta\cosh\theta + \cosh 2\theta\sinh\theta)\sinh 2\theta$
$= (2sc \times c + (2c^2 - 1)s)2sc$
$= 2(4c^2 - 1)s^2c$
$= 2(4c^2 - 1)(c^2 - 1)c$
$= 8c^5 - 10c^3 + 2c$
Combining the results
$\cosh 5\theta = 8c^5 - 10c^3 + 3c + 8c^5 - 10c^3 + 2c$
$= 16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta$

> In a complicated calculation like this, it is sensible to use the abbreviated notation suggested here but, if you intend to use a notation like this, you should state the notation in the solution so that the marker knows what you are doing.

> You use the 'double angle' for hyperbolics $\cosh 2\theta = 2\cosh^2\theta - 1$ and $\sinh 2\theta = 2\sinh\theta\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$. The signs in these formulae can be worked out using **Osborn's rule**.

b $2\cosh 5x + 10\cosh 3x + 20\cosh x = 243$,
Letting $\cosh x = c$ and using the results in **a**
$32c^5 - 40c^3 + 10c + 40c^3 - 30c + 20c = 243$
$$c^5 = \frac{243}{32} \Rightarrow c = \frac{3}{2}$$
$$x = \pm\text{arcosh}\,\frac{3}{2} \approx \pm 0.96$$

> You can use an inverse hyperbolic button on your calculator to find arcosh $\frac{3}{2}$.

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Review Exercise 1
Exercise A, Question 17

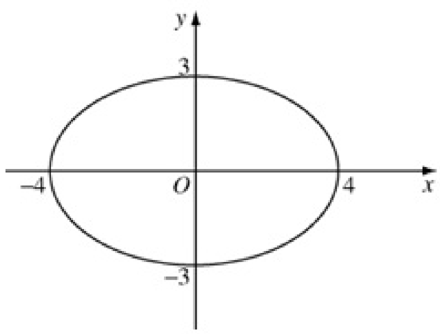
Question:

An ellipse has equation $\frac{x^2}{16}+\frac{y^2}{9}=1$.

a Sketch the ellipse.
b Find the value of the eccentricity e.
c State the coordinates of the foci of the ellipse. **[E]**

Solution:

a



When you draw a sketch, you should show the important features of the curve. When drawing an ellipse, you should show that it is a simple closed curve and indicate the coordinates of the points where the curve intersects the axes.

b $b^2=a^2\left(1-e^2\right)$

$9=16\left(1-e^2\right)=16-16e^2$

$e^2=\frac{16-9}{16}=\frac{7}{16}$

$e=\frac{\sqrt{7}}{4}$

c The coordinates of the foci are given by

$(\pm ae,0)=\left(\pm 4\times\frac{\sqrt{7}}{4},0\right)=(\pm\sqrt{7},0)$

The formula you need for calculating the eccentricity and the coordinates of the foci are given in the Edexcel formula booklet you are allowed to use in the examination. You should be familiar with the formulae in that booklet. You should quote any formulae you use in your solution.

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Review Exercise 1
Exercise A, Question 18

Question:

The hyperbola H has equation $\frac{x^2}{16} - \frac{y^2}{4} = 1$. Find

a the value of the eccentricity of H,
b the distance between the foci of H.

The ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

c Sketch H and E on the same diagram, showing the coordinates of the points where each curve crosses the axes. **[E]**

Solution:

a $b^2 = a^2(e^2 - 1)$

$4 = 16(e^2 - 1) = 16e^2 - 16$

$e^2 = \frac{16+4}{16} = \frac{20}{16} = \frac{5}{4}$

$e = \frac{\sqrt{5}}{2}$

The formula for calculating the eccentricity is $b^2 = a^2(e^2 - 1)$. It is important not to confuse this with the formula for calculating the eccentricity of an ellipse $b^2 = a^2(1 - e^2)$.

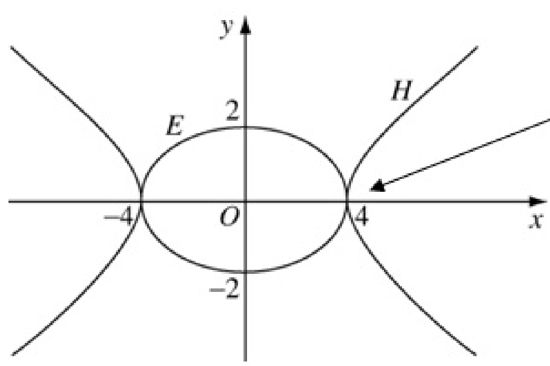
b The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 4 \times \frac{\sqrt{5}}{2}, 0\right) = (\pm 2\sqrt{5}, 0)$

The distance between the foci is $4\sqrt{5}$.

The formulae for the foci of an ellipse and a hyperbola are the same $(\pm ae, 0)$.

c



In this sketch, you should show where the curves cross the axes. Label which curve is H and which is E. These two curves touch each other on the x-axis.

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Review Exercise 1
Exercise A, Question 19

Question:

The ellipse D has equation $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the ellipse E has equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

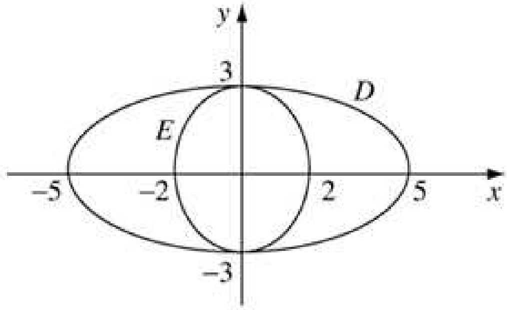
a Sketch D and E on the same diagram, showing the coordinates of the points where each curve crosses the axes.

The point S is a focus of D and the point T is a focus of E.

b Find the length of ST. **[E]**

Solution:

a



b For D

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2) = 25 - 25e^2$

$e^2 = \frac{25-9}{25} = \frac{16}{25}$

$e = \frac{4}{5}$

As the coordinates of a focus of D are $(ae, 0)$, you first need to find the eccentricity of the ellipse using $b^2 = a^2(1-e^2)$ with $a = 5$ and $b = 3$.

For S $(ae, 0) = \left(5 \times \frac{4}{5}, 0\right) = (4, 0)$

As an ellipse has two foci, you could choose either for S and there are also two possible choices for T. The symmetries of the diagram show that you would always get the same distance for ST whichever you choose. It does not matter which you choose but it is sensible to choose the positive coordinate.

For E

$b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) = 9 - 9e^2$

$e^2 = \frac{9-4}{9} = \frac{5}{9}$

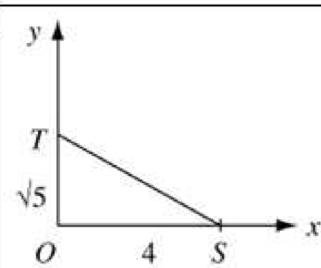
$e = \frac{\sqrt{5}}{3}$

The major axis for ellipse E is along the y-axis, so its foci have coordinates $(0, \pm ae)$. You find the eccentricity of E using $b^2 = a^2(1-e^2)$ with $a = 3$ and $b = 2$, a is always the semi-major axis and b the semi-minor axis, so $a > b$.

For T $(0, ae) = \left(0, 3 \times \frac{\sqrt{5}}{3}\right) = (0, \sqrt{5})$

$ST^2 = 4^2 + (\sqrt{5})^2 = 21$

$ST = \sqrt{21}$



As the focus of D is on the x-axis and the focus of E is on the y-axis, you find the distance between them using Pythagoras' Theorem.

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Review Exercise 1
Exercise A, Question 20

Question:

An ellipse, with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$, has foci S and S'.

a Find the coordinates of the foci of the ellipse.

b Using the focus-directrix property of the ellipse, show that, for any point P on the ellipse,
$SP+S'P=6$. **[E]**

Solution:

a $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) = 9 - 9e^2$

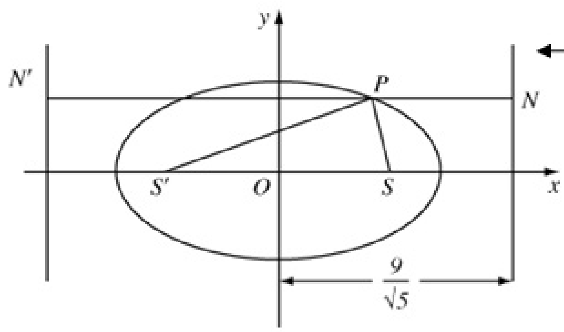
$e^2 = \frac{9-4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

As the coordinates of the foci of an ellipse are $(\pm ae, 0)$, you first need to find the eccentricity of the ellipse using $b^2 = a^2(1-e^2)$ with $a = 3$ and $b = 2$.

The coordinates of the foci are given by

$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = (\pm\sqrt{5}, 0)$

b



In this question, you are not asked to draw a diagram but with questions on coordinate geometry it is usually a good idea to sketch a diagram so you can see what is going on.

The equations of the directrices are $x = \pm\frac{a}{e}$.

$x = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$

Let the line through P parallel to the x-axis intersect the directrices at N and N', as shown in the diagram

$N'N = 2 \times \frac{9}{\sqrt{5}} = \frac{18}{\sqrt{5}}$

If you introduce points, like N and N' here, you should define them in your solution and mark them on your diagram. This helps the examiner follow your solution.

The focus directrix property of the ellipse gives that

$SP = ePN$ and $S'P = ePN'$

$SP + S'P = ePN + ePN'$

$= e(PN + PN') = eN'N$

$= \frac{\sqrt{5}}{3} \times \frac{18}{\sqrt{5}} = 6$, as required.

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Review Exercise 1
Exercise A, Question 21

Question:

a Find the eccentricity of the ellipse with equation $3x^2 + 4y^2 = 12$.

b Find an equation of the tangent to the ellipse with equation $3x^2 + 4y^2 = 12$ at the point with coordinates $\left(1, \frac{3}{2}\right)$.

This tangent meets the y-axis at G. Given that S and S' are the foci of the ellipse,

c find the area of $\Delta SS'G$. **[E]**

Solution:

a $3x^2+4y^2=12$

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

$$b^2=a^2\left(1-e^2\right)$$

$$3=4\left(1-e^2\right)=4-4e^2$$

$$e^2=\frac{4-3}{4}=\frac{1}{4}\Rightarrow e=\frac{1}{2}$$

You divide this equation by 12. Comparing the result with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a^2=4$ and $b^2=3$ and you use $b^2=a^2\left(1-e^2\right)$ to calculate e.

b $3x^2+4y^2=12$

Differentiate implicitly with respect to x

$$6x+8y\frac{dy}{dx}=0$$

Differentiating implicitly using the chain rule, $\frac{d}{dx}\left(4y^2\right)=\frac{dy}{dx}\frac{d}{dy}\left(4y^2\right)=\frac{dy}{dx}\times 8y$.

$$\frac{dy}{dx}=-\frac{6x}{8y}=-\frac{3x}{4y}$$

At $\left(1,\frac{3}{2}\right)$

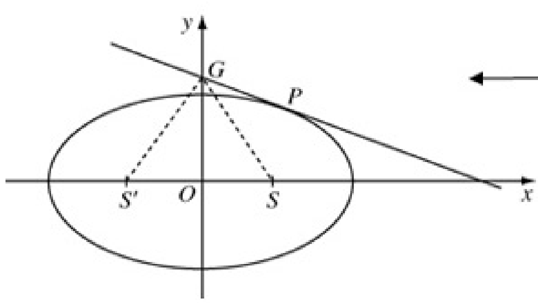
$$\frac{dy}{dx}=\frac{-3\times 1}{4\times\frac{3}{2}}=-\frac{1}{2}$$

Using $y-y_1=m\left(x-x_1\right)$, an equation of the tangent is

$$y-\frac{3}{2}=-\frac{1}{2}(x-1)=-\frac{1}{2}x+\frac{1}{2}$$

$$y=-\frac{1}{2}x+2$$

c



Sketching a diagram makes it clear that the area of the triangle is to be found using the standard expression $\frac{1}{2}\text{base}\times\text{height}$ with the base $S'S$ and the height OG.

The coordinates of S are

$$(ae,0)=\left(2\times\frac{1}{2},0\right)=(1,0)$$

By symmetry, the coordinates of S' are $(-1,0)$. The y-coordinate of G is given by

$$y=0+2=2$$

You find the y-coordinate of G by substituting $x=0$ into the answer to part a.

$$\Delta SS'G=\frac{1}{2}\text{base}\times\text{height}$$

$$=\frac{1}{2}S'S\times OG$$

$$=\frac{1}{2}2\times 2=2$$

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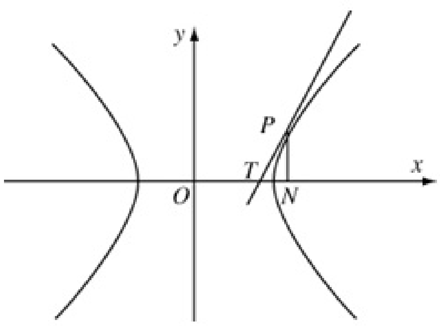
Review Exercise 1
Exercise A, Question 22

Question:

The point P lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and N is the foot of the perpendicular from P onto the x-axis. The tangent to the hyperbola at P meets the x-axis at T.
Show that $OT \cdot ON = a^2$, where O is the origin. **[E]**

Solution:



To find the coordinates of T, it is easiest to carry out your calculation in terms of a parameter. As the question specifies no particular parametric form, you can choose your own. The hyperbolic form has been used here but (a sec t, b tan t) would work as well and there are other possible alternatives.

Let the point P have coordinates (a cosh t, b sinh t)

To find an equation of the tangent PT,

$$\frac{dx}{dt} = a\sinh t, \frac{dy}{dt} = b\cosh t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{b\cosh t}{a\sinh t}$$

Using $y - y_1 = m(x - x_1)$

$$y - b\sinh t = \frac{b\cosh t}{a\sinh t}(x - a\cosh t)$$

To find the x-coordinate of T, you substitute $y = 0$ into a equation of the tangent at P, so first you must obtain an equation for the tangent.

$$ay\sinh t - ab\sinh^2 t = bx\cosh t - ab\cosh^2 t$$

$$ay\sinh t = bx\cosh t - ab(\cosh^2 t - \sinh^2 t)$$

$$= bx\cosh t - ab$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$.

For T, $y = 0$

$$bx\cosh t = ab \Rightarrow x = \frac{a}{\cosh t}$$

The coordinates of N are (a cosh t, 0)

$$OT \cdot ON = \frac{a}{\cosh t} \times a\cosh t = a^2, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 23

Question:

The hyperbola C has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

a Show that an equation of the normal to C at the point P $(a\sec t, b\tan t)$ is
$ax\sin t + by = (a^2 + b^2)\tan t$.

The normal to C at P cuts the x-axis at the point A and S is a focus of C. Given that the eccentricity of C is $\frac{3}{2}$, and that $OA = 3OS$, where O is the origin,

b determine the possible values of t, for $0 \le t \le 2\pi$. **[E]**

Solution:

a $\frac{dx}{dt} = a\sec t\tan t, \frac{dy}{dt} = b\sec^2 t$

$\frac{dy}{dx} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t} = \frac{b}{a\sin t}$

Using $mm' = -1$, the gradient of the normal is given by $m' = -\frac{a\sin t}{b}$

An equation of the normal is

$$y - y_1 = m'(x - x_1)$$

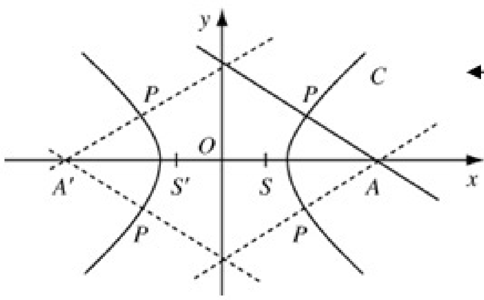
$$y - b\tan t = -\frac{a\sin t}{b}(x - a\sec t)$$

$$by - b^2\tan t = -ax\sin t + a^2\tan t$$

$ax\sin t + by = (a^2 + b^2)\tan t$, as required

To find the gradient of the tangent, you use a version of the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

b



A diagram is essential here. Without it, you would be unlikely to see that there are four possible points where $OA = 3OS$. There are two to the right of the y-axis, corresponding to the focus S with coordinates $(ae, 0)$, and two to the left of the y-axis, corresponding to the focus, here marked S', with coordinates $(-ae, 0)$.

The x-coordinate of A is given by

$ax\sin t + 0 = (a^2 + b^2)\tan t$

$$x = \frac{a^2 + b^2}{a} \times \frac{\tan t}{\sin t} = \frac{a^2 + b^2}{a\cos t}$$

Hence $OA = \frac{a^2 + b^2}{a\cos t}$

Using $b^2 = a^2(e^2 - 1)$ with $e = \frac{3}{2}$

$b^2 = a^2\left(\frac{9}{4} - 1\right) = \frac{5a^2}{4}$

and $OA = \frac{a^2 + b^2}{a\cos t} = \frac{a^2 + \frac{5a^2}{4}}{a\cos t} = \frac{9a}{4\cos t}$

You need to eliminate b from the length OA to obtain a solvable equation in t from the condition $OA = 3AS$.

As $e = \frac{3}{2}$,

$$OS = ae = \frac{3a}{2}$$

$$OA = 3OS$$

$$\frac{9a}{4\cos t} = \frac{9a}{2} \Rightarrow \cos t = \frac{1}{2}$$

$$t = \frac{\pi}{3}, \frac{5\pi}{3}$$

These values give two points P, $(2a, \sqrt{3}b)$ and $(2a, -\sqrt{3}b)$.

These are the solutions in the first and fourth quadrants.
From the diagram, by symmetry, there are also solutions in the second and third quadrants giving

$$t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

These correspond to the two points $(-2a, \sqrt{3}b)$ and $(-2a, -\sqrt{3}b)$ where $\cos t = -\frac{1}{2}$.

The possible values of t are

$$t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

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Review Exercise 1
Exercise A, Question 24

Question:

An ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are constants and $a > b$.

a Find an equation of the tangent at the point $P(a\cos t, b\sin t)$.

b Find an equation of the normal at the point $P(a\cos t, b\sin t)$.

The normal at P meets the x-axis at the point Q. The tangent at P meets the y-axis at the point R.

c Find, in terms of a, b and t, the coordinates of M, the mid-point of QR.

Given that $0 < t < \frac{\pi}{2}$,

d Show that, as t varies, the locus of M has equation $\left(\frac{2ax}{a^2-b^2}\right)^2+\left(\frac{b}{2y}\right)^2=1$. **[E]**

Solution:

a $x = a\cos t,\quad y = b\sin t$

$$\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = b\cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{b\cos t}{a\sin t}$$

For the tangent

$$y - y_1 = m(x - x_1)$$

$$y - b\sin t = -\frac{b\cos t}{a\sin t}(x - a\cos t)$$

$$ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$$

$$ay\sin t + bx\cos t = ab(\sin^2 t + \cos^2 t)$$

$$ay\sin t + bx\cos t = ab$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However the calculation in part c will be easier if you simplify the equation at this stage using $\sin^2 t + \cos^2 t = 1$.

b As $\frac{dy}{dx} = -\frac{b\cos t}{a\sin t}$, using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin t}{b\cos t}$$

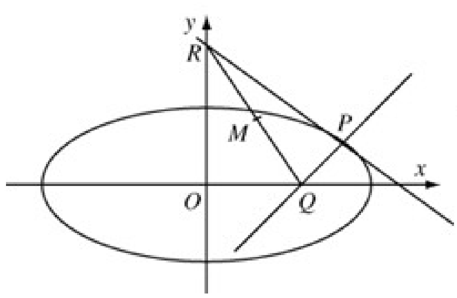
$$y - y_1 = m'(x - x_1)$$

$$y - b\sin t = \frac{a\sin t}{b\cos t}(x - a\cos t)$$

$$by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$$

$$ax\sin t - by\cos t = (a^2 - b^2)\sin t\cos t$$

c



The condition $0 < t < \frac{\pi}{2}$ implies that P is in the first quadrant.

Substituting $y = 0$ into the answer to part **b**

$$ax\sin t = (a^2 - b^2)\sin t\cos t \Rightarrow x = \frac{a^2 - b^2}{a}\cos t$$

You find the x-coordinate of Q by substituting $y = 0$ into the equation you found for the normal in part **b** and solving for x.

The coordinates of Q are $\left(\frac{a^2-b^2}{a}\cos t, 0\right)$

Substituting $x=0$ into the answer to part **a**

$ay\sin t = ab \Rightarrow y = \frac{b}{\sin t}$

You find the y-coordinate of R by substituting $x=0$ into the equation you found for the tangent in part **a** and solving for y.

The coordinates of R are $\left(0, \frac{b}{\sin t}\right)$

The coordinates of M are given by

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{a^2-b^2}{2a}\cos t, \frac{b}{2\sin t}\right)$$

d If the coordinates of M are (x, y) then $x = \frac{a^2-b^2}{2a}\cos t \Rightarrow \cos t = \frac{2ax}{a^2-b^2}$ and

$y = \frac{b}{2\sin t} \Rightarrow \sin t = \frac{b}{2y}$

As $\cos^2 t + \sin^2 t = 1$, the locus of

M is $\left(\frac{2ax}{a^2-b^2}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$, as required

$x = \frac{a^2-b^2}{2a}\cos t$ and $y = \frac{b}{2\sin t}$ are the parametric equations of the locus of M. To find the Cartesian equation, you must eliminate t. The form of the answer given in the question gives you a hint that you can use the identity $\cos^2 t + \sin^2 t = 1$ to do this.

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Review Exercise 1
Exercise A, Question 25

Question:

The points S_1 and S_2 have Cartesian coordinates $\left(-\frac{a}{2}\sqrt{3}, 0\right)$ and $\left(\frac{a}{2}\sqrt{3}, 0\right)$ respectively.

a Find a Cartesian equation of the ellipse which has S_1 and S_2 as its two foci, and a semi-major axis of length a.

b Write down an equation of a directrix of this ellipse.

Given that parametric equations of this ellipse are
$x = a\cos\varphi, y = b\sin\varphi,$

c express b in terms of a.

The point P is given by $\varphi = \frac{\pi}{4}$ and the point Q by $\varphi = \frac{\pi}{2}$.

d Show that an equation of the chord PQ is
$(\sqrt{2} - 1)\, x + 2y - a = 0.$ **[E]**

Solution:

a S_2 has coordinates $\left(\frac{a}{2}\sqrt{3}, 0\right)$

Hence

$$e = \frac{\sqrt{3}}{2}$$

$$b^2 = a^2(1 - e^2)$$

$$= a^2\left(1 - \frac{3}{4}\right) = \frac{a^2}{4} \quad *$$

Comparing $\left(\frac{a}{2}\sqrt{3}, 0\right)$ with the formula for the focus $(ae, 0)$, $e = \frac{\sqrt{3}}{2}$.

An equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

You are given that a is the semi-major axis, so a can be left in the equation. The data in the question does not include b, so b must be replaced.

Using *

$$\frac{x^2}{a^2} + \frac{y^2}{\frac{a^2}{4}} = 1$$

The required equation is

$$\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$$

$$x^2 + 4y^2 = a^2$$

b Equations of the directrices are

$$x = \pm\frac{a}{e} = \pm\frac{a}{\frac{\sqrt{3}}{2}} = \pm\frac{2a}{\sqrt{3}}$$

c From * above, $b = \frac{a}{2}$

d For Q

$$\left(a\cos\phi, \frac{1}{2}a\sin\varphi\right) = \left(a\cos\frac{\pi}{4}, \frac{1}{2}a\sin\frac{\pi}{4}\right)$$

$$= \left(\frac{a}{\sqrt{2}}, \frac{a}{2\sqrt{2}}\right) = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}\right)$$

For P

$$\left(a\cos\phi, \frac{1}{2}a\sin\varphi\right) = \left(a\cos\frac{\pi}{2}, \frac{1}{2}a\sin\frac{\pi}{2}\right)$$

$$= \left(0, \frac{a}{2}\right)$$

For PQ

$$\frac{y-\frac{a}{2}}{\frac{a\sqrt{2}}{4}-\frac{a}{2}}=\frac{x-0}{\frac{a\sqrt{2}}{2}-0}$$

Using the formula from module C1 for a line, $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$.

$$\frac{4y-2a}{\sqrt{2}-2}=\frac{2x}{\sqrt{2}}$$

$$4\sqrt{2}y-2\sqrt{2}a=\left(2\sqrt{2}-4\right)x$$

$$\left(4-2\sqrt{2}\right)x+4\sqrt{2}y-2\sqrt{2}a=0$$

The a cancels throughout the denominators of this equation. On the left-hand side $\frac{4\left(y-\frac{a}{2}\right)}{4\left(\frac{\sqrt{2}}{4}-\frac{1}{2}\right)}=\frac{4y-2a}{\sqrt{2}-2}$.

Dividing throughout by $2\sqrt{2}$

$\left(\sqrt{2}-1\right)x+2y-a=0$, as required.

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Review Exercise 1
Exercise A, Question 26

Question:

Show that the equations of the tangents with gradient m to the hyperbola with equation

$x^2 - 4y^2 = 4$

are

$y = mx \pm \sqrt{(4m^2 - 1)}$, where $|m| > \frac{1}{2}$. **[E]**

Solution:

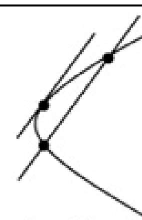
Let the equation of the tangent be $y = mx + c$

Eliminating y between $y = mx + c$ and $x^2 - 4y^2 = 4$

$$x^2 - 4(mx + c)^2 = 4$$

$$x^2 - 4m^2x^2 - 8mcx - 4c^2 = 4$$

$$(4m^2 - 1)x^2 + 8mcx + 4(c^2 + 1) = 0 \quad *$$



If the line was a chord, it would cut the curve in two distinct points and this equation would have a positive discriminant. As the line is a tangent it touches the curve at just one point and this equation has a repeated root. The discriminant is zero.

As the line is a tangent, equation * has repeated roots

'$b^2 - 4ac = 0$'

$$64m^2c^2 - 16(4m^2 - 1)(c^2 + 1) = 0$$

$$64m^2c^2 - 64m^2c^2 - 64m^2 + 16c^2 + 16 = 0$$

$$16c^2 = 64m^2 - 16$$

$$c^2 = 4m^2 - 1 \Rightarrow c = \pm\sqrt{(4m^2 - 1)}$$

The equation of the tangent is

$y = mx \pm \sqrt{(4m^2 - 1)}$, where $|m| > \frac{1}{2}$, as required.

If $|m| < \frac{1}{2}$, then $\sqrt{(4m^2 - 1)}$ would be the square root of a negative number and there would be no real answer. The cases $m = \pm\frac{1}{2}$ are interesting. For these values the equations are $y = \pm\frac{1}{2}x$. These are the asymptotes of the hyperbola and do not touch it at any point with finite coordinates. Asymptotes can be thought of as tangents to the curve 'at infinity'.

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Review Exercise 1
Exercise A, Question 27

Question:

The line with equation $y = mx + c$ is a tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

a Show that $c^2 = a^2m^2 + b^2$.

b Hence, or otherwise, find the equations of the tangents from the point (3, 4) to the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$. **[E]**

Solution:

a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

Multiply this equation throughout by a^2b^2. Then multiply out the bracket and collect the terms together as a quadratic in x.

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

As the line is a tangent this equation has repeated roots

'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(a^2m^2 + b^2)a^2(c^2 - b^2) = 0$$

$$a^2m^2c^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$$

$$\cancel{a^2m^2c^2} - \cancel{a^2m^2c^2} + a^2m^2b^2 - b^2c^2 + b^4 = 0$$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

$c^2 = a^2m^2 + b^2$, as required.

b $(3,4) \in y = mx + c$

Hence $4 = 3m + c \Rightarrow c = 4 - 3m$ ①

The tangents have equations of the form $y = mx + c$ and $x = 3, y = 4$ must satisfy this relation.

For this ellipse, $a = 4$ and $b = 5$ and the result in part a becomes

$c^2 = 16m^2 + 25$ ②

Substituting ① into ②

$$(4 - 3m)^2 = 16m^2 + 25$$

$$16 - 24m + 9m^2 = 16m^2 + 25$$

$$7m^2 + 24m + 9 = (m+3)(7m+3) = 0$$

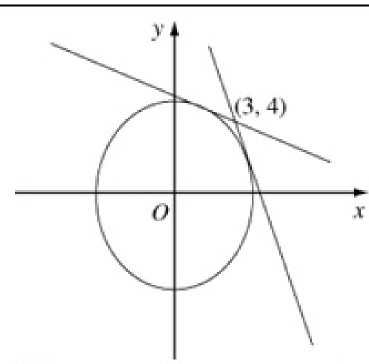
$$m = -3, -\frac{3}{7}$$

If $m = -3$, $c = 4 - 3m = 4 + 9 = 13$

If $m = -\frac{3}{7}$, $c = 4 - 3m = 4 + \frac{9}{7} = \frac{37}{7}$

The equations of the tangents are

$y = -3x + 13$ and $y = -\frac{3}{7}x + \frac{37}{7}$



There are two tangents to the ellipse which pass through (3, 4). Both have negative gradients.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 28

Question:

The ellipse E has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line L has equation $y=mx+c$, where $m>0$ and $c>0$.

a Show that, if L and E have any points of intersection, the x-coordinates of these points are the roots of the equation $(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$.

Hence, given that L is a tangent to E,

b show that $c^2=b^2+a^2m^2$.

The tangent L meets the negative x-axis at the point A and the positive y-axis at the point B, and O is the origin.

c Find, in terms of m, a and b, the area of the triangle OAB.

d Prove that, as m varies, the minimum area of the triangle OAB is ab.

e Find, in terms of a, the x-coordinate of the point of contact of L and E when the area of the triangle is a minimum. **[E]**

Solution:

a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0, \text{ as required}$$

Multiply this equation throughout by a^2b^2. Then multiply out the bracket and collect the terms together as a quadratic in x.

b As the line is a tangent the result of part **a** has repeated roots

'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(b^2 + a^2m^2)a^2(c^2 - b^2) = 0$$

$$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0$$

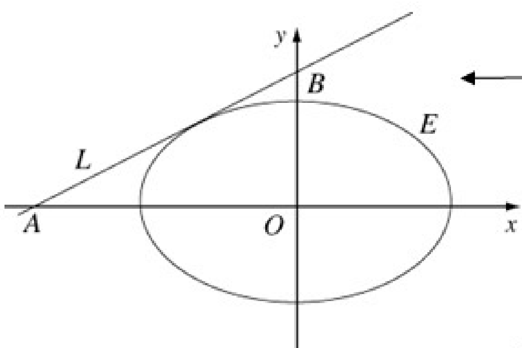
$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$$

$c^2 = a^2m^2 + b^2$, as required.

Divide this equation throughout by $4a^2$.

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

c



As $c^2 = a^2m^2 + b^2$, $y = mx + c$ could have the forms $y = mx \pm \sqrt{(b^2 + a^2m^2)}$. However, the question specifies that the tangent crosses the positive y-axis. As the line has a positive y intercept, you can reject the negative possibility.

An equation of L is $y = mx + \sqrt{(b^2 + a^2m^2)}$

For A, $y = 0$

$$0 = mx + \sqrt{(b^2 + a^2m^2)} \Rightarrow x = -\frac{\sqrt{(b^2 + a^2m^2)}}{m}$$

Hence $OA = \frac{\sqrt{(b^2 + a^2m^2)}}{m}$

For B, $x = 0$

$y = \sqrt{(b^2 + a^2m^2)}$

Hence $OB = \sqrt{(b^2 + a^2m^2)}$

The area of triangle OAB, T say, is given by

$$T = \frac{1}{2}OA \times OB = \frac{1}{2}\frac{\sqrt{\left(b^2 + a^2m^2\right)}}{m}\sqrt{\left(b^2 + a^2m^2\right)}$$

$$= \frac{b^2 + a^2m^2}{2m}$$

d $T = \dfrac{b^2 + a^2m^2}{2m} = \dfrac{1}{2}b^2m^{-1} + \dfrac{1}{2}a^2m$

For a minimum

$$\frac{dT}{dm} = -\frac{1}{2}b^2m^{-2} + \frac{1}{2}a^2 = 0$$

$$\frac{b^2}{m^2} = a^2 \Rightarrow m^2 = \frac{b^2}{a^2}$$

As L has a positive gradient

The diagram shows that the tangent has a positive gradient and so the possible value $-\dfrac{b}{a}$ can be ignored.

$$m = \frac{b}{a}$$

$$\frac{d^2T}{dm^2} = b^2m^{-3} = \frac{b^2}{m^3}$$

At $m = \dfrac{b}{a}, \dfrac{d^2T}{dm^2} = \dfrac{b^2}{m^3} = \dfrac{a^3}{b} > 0$ and so this gives a minimum value of

$$T = \frac{b^2 + a^2\left(\frac{b}{a}\right)^2}{2\left(\frac{b}{a}\right)} = \frac{2b^2}{2\left(\frac{b}{a}\right)} = ab \text{, as required.}$$

e At $m = \dfrac{b}{a}, c^2 = a^2m^2 + b^2 = a^2\left(\dfrac{b}{a}\right)^2 + b^2 = 2b^2$

Substituting $m = \dfrac{b}{a}$ and $c = \sqrt{2}b$ into the result in part **a**

$$\left(b^2 + a^2 \times \frac{b^2}{a^2}\right)x^2 + 2a^2 \times \frac{b}{a} \times \sqrt{2}bx + a^2\left(2b^2 - b^2\right) = 0$$

$$2b^2x^2 + 2\sqrt{2}ab^2x + a^2b^2 = 0$$

Divide this equation throughout by b^2.

$$2x^2 + 2\sqrt{2}ax + a^2 = 0$$

$$\left(\sqrt{2}x + a\right)^2 = 0$$

$$x = -\frac{a}{\sqrt{2}}$$

As the line is a tangent, this quadratic must factorise to a complete square. If you cannot see the factors, you can use the quadratic formula.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 29

Question:

a Find the eccentricity of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

b Find also the coordinates of both foci and equations of both directrices of this ellipse.

c Show that an equation for the tangent to this ellipse at the point $P(3\cos\theta, 2\sin\theta)$ is $\frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1$.

d Show that, as θ varies, the foot of the perpendicular from the origin to the tangent at P lies on the curve $(x^2+y^2)^2=9x^2+4y^2$. **[E]**

Solution:

a $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) = 9 - 9e^2$

$e^2 = \frac{9-4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

> The formulae you need for calculating the eccentricity, the coordinates of the foci, and the equations of the directrices are given in the Edexcel formula booklet you are allowed to use in the examination. However, it wastes time checking your textbook every time you need to use these formulae and it is worthwhile remembering them. **Remember** to quote any formulae you use in your solution.

b The coordinates of the foci are

$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = (\pm\sqrt{5}, 0)$

The equations of the directrices are

$x = \pm\frac{a}{e} = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$

c $x = 3\cos\theta, \quad y = 2\sin\theta$

$\frac{dx}{d\theta} = -3\sin\theta, \frac{dy}{d\theta} = 2\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{2\cos\theta}{3\sin\theta}$

$y - y_1 = m(x - x_1)$

$y - 2\sin\theta = -\frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

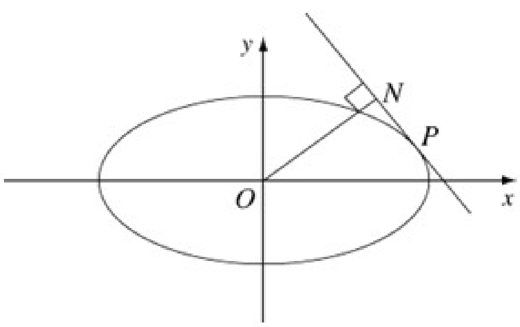
$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

$2x\cos\theta + 3y\sin\theta = 6(\cos^2\theta + \sin^2\theta) = 6$

> Divide this line throughout by 6.

$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$, as required

d



Let the foot of the perpendicular from O to the tangent at P be N.
Using $mm' = -1$, the gradient of ON is given by

$m' = -\frac{1}{\frac{dy}{dx}} = \frac{3\sin\theta}{2\cos\theta}$

An equation of ON is $y = \frac{3\sin\theta}{2\cos\theta}x$ *

Eliminating y between equation * and the answer to part c

$$\frac{x\cos\theta}{3} + \frac{\sin\theta}{2}\left(\frac{3\sin\theta}{2\cos\theta}x\right) = 1$$

$$x\left(\frac{4\cos^2\theta + 9\sin^2\theta}{12\cos\theta}\right) = 1$$

$$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$ and $y = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$ are parametric equations of the locus. Eliminating θ between them to obtain a Cartesian equation is not easy and you will need to use the printed answer to help you.

Substituting this expression for x into equation *

$$y = \frac{3\sin\theta}{2\cos\theta} \times \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta} = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$$x^2 + y^2 = \left(\frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2 + \left(\frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2$$

$$= \frac{144\cos^2\theta + 324\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{36(4\cos^2\theta + 9\sin^2\theta)}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \frac{36}{4\cos^2\theta + 9\sin^2\theta}$$

$$9x^2 + 4y^2 = \frac{9\times144\cos^2\theta + 4\times324\sin^2\theta}{\left(4\cos^2\theta + 9\sin^2\theta\right)^2}$$

$$= \frac{1296\cos^2\theta + 1296\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{1296}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \left(\frac{36}{4\cos^2\theta + 9\sin^2\theta}\right)^2 = (x^2 + y^2)^2$$

The locus of N is $(x^2 + y^2)^2 = 9x^2 + 4y^2$, as required.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 30

Question:

a Show that the hyperbola $x^2 - y^2 = a^2, a > 0$, has eccentricity equal to $\sqrt{2}$.

b Hence state the coordinates of the focus S and an equation of the corresponding directrix L, where both S and L lie in the region $x > 0$.

The perpendicular from S to the line $y = x$ meets the line $y = x$ at P and the perpendicular from S to the line $y = -x$ meets the line $y = -x$ at Q.

c Show that both P and Q lie on the directrix L and give the coordinates of P and Q.

Given that the line SP meets the hyperbola at the point R,

d prove that the tangent at R passes through the point Q. **[E]**

Solution:

a $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

$b^2=a^2\left(e^2-1\right)$

For this hyperbola $b^2=a^2$

$a^2=a^2\left(e^2-1\right)\Rightarrow 1=e^2-1\Rightarrow e^2=2$

$e=\sqrt{2}$, as required.

> $x^2-y^2=a^2\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{a^2}=1$. This is an hyperbola in which $a=b$.

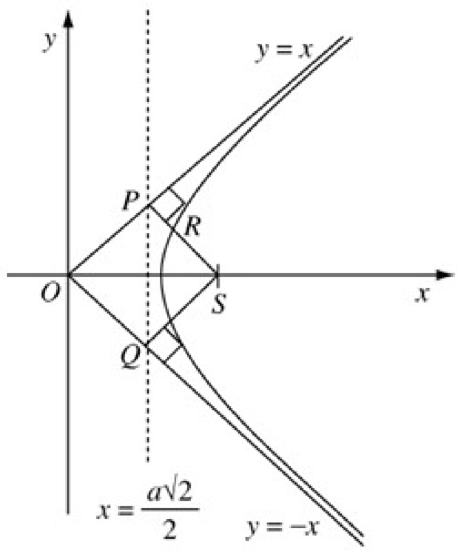
b The coordinates of S are

$(ae,0)=\left(a\sqrt{2},0\right)$

An equation of L is

$x=\frac{a}{e}=\frac{a}{\sqrt{2}}=\frac{a\sqrt{2}}{2}$

c



SP is perpendicular to $y=x$, so its gradient is -1. An equation of SP is

$y=-1\left(x-a\sqrt{2}\right)=-x+a\sqrt{2}$

$y+x=a\sqrt{2}$

SP meets $y=x$ where

$x+x=a\sqrt{2}\Rightarrow x=\frac{a\sqrt{2}}{2}$

Hence P is on the directrix L.

SQ is perpendicular to $y=-x$, so its gradient is 1.

> The asymptotes to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$. These formulae are given in the Edexcel formulae booklet. With this hyperbola $a=b$ and the asymptotes are $y=\pm x$. This question is about the intersection of line with the asymptotes. The lines $y=x$ and $y=-x$ are perpendicular to each other and a hyperbola with perpendicular asymptotes is called a rectangular hyperbola. In Module FP1, you studied another rectangular hyperbola, $xy=c^2$.

An equation of SQ is

$y = 1(x - a\sqrt{2}) = x - a\sqrt{2}$

$y = x - a\sqrt{2}$

SQ meets $y = -x$ where $-x = x - a\sqrt{2} \Rightarrow x = \frac{a\sqrt{2}}{2}$

Hence Q is on the directrix L.
Both P and Q lie on the directrix L.

The coordinates of P are $\left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}\right)$.

The coordinates of Q are $\left(\frac{a\sqrt{2}}{2}, -\frac{a\sqrt{2}}{2}\right)$.

d SP: $y + x = a\sqrt{2}$ ①

Hyperbola $x^2 - y^2 = a^2$ ②

From ① $y = a\sqrt{2} - x$ ③

Substitute ③ into ②

$$x^2 - (a\sqrt{2} - x)^2 = a^2$$

$$x^2 - 2a^2 + 2\sqrt{2}ax - x^2 = a^2$$

$$2\sqrt{2}ax = 3a^2 \Rightarrow x = \frac{3a}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}a$$

To find the coordinates of R, you solve equations ① and ② simultaneously.

Substituting for x in ③

$$y = a\sqrt{2} - \frac{3\sqrt{2}}{4}a = \frac{\sqrt{2}}{4}a$$

The coordinates of R are $\left(\frac{3\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a\right)$

To find the tangent to the hyperbola at R

$$x^2 - y^2 = a^2$$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Differentiating the equation of the hyperbola implicitly with respect to x.

At R

$$\frac{dy}{dx} = \frac{x}{y} = \frac{\frac{3\sqrt{2}}{4}a}{\frac{\sqrt{2}}{4}a} = 3$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{2}}{4}a = 3\left(x - \frac{3\sqrt{2}}{4}a\right) = 3x - \frac{9\sqrt{2}}{4}a$$

$$y = 3x - 2\sqrt{2}a$$

This is the equation of the tangent to the hyperbola at R. To establish that R passes through Q, you substitute the x-coordinate of Q into this equation and show that this gives the y-coordinate of Q.

At $x = \frac{a\sqrt{2}}{2}, y = 3\left(\frac{a\sqrt{2}}{2}\right) - 2\sqrt{2}a = -\frac{a\sqrt{2}}{2}$

This is the y-coordinate of Q.
Hence the tangent at R passes through Q.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 31

Question:

a Show that an equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$.

The normal at P cuts the x-axis at G.

b Show that the coordinates of M, the mid-point of PG, are

$$\left[\left(\frac{2a^2-b^2}{2a}\right)\cos\theta, \left(\frac{b}{2}\right)\sin\theta\right]$$

c Show that, as θ varies, the locus of M is an ellipse and determine the equation of this locus.

Given that the normal at P meets the y-axis at H and that O is the origin,

d show that, if $a > b$, area ΔOMG : area$\Delta OGH = b^2 : 2(a^2-b^2)$. **[E]**

Solution:

a $x = a\cos\theta, \quad y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{b\cos\theta}{a\sin\theta}$$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin\theta}{b\cos\theta}$$

$$y - y_1 = m'(x - x_1)$$

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

Divide this equation throughout by $\sin\theta\cos\theta$.

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$, as required

b Substituting $y = 0$ in the result to part **a**

$$ax\sec\theta = a^2 - b^2$$

You find the x-coordinate of G by substituting $y = 0$ into the equation of the normal at P and solving the resulting equation for x.

$$x = \frac{a^2 - b^2}{a}\cos\theta$$

$$P: (a\cos\theta, b\sin\theta), G: \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right)$$

The coordinates (x_M, y_M) of M the mid-point of PG are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

$$x_M = \frac{a\cos\theta + \frac{a^2 - b^2}{a}\cos\theta}{2}$$

$$= \frac{\cos\theta}{2}\left(\frac{a^2 + a^2 - b^2}{a}\right) = \left(\frac{2a^2 - b^2}{2a}\right)\cos\theta$$

Hence, the coordinates of M are

$\left[\left(\frac{2a^2 - b^2}{2a}\right)\cos\theta, \left(\frac{b}{2}\right)\sin\theta\right]$, as required

c For M

$$x = \left(\frac{2a^2 - b^2}{2a}\right)\cos\theta, y = \left(\frac{b}{2}\right)\sin\theta$$

$$\cos\theta = \frac{x}{\left(\frac{2a^2-b^2}{2a}\right)}, \sin\theta = \frac{y}{\left(\frac{b}{2}\right)}$$

$$\cos^2\theta + \sin^2\theta = 1$$

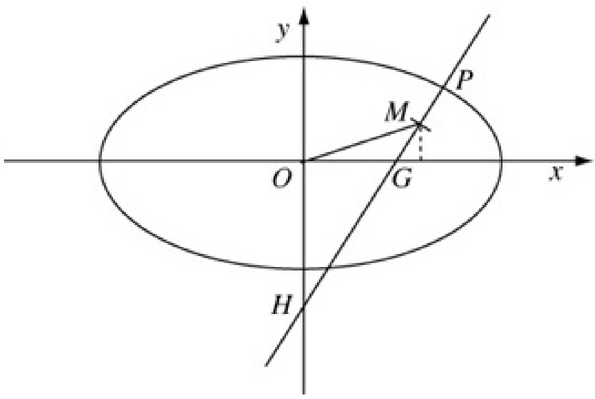
$$\frac{x^2}{\left(\frac{2a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{b}{2}\right)^2} = 1$$

This is an ellipse. A Cartesian equation of this ellipse is

$$\frac{4a^2x^2}{\left(2a^2-b^2\right)^2} + \frac{4y^2}{b^2} = 1$$

Any curve with an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse. If you are asked to show that a locus is an ellipse, it is sufficient to show that it has a Cartesian equation of this form.

d



Substituting $x = 0$ into the equation of the normal

$$-by\operatorname{cosec}\theta = a^2 - b^2 \Rightarrow y = -\frac{a^2-b^2}{b}\sin\theta$$

Hence $OH = \frac{a^2-b^2}{b}\sin\theta$.

$$\frac{\text{area}\Delta OMG}{\text{area}\Delta OGH} = \frac{y-\text{coordinate of } M}{OH}$$

$$= \frac{\left(\frac{b}{2}\right)\sin\theta}{\frac{a^2-b^2}{b}\sin\theta}$$

$$= \frac{b^2}{2\left(a^2-b^2\right)}, \text{ as required}$$

The triangles OMG and OGH can be looked at as having the same base OG. As the area of a triangle is $\frac{1}{2}\times\text{base}\times\text{height}$, triangles with the same base will have areas proportional to their heights. The height of the triangle OGM is shown by a dotted line in the diagram and is given by the y-coordinate of M.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 32

Question:

a Find equations for the tangent and normal to the rectangular hyperbola $x^2 - y^2 = 1$, at the point P with coordinates $(\cosh t, \sinh t), t > 0$.

The tangent and normal intersect the x-axis at T and G respectively. The perpendicular from P to the x-axis meets an asymptote in the first quadrant at Q.

b Show that GQ is perpendicular to this asymptote.

The normal intercepts the y-axis at R.

c Show that R lies on the circle with centre at T and radius TG. **[E]**

Solution:

a To find an equation of the tangent at P.

$x = \cosh t, \quad y = \sinh t$

$\frac{dx}{dt} = \sinh t, \quad \frac{dy}{dt} = \cosh t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cosh t}{\sinh t}$

Using $y - y_1 = m(x - x_1)$

$$y - \sinh t = \frac{\cosh t}{\sinh t}(x - \cosh t)$$

$$y \sinh t - \sinh^2 t = x \cosh t - \cosh^2 t$$

$$y \sinh t = x \cosh t - (\cosh^2 t - \sinh^2 t)$$

$$= x \cosh t - 1$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$.

$x \cosh t - y \sinh t = 1$ ①

To find the equation of the normal at P

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{\sinh t}{\cosh t}$$

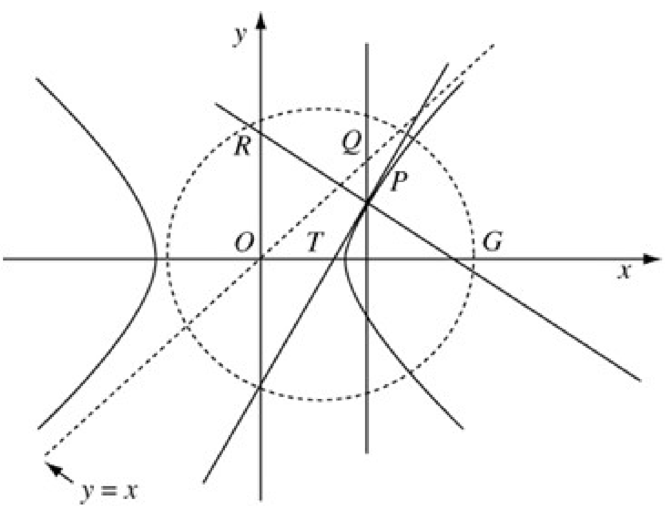
$$y - y_1 = m'(x - x_1)$$

$$y - \sinh t = -\frac{\sinh t}{\cosh t}(x - \cosh t)$$

$$y \cosh t - \sinh t \cosh t = -x \sinh t + \sinh t \cosh t$$

$$x \sinh t + y \cosh t = 2 \sinh t \cosh t$$ ②

b



Substitute $y=0$ into ②

$x\sinh t = 2\sinh t\cosh t$

$x = 2\cosh t$

The coordinates of G are $(2\cosh t, 0)$.

To find the coordinates of G, you substitute $y=0$ into the equation of the normal found in part **a**.

The x-coordinate of Q is $\cosh t$.

The asymptote in the first quadrant has equation $y=x$.

Hence the coordinates of Q are $(\cosh t, \cosh t)$.

The gradient of GQ is given by $\frac{y_1-y_2}{x_1-x_2}=\frac{0-\cosh t}{2\cosh t-\cosh t}=-1$

As the gradient of $y=x$ is 1 and $1\times -1=-1$, GQ is perpendicular to the asymptote.

The asymptotes to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$. These formulae are given in the Edexcel formulae booklet. With this hyperbola $a=b=1$ and the asymptotes are $y=\pm x$. The asymptote in the first quadrant has equation $y=x$.

c Substitute $y=0$ into ①

$x\cosh t=1 \Rightarrow x=\frac{1}{\cosh t}$

The coordinates of T are $\left(\frac{1}{\cosh t}, 0\right)$.

To find the coordinates of T, you substitute $y=0$ into the equation of the tangent found in part **a**.

Substitute $x=0$ into ②

$y\cosh t = 2\sinh t\cosh t \Rightarrow y = 2\sinh t$

The coordinates of R are $(0, 2\sinh t)$

To find the coordinates of R, you substitute $x=0$ into the equation of the normal found in part **a**.

$TG = 2\cosh t-\frac{1}{\cosh t}$

$$TR^2 = OR^2+OT^2=(2\sinh t)^2+\left(\frac{1}{\cosh t}\right)^2$$

$$=4\sinh^2 t+\frac{1}{\cosh^2 t}=4(\cosh^2 t-1)+\frac{1}{\cosh^2 t}$$

$$=4\cosh^2 t-4+\frac{1}{\cosh^2 t}$$

$$=\left(2\cosh t-\frac{1}{\cosh t}\right)^2=TG^2$$

If a circle can be drawn through R with centre T and radius TG then TR must also be a radius of the circle. So you can solve the problem by showing that TR and TG have the same length.

Hence $TR=TG$ and R lies on the circle with centre at T and radius TG.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 33

Question:

a Find the equations for the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.

b If these lines meet the y-axis at P and Q respectively, show that the circle described on PQ as diameter passes through the foci of the hyperbola. **[E]**

Solution:

a To find the equation of the tangent at $(a\sec\theta, b\tan\theta)$

$$x = a\sec\theta,\quad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \frac{dy}{dt} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a\sin\theta}$$

$$y - y_1 = m(x - x_1)$$

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However, the calculation in part **b** will be easier if you simplify the equation at this stage.

$$ay\sin\theta - \frac{ab\sin^2\theta}{\cos\theta} = bx - ab\sec\theta$$

$$bx - ay\sin\theta = ab\left(\frac{1-\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta}$$

$$bx - ay\sin\theta = ab\cos\theta \quad ①$$

To find the equation of the normal at $(a\sec\theta, b\tan\theta)$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{a\sin\theta}{b}$$

$$y - y_1 = m'(x - x_1)$$

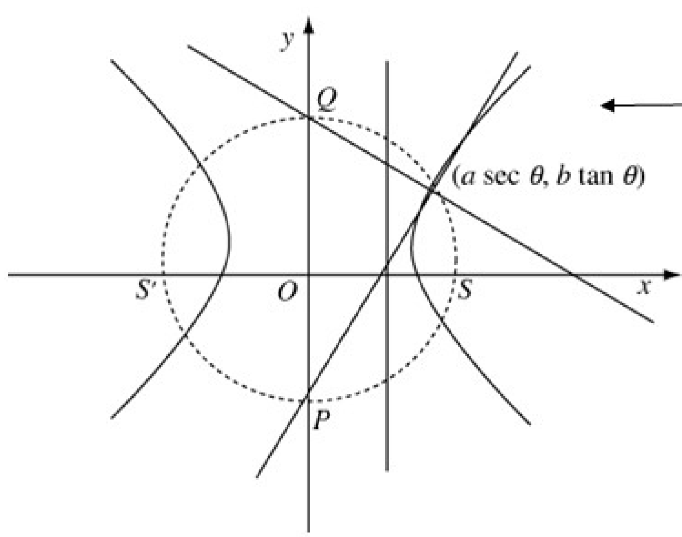
$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

When you multiply the bracket out, $\sin\theta\sec\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\tan\theta$$

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta \quad ②$$

b



This problem will be solved using the property that the angle in a semi-circle is a right angle and you need to show that PS and QS are perpendicular. All five of the points, P, Q, $(a\sec\theta, b\tan\theta)$ and the two foci lie on the same circle.

Substitute $x = 0$ into ①
$-ay\sin\theta = ab\cos\theta \Rightarrow y = -b\cot\theta$

To find the coordinates of P, you substitute $x = 0$ into the equation of the tangent found in part **a**.

The coordinates of P are $(0, -b\cot\theta)$.
Substitute $x = 0$ into ②

$by = (a^2 + b^2)\tan\theta \Rightarrow y = \frac{a^2+b^2}{b}\tan\theta$

To find the coordinates of Q, you substitute $x = 0$ into the equation of the normal found in part **a**.

The coordinates of Q are $\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$.

The focus S has coordinates $(ae, 0)$

The gradient of PS is given by $m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-b\cot\theta - 0}{0 - ae} = \frac{b}{ae}\cot\theta$

The gradient of QS is given by

$$m' = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{a^2+b^2}{b}\tan\theta - 0}{0 - ae} = \frac{-(a^2+b^2)}{abe}\tan\theta$$

$$mm' = \frac{b}{ae}\cot\theta \times -\frac{a^2+b^2}{abe}\tan\theta = -\frac{a^2+b^2}{a^2e^2}$$

The formula for the eccentricity is

$b^2 = a^2(e^2 - 1)$

$b^2 = a^2e^2 - a^2 \Rightarrow a^2e^2 = a^2 + b^2$

Hence $mm' = -\frac{a^2+b^2}{a^2e^2} = -\frac{a^2+b^2}{a^2+b^2} = -1$

So PS is perpendicular to QS and $\angle PSQ = 90°$.
By the converse of the theorem that the angle in a semi-circle is a right angle, the circle described on PQ as diameter passes through the focus S.
By symmetry, the circle also passes through the focus S'.

There is no need to repeat the calculations for PS' and QS'. It is evident from the diagram that the whole diagram is symmetrical about the y-axis, so, if the circle passes through S, it passes through S'. It is quite acceptable to appeal to symmetry to complete your proof.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 34

Question:

Given that $r > a > 0$ and $0 < \arcsin\left(\frac{a}{r}\right) < \frac{\pi}{2}$, show that

$$\frac{d}{dr}\left[\arcsin\left(\frac{a}{r}\right)\right] = -\frac{a}{r\sqrt{(r^2 - a^2)}}$$ **[E]**

Solution:

Let $y = \arcsin\left(\frac{a}{r}\right)$

Let $u = \frac{a}{r} = ar^{-1}$

$y = \arcsin u$

$\frac{dy}{dr} = \frac{dy}{du} \times \frac{du}{dr}$

$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}}$

$\frac{du}{dr} = -ar^{-2} = -\frac{a}{r^2}$

Hence $\frac{dy}{dr} = \frac{1}{\sqrt{(1-u^2)}} \times -\frac{a}{r^2} = -\frac{a}{r^2\sqrt{\left(1-\frac{a^2}{r^2}\right)}}$

$= -\frac{a}{r\sqrt{(r^2 - a^2)}}$, as required

You can use the chain rule to differentiate $\arcsin\left(\frac{a}{r}\right)$.

You take one of the rs inside the square root sign in the denominator. In detail

$r^2\sqrt{\left(1-\frac{a^2}{r^2}\right)} = r\sqrt{r^2}\sqrt{\left(1-\frac{a^2}{r^2}\right)}$

$= r\sqrt{\left(r^2\left(1-\frac{a^2}{r^2}\right)\right)} = r\sqrt{(r^2 - a^2)}$.

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Review Exercise 1
Exercise A, Question 35

Question:

Given that $y = (\arcsin x)^2$,

a prove that $(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y$,

b deduce that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$. **[E]**

Solution:

a $y = (\arcsin x)^2$

Let $u = \arcsin x$

$y = u^2$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = 2u$

$\frac{du}{dx} = \frac{1}{\sqrt{(1-x^2)}}$

This result is in the Edexcel formula booklet, which is provided for use with the paper. It is a good idea to quote any formulae you use in your solution.

Hence

$$\frac{dy}{dx} = 2u \times \frac{1}{\sqrt{(1-x^2)}} = \frac{2\arcsin x}{\sqrt{(1-x^2)}}$$

$$\sqrt{(1-x^2)}\frac{dy}{dx} = 2\arcsin x$$

Square both sides of this solution and use the given $y = (\arcsin x)^2$ to complete the solution.

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\arcsin x)^2$$

$$= 4y, \text{ as required}$$

b Differentiating the result of part **a** implicitly with respect to x

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4\frac{dy}{dx}$$

Using the chain rule $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) = 2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}$.

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 2$$

Divide the equation throughout by $2\frac{dy}{dx}$.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2, \text{ as required}$$

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Review Exercise 1
Exercise A, Question 36

Question:

a Show that, for $x = \ln k$, where k is a positive constant, $\cosh 2x = \frac{k^4+1}{2k^2}$.

b Given that $f(x) = px - \tanh 2x$, where p is a constant, find the value of p for which f(x) has a stationary value at $x = \ln 2$, giving your answer as an exact fraction. **[E]**

Solution:

a $\cosh 2x = \frac{e^{2x}+e^{-2x}}{2} = \frac{e^{2\ln k}+e^{-2\ln k}}{2}$

$= \frac{e^{\ln k^2}+e^{\ln\frac{1}{k^2}}}{2} = \frac{1}{2}\left(k^2+\frac{1}{k^2}\right)$

$= \frac{1}{2}\left(\frac{k^4+1}{k^2}\right) = \frac{k^4+1}{2k^2}$, as required

Using the law of logarithms $n\ln x = \ln x^n$, with $n = -2$, $-2\ln k = \ln k^{-2} = \ln\frac{1}{k^2}$.

b $f(x) = px - \tanh 2x$

For a stationary value

$f'(x) = p - 2\text{sech}^2 2x = 0$

$p = 2\text{sech}^2 2x = \frac{2}{\cosh^2 2x}$

Using the result of part **a** with $k = 2$

If $x = \ln 2$

$\cosh 2x = \frac{2^4+1}{2\times 2^2} = \frac{17}{8}$

Hence

$p = \frac{2}{\left(\frac{17}{8}\right)^2} = \frac{128}{289}$

There is no 'hence' in this question but using the result in part **a** shortens the working. The question requires an exact fraction for the answer and you should not use a calculator other than, possibly, for multiplying and dividing fractions.

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Review Exercise 1
Exercise A, Question 37

Question:

The curve with equation $y = -x + \tanh 4x, x \geq 0$, has a maximum turning point A.

a Find, in exact logarithmic form, the x-coordinate of A.

b Show that the y-coordinate of A is $\frac{1}{4}\{2\sqrt{3} - \ln(2+\sqrt{3})\}$. **[E]**

Solution:

a $y = -x + \tanh 4x$

$$\frac{dy}{dx} = -1 + 4\operatorname{sech}^2 4x = 0$$

$$\operatorname{sech}^2 4x = \frac{1}{4} \Rightarrow \cosh^2 4x = 4$$

$$\cosh 4x = 2$$

$$4x = \operatorname{arcosh} 2 = \ln\left(2+\sqrt{3}\right)$$

$$x = \frac{1}{4}\ln\left(2+\sqrt{3}\right)$$

As $\cosh x \geq 1$ for all real x, $\cosh 4x = -2$ is impossible.

For $x \geq 0$, there is only one value of x which gives a stationary value. The question tells you that the curve has a maximum point so, in this question, you need not show that this point is a maximum by, for example, examining the second derivative.

b $\tanh^2 4x = 1 - \operatorname{sech}^2 4x = 1 - \frac{1}{4} = \frac{3}{4}$

You need a value for $\tanh 4x$ and this is easiest found using the hyperbolic identity $\operatorname{sech}^2 x = 1 - \tanh^2 x$.

As $x \geq 0$, $\tanh 4x = \frac{\sqrt{3}}{2}$

At $x = \frac{1}{4}\ln\left(2+\sqrt{3}\right)$

$$y = -x + \tanh 4x = -\frac{1}{4}\ln\left(2+\sqrt{3}\right) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left\{2\sqrt{3} - \ln\left(2+\sqrt{3}\right)\right\}, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 38

Question:

The curve C has equation $y = \text{arcsec } e^x, x > 0, 0 \leq y < \frac{1}{2}\pi$.

a Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{(e^{2x} - 1)}}$.

b Sketch the graph of C

The point A on C has x-coordinate $\ln 2$. The tangent to C at A intersects the y-axis at the point B.

c Find the exact value of the y-coordinate of B. **[E]**

Solution:

a $y = \text{arcsec}\, e^x$

$\sec y = e^x$

Differentiating implicitly with respect to x

$$\sec y \tan y \frac{dy}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{e^x}{\sec y \tan y}$$

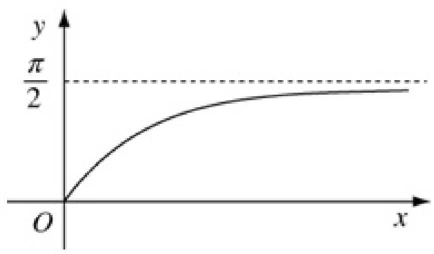
As $\sec y = e^x, \tan^2 y = \sec^2 y - 1 = e^{2x} - 1$

$$\tan y = \sqrt{(e^{2x} - 1)}$$

$$\frac{dy}{dx} = \frac{e^x}{e^x \sqrt{(e^{2x}-1)}} = \frac{1}{\sqrt{(e^{2x}-1)}}, \text{ as required.}$$

$\tan y = -\sqrt{(e^{2x}-1)}$ is, in general, possible. In this case, the question specifies that $x > 0$ and $0 < y < \frac{1}{2}\pi$ and, with these ranges, $\text{arcsec}\, e^x$ is an increasing function of x and so $\frac{dy}{dx}$ is positive ($\tan y$ is positive).

b



In your sketch, you must show any important features of the curve. In this case, you need to show that the curve starts at the origin and that the line $y = \frac{\pi}{2}$ is an asymptote to the curve.

c At $x = \ln 2$, the gradient of the curve is given by

$$m = \frac{dy}{dx} = \frac{1}{\sqrt{(e^{2x}-1)}} = \frac{1}{\sqrt{(e^{2\ln 2}-1)}}$$

$$= \frac{1}{\sqrt{(e^{\ln 4}-1)}} = \frac{1}{\sqrt{(4-1)}} = \frac{1}{\sqrt{3}}$$

At $x = \ln 2$,

$$y = \text{arcsec}\, e^x = \text{arcsec}\, e^{\ln 2} = \text{arcsec} 2 = \frac{\pi}{3}$$

$\text{arcsec} 2 = \arccos \frac{1}{2} = \frac{\pi}{3}$. In questions involving calculus you must use radians.

An equation of the tangent is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{3} = \frac{1}{\sqrt{3}}(x - \ln 2)$$

There is no need to simplify this equation. You only need to find the value of y at $x = 0$.

At B, $x = 0$

$$y = \frac{\pi}{3} - \frac{\ln 2}{\sqrt{3}} = \frac{1}{3}\left(\pi - \sqrt{3} \ln 2\right)$$

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Review Exercise 1
Exercise A, Question 39

Question:

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{(x^2-2x+17)}} \right) dx$, giving your answer as an exact logarithm. **[E]**

Solution:

$x^2-2x+17 = x^2-2x+1+16 = (x-1)^2+4^2$

It is usually a good idea to begin any integration involving the square root of a quadratic by completing the square.

Hence

$$\int_1^4 \frac{1}{\sqrt{(x^2-2x+17)}} dx = \int_1^4 \frac{1}{\sqrt{((x-1)^2+4^2)}} dx$$

$$= \left[\operatorname{arsinh} \frac{x-1}{4} \right]_1^4 = \operatorname{arsinh} \frac{3}{4} - \operatorname{arsinh} 0$$

$$= \ln\left(\frac{3}{4} + \sqrt{\left(\frac{9}{16}+1 \right)} \right) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{25}{16} \right)} \right)$$

$$= \ln\left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$$

This is a direct application of the formula

$$\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \operatorname{arsinh}\left(\frac{x}{a} \right)$$

which is given in the Edexcel formulae booklet. You would need to be careful to adapt this formula correctly if the coefficient of x^2 in the quadratic was not 1.

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Review Exercise 1
Exercise A, Question 40

Question:

Evaluate $\int_1^3 \frac{1}{\sqrt{(x^2+4x-5)}}\,dx$, giving your answer as an exact logarithm. **[E]**

Solution:

$x^2+4x-5=x^2+4x+4-9=(x+2)^2-3^2$

Hence

$$\int_1^3 \frac{1}{\sqrt{(x^2+4x-5)}}\,dx=\int_1^3 \frac{1}{\sqrt{((x+2)^2-3^2)}}\,dx$$

$$=\left[\operatorname{arcosh}\frac{x+2}{3}\right]_1^3=\operatorname{arcosh}\frac{5}{3}-\operatorname{arcosh}1$$

$$=\ln\left(\frac{5}{3}+\sqrt{\left(\frac{25}{9}-1\right)}\right)=\ln\left(\frac{5}{3}+\sqrt{\left(\frac{16}{9}\right)}\right)$$

$$=\ln\left(\frac{5}{3}+\frac{4}{3}\right)=\ln 3$$

$\operatorname{arcosh}1=0$

To obtain the answer as an exact logarithm, you can use the formula $\operatorname{arcosh}x=\ln\left(x+\sqrt{(x^2-1)}\right)$. If you forget this, or can't remember the sign, you can find it in the Edexcel formulae booklet which is provided for use with the paper. This booklet contains many of the formulae needed for the calculus topics in the FP3 module.

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 41

Question:

Use the substitution $x = \frac{a}{\sinh\theta}$, where a is a constant, to show that, for $x > 0, a > 0$,

$$\int \frac{1}{x\sqrt{(x^2 + a^2)}}\,dx = -\frac{1}{a}\operatorname{arsinh}\left(\frac{a}{x}\right) + \text{constant}.$$ **[E]**

Solution:

$$x = \frac{a}{\sinh\theta} = a(\sinh\theta)^{-1}$$

$$\frac{dx}{d\theta} = -a(\sinh\theta)^{-2}\cosh\theta = -\frac{a\cosh\theta}{\sinh^2\theta}$$

When substituting remember to substitute for the dx as well as the rest of the integral.

$$\int \frac{1}{x\sqrt{(x^2+a^2)}}\,dx = \int \frac{1}{\frac{a}{\sinh\theta}\sqrt{\left(\frac{a^2}{\sinh^2\theta} + a^2\right)}} \times \frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{\frac{-a\cosh\theta}{\sinh^2\theta}}{\frac{a^2\sqrt{1+\sinh^2\theta}}{\sinh^2\theta}}\,d\theta = \frac{-1}{a}\int \frac{\cosh\theta}{\cos\theta}\,d\theta$$

Use $1 + \sinh^2\theta = \cosh^2\theta$ to simplify this expression.

$$= -\frac{1}{a}\int \frac{\cosh\theta}{\cosh\theta}\,d\theta = -\frac{1}{a}\int 1\,d\theta$$

$$= -\frac{1}{a}\theta + \text{constant}$$

$$= -\frac{1}{a}\operatorname{arsinh}\left(\frac{a}{x}\right) + \text{constant, as required.}$$

As $x = \frac{a}{\sinh\theta}$, then $\sinh\theta = \frac{a}{x}$ and $\theta = \operatorname{arsinh}\left(\frac{a}{x}\right)$.

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Review Exercise 1
Exercise A, Question 42

Question:

a Prove that the derivative of artanh x, $-1 < x < 1$, is $\frac{1}{1-x^2}$.

b Find $\int \text{artanh}\,x \, dx$. **[E]**

Solution:

a Let $y = \text{artanh}\,x$

$\tanh y = x$

Differentiate implicitly with respect to x

$$\text{sech}^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y}$$

$$= \frac{1}{1-x^2}, \text{ as required}$$

To differentiate a function f(y) with respect to x you use a version of the chain rule $\frac{d}{dx}(f(y)) = f'(y) \times \frac{dy}{dx}$.

b Using integration by parts and the result in part **a**

$$\int \text{artanh}\,x \, dx = \int 1 \times \text{artanh}\,x \, dx$$

$$= x\,\text{artanh}\,x - \int \frac{x}{1-x^2} dx$$

$$= x\,\text{artanh}\,x + \frac{1}{2}\ln(1-x^2) + A$$

You use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ with $u = \text{artanh}\,x$ and $\frac{dv}{dx} = 1$. You know $\frac{du}{dx}$ from part **a**.

This solution uses the result $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. So $\int \frac{-2x}{1-x^2} dx = \ln(1-x^2)$ and you multiply this by $-\frac{1}{2}$ to complete the solution. This is a question where there are a number of possible alternative forms of the answer.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 43

Question:

a Find $\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx$.

b Find, to 3 decimal places, the value of $\int_0^{0.3} \frac{1+x}{\sqrt{(1-4x^2)}}\,dx$. **[E]**

Solution:

a $\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx = \int \frac{1}{\sqrt{(1-4x^2)}}\,dx + \int \frac{x}{\sqrt{(1-4x^2)}}\,dx$

You must treat this integral as two separate integrals added together. Both integrals have been solved here using substitution. This is a safe method of solution but you may be able to shorten the working by adapting standard formulae or inspection.

Let $2x = \sin\theta$, then $2\frac{dx}{d\theta} = \cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\cos\theta$

$$\int \frac{1}{\sqrt{(1-4x^2)}}\,dx = \int \frac{1}{\sqrt{(1-\sin^2\theta)}}\frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{1}{\cos\theta} \times \frac{1}{2}\cos\theta\,d\theta = \int \frac{1}{2}\,d\theta$$

$$= \frac{1}{2}\theta + A = \frac{1}{2}\arcsin 2x + A$$

Let $u^2 = 1-4x^2$, then differentiating implicitly with respect to x

$$2u\frac{du}{dx} = -8x \Rightarrow x\frac{dx}{du} = -\frac{1}{4}u$$

$$\int \frac{x}{\sqrt{(1-4x^2)}}\,dx = \int \frac{1}{u} \times x\frac{dx}{du}\,du = \int \frac{1}{u} \times -\frac{1}{4}u\,du$$

$$= \int -\frac{1}{4}\,du = -\frac{1}{4}u + B = -\frac{1}{4}\sqrt{(1-4x^2)} + B$$

Combining the integrals

$$\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx = \frac{1}{2}\arcsin 2x - \frac{1}{4}\sqrt{(1-4x^2)} + C$$

b $$\int_0^{0.3} \frac{1+x}{\sqrt{(1-4x^2)}}\,dx = \left[\frac{1}{2}\arcsin 2x - \frac{1}{4}\sqrt{(1-4x^2)}\right]_0^{0.3}$$

$$= \frac{1}{2}\arcsin 0.6 - \frac{1}{4}\sqrt{(1-4\times 0.09)} - \left(0 - \frac{1}{4}\right)$$

You can use your calculator at any stage to evaluate this definite integral. The calculator must be in radian mode.

$$= \frac{1}{2}\arcsin 0.6 - 0.2 + \frac{1}{4}$$

$$= 0.372 \quad (3\text{ d.p.})$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 44

Question:

a Given that $y = \arctan 3x$, and assuming the derivative of $\tan x$, prove that $\frac{dy}{dx} = \frac{3}{1+9x^2}$.

b Show that $\int_0^{\frac{\sqrt{3}}{3}} 6x \arctan 3x = \frac{1}{9}(4\pi - 3\sqrt{3})$. **[E]**

Solution:

a $y = \arctan 3x$

$\tan y = 3x$

Differentiating implicitly with respect to x

$$\sec^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\sec^2 y} = \frac{3}{1+\tan^2 y}$$

$$= \frac{3}{1+9x^2}, \text{ as required}$$

b Using integration by parts and the result in part **a**

$$\int 6x\arctan 3x \, dx = 3x^2\arctan 3x - \int 3x^2 \times \frac{3}{1+9x^2} dx$$

$$= 3x^2\arctan 3x - \int \frac{9x^2+1-1}{1+9x^2} dx$$

$$= 3x^2\arctan 3x - \int 1\, dx + \int \frac{1}{1+9x^2} dx$$

$$= 3x^2\arctan 3x - x + \frac{1}{3}\arctan 3x$$

You use $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ with $u = \text{artanh}\, 3x$ and $\frac{dv}{dx} = 6x$.

You know $\frac{du}{dx}$ from part **a**.

You have to integrate $\frac{9x^2}{1+9x^2}$. As the degree of the numerator is equal to the degree of the denominator, you must divide the denominator into the numerator before integrating.

The adaptation of the formula given in the Edexcel formulae booklet, $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan\left(\frac{x}{a}\right)$ to this integral is not straightforward.

$$\int \frac{1}{1+9x^2}dx = \frac{1}{9}\int \frac{1}{\frac{1}{9}+x^2}dx$$

$$= \frac{1}{9}\times\frac{1}{\frac{1}{3}}\arctan\left(\frac{x}{\frac{1}{3}}\right) = \frac{1}{3}\arctan 3x.$$

You may prefer to find such an integral using the substitution $3x = \tan\theta$.

$$\left[3x^2\arctan 3x - x + \frac{1}{3}\arctan 3x\right]_0^{\frac{\sqrt{3}}{3}}$$

$$= 3\times\left(\frac{\sqrt{3}}{3}\right)^2 \arctan\sqrt{3} - \frac{\sqrt{3}}{3} + \frac{1}{3}\arctan\sqrt{3}$$

$$= \frac{4}{3}\arctan\sqrt{3} - \frac{\sqrt{3}}{3}$$

$$= \frac{4}{3}\times\frac{\pi}{3} - \frac{\sqrt{3}}{3} = \frac{1}{9}\left(4\pi - 3\sqrt{3}\right), \text{ as required}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 45

Question:

a Starting from the definition of sinh x in terms of e^x, prove that
$\text{arsinh}x = \ln[x + \sqrt{(x^2+1)}]$.

b Prove that the derivative of arsinh x is $(1+x^2)^{-\frac{1}{2}}$.

c Show that the equation $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$ is satisfied when
$y = (\text{arsinh}x)^2$.

d Use integration by parts to find $\int_0^1 \text{arsinh}x \, dx$, giving your answer in terms of a natural logarithm. **[E]**

Solution:

a Let $y = \text{arsinh}x$ then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$$2x = e^y - e^{-y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x + \sqrt{(4x^2+4)}}{2}$$

$$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

Taking the natural logarithms of both sides, $y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

b $y = \text{arsinh}x$

$\sinh y = x$

Differentiating implicitly with respect to x

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\cosh^2 y = 1 + \sinh^2 y = 1 + x^2 \Rightarrow \cosh y = \sqrt{(1+x^2)}$$

Hence $\dfrac{d}{dx}(\text{arsinh}x) = \dfrac{1}{\sqrt{(1+x^2)}} = (1+x^2)^{-\frac{1}{2}}$, as required.

arsinh x is an increasing function of x for all x. So its gradient is always positive and you need not consider the negative square root.

c $y = (\text{arsinh}x)^2$

$$\frac{dy}{dx} = 2\text{arsinh}x(1+x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2(1+x^2)^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + 2\text{arsinh}x \times \left(-\frac{1}{2}\right)(2x)(1+x^2)^{-\frac{3}{2}}$$

$$= 2(1+x^2)^{-1} - 2x\text{arsinh}x(1+x^2)^{-\frac{3}{2}}$$

You use the product rule for differentiation $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ with $u = 1 \;\; 2\text{arsinh}\,x$ and $v = (1+x^2)^{-\frac{1}{2}}$.

Substituting for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ into

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2$$

$$= (1+x^2)\left(2(1+x^2)^{-1} - 2x\text{arsinh}x(1+x^2)^{-\frac{3}{2}}\right) + x \times 2\text{arsinh}x(1+x^2)^{-\frac{1}{2}} - 2$$

$$= 2 - 2x\text{arsinh}x(1+x^2)^{-\frac{1}{2}} + 2x\text{arsinh}x(1+x^2)^{-\frac{1}{2}} - 2$$

$= 0$, as required.

d $\displaystyle\int_0^1 \text{arsinh}x\,dx = \int_0^1 1 \times \text{arsinh}x\,dx$

$$= [x\text{arsinh}x]_0^1 - \int_0^1 \frac{x}{\sqrt{(1+x^2)}}dx$$

$$= \text{arsinh}1 - \left[\sqrt{(1+x^2)}\right]_0^1$$

$$= \ln(1+\sqrt{2}) - \sqrt{2} + 1$$

$\dfrac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right) = \dfrac{1}{2} \times 2x \times (1+x^2)^{-\frac{1}{2}} = \dfrac{x}{\sqrt{(1+x^2)}}$

so $\displaystyle\int \frac{x}{\sqrt{(1+x^2)}}dx = \sqrt{(1+x^2)}$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 46

Question:

a Using the substitution $u = e^x$, find $\int \operatorname{sech} x \, dx$.

b Sketch the curve with equation $y = \operatorname{sech} x$.

The finite region R is bounded by the curve with equation $y = \operatorname{sech} x$, the lines $x = 2, x = -2$ and the x-axis.

c Using your result from **a**, find the area of R, giving your answer to 3 decimal places. **[E]**

Solution:

a $u = e^x \Rightarrow \frac{du}{dx} = e^x = u$

Hence

$$\frac{dx}{du} = \frac{1}{u}$$

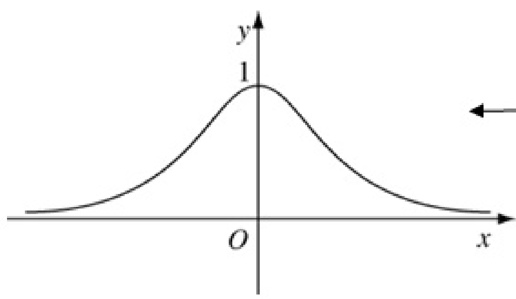
$$\int \text{sech}\,x\,dx = \int \frac{2}{e^x + e^{-x}} \times \frac{dx}{du}\,du$$

$$= \int \frac{2}{u + \frac{1}{u}} \times \frac{1}{u}\,du = \int \frac{2}{u^2 + 1}\,du$$

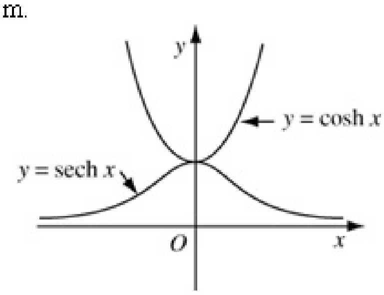
$$= 2\arctan u + A$$

$$= 2\arctan\left(e^x\right) + A$$

b



The specification requires you to know the graphs of cosh and sech. The sketch below illustrates the relation between them.



c Using the symmetry of the curve in **b**, the area, A, of R is given by

$$A = 2\int_0^2 \text{sech}\,x\,dx = \left[4\arctan\left(e^x\right)\right]_0^2$$

$$= 4\arctan\left(e^2\right) - 4\arctan 1$$

$$= 4\arctan\left(e^2\right) - \pi = 2.604 \quad \text{(3 d.p.)}$$

The curve is symmetric, so that the area bounded by the lines $x = -2$ and $x = 2$ is twice the area between the y-axis and the line $x = 2$.

Using the result from part **a**.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 47

Question:

a Prove that $\text{arsinh}x = \ln[x + \sqrt{(x^2+1)}]$.

b i Find, to 3 decimal places, the coordinates of the stationary points on the curve with equation $y = x - 2\text{arsinh}x$.

ii Determine the nature of each stationary point.

iii Hence, sketch the curve with equation $y = x - 2\text{arsinh}x$.

c Evaluate $\int_{-2}^{0}(x - 2\text{arsinh}x)\,dx$. **[E]**

Solution:

a Let $y = \text{arsinh}x$ then $x = \sinh y = \frac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x + \sqrt{(4x^2+4)}}{2}$

$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The negative sign can be ignored in the quadratic formula as it gives e^y negative less possible.

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

The specification requires you to prove this and similar results. Your preparation for the examination should include learning how to prove the formulae which express arsinh x, arcosh x and artanh x as natural logarithms.

b i $y = x - 2\text{arsinh}x$

$\frac{dy}{dx} = 1 - \frac{2}{\sqrt{(1+x^2)}} = 0$

$\sqrt{(1+x^2)} = 2 \Rightarrow 1 + x^2 = 4 \Rightarrow x = \pm\sqrt{3}$

At $x = \sqrt{3}$,

$y = \sqrt{3} - 2\text{arsinh}\sqrt{3} = \sqrt{3} - 2\ln\left(\sqrt{3} + \sqrt{(3+1)}\right)$

$= \sqrt{3} - 2\ln\left(2+\sqrt{3}\right) = -0.902$ (3 d.p.)

At $x = -\sqrt{3}$,

$y = -\sqrt{3} - 2\text{arsinh}\left(-\sqrt{3}\right) = -\sqrt{3} - 2\ln\left(-\sqrt{3} + \sqrt{(3+1)}\right)$

$= -\sqrt{3} - 2\ln\left(2-\sqrt{3}\right) = 0.902$ (3 d.p.)

To 3 decimal places the coordinates of the stationary points are $(1.732, -0.902), (-1.732, 0.902)$.

ii $\frac{dy}{dx} = 1 - 2\left(1+x^2\right)^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = -2\left(-\frac{1}{2}\right)(2x)\left(1+x^2\right)^{-\frac{3}{2}}$

$= \frac{2x}{\left(1+x^2\right)^{\frac{3}{2}}}$

At $x = \sqrt{3}$,

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{3}}{(1+3)^{\frac{3}{2}}} = \frac{\sqrt{3}}{4} > 0 \Rightarrow \text{minimum}$$

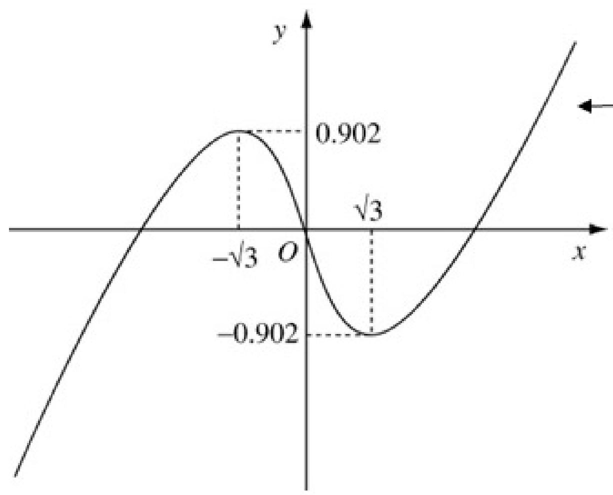
At $x = -\sqrt{3}$,

$$\frac{d^2y}{dx^2} = \frac{-2\sqrt{3}}{(1+3)^{\frac{3}{2}}} = -\frac{\sqrt{3}}{4} < 0 \Rightarrow \text{maximum}$$

These calculations show you that the curve has a maximum point in the second quadrant and a minimum point in the fourth quadrant. This helps you to sketch the graph correctly.

Hence $(1.732, -0.902)$ is a minimum point and $(-1.732, 0.902)$ is a maximum point.

iii



The curve has rotational symmetry about the origin.

c $\int \text{arsinh}\,x\,dx = \int 1 \times \text{arsinh}\,x\,dx$

$$= x\,\text{arsinh}\,x - \int \frac{x}{\sqrt{(1+x^2)}}\,dx$$

$$= x\,\text{arsinh}\,x - \sqrt{(1+x^2)}$$

Integrating arsinh x is not easy in itself and it is a good idea to work this out separately before attempting the whole integral. You integrate arsinh x using parts.

Hence $\int_{-2}^{0} (x - 2\,\text{arsinh}\,x)\,dx$

$$= \left[\frac{x^2}{2} - 2x\,\text{arsinh}\,x + 2\sqrt{(1+x^2)}\right]_{-2}^{0}$$

$$= (2) - (2 + 4\,\text{arsinh}\,(-2) + 2\sqrt{5})$$

$$= -4\ln(-2+\sqrt{5}) - 2\sqrt{5}$$

$$= 1.302 \text{ (3 d.p.)}$$

This exact answer is an acceptable answer to the question but reference to the graph shows the answer should be positive. This is not obvious from the expression and it is worthwhile evaluating it to check your work.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 48

Question:

Use the substitution $e^x = t - \frac{3}{5}$, or otherwise, to find $\int \frac{1}{3+5\cosh x} dx$. **[E]**

Solution:

If $e^x = t - \frac{3}{5}$,

then $e^x \frac{dx}{dt} = 1$

and $e^{2x} = \left(t - \frac{3}{5}\right)^2 = t^2 - \frac{6}{5}t + \frac{9}{25}$

Differentiate the equation implicitly with respect to t.

$$\int \frac{1}{3+5\cosh x} dx = \int \frac{1}{3+5\left(\frac{e^x+e^{-x}}{2}\right)} dx$$

$$= \int \frac{2e^x}{6e^x+5e^{2x}+5} dx$$

Multiply the numerator and denominator of the right hand side of this equation by $2e^x$.

$$= \int \frac{2}{5e^{2x}+6e^x+5}\left(e^x \frac{dx}{dt}\right) dt$$

$$= \int \frac{2}{5\left(t^2-\frac{6}{5}t+\frac{9}{25}\right)+6\left(t-\frac{3}{5}\right)+5}(1)\, dt$$

$$= \int \frac{2}{5t^2-6t+\frac{9}{5}+6t-\frac{18}{5}+5} dt$$

$$= \int \frac{2}{5t^2+\frac{16}{5}} dt = \frac{2}{5}\int \frac{1}{t^2+\frac{16}{25}} dt$$

$$= \frac{2}{5} \times \frac{1}{\frac{4}{5}} \arctan\left(\frac{t}{\frac{4}{5}}\right) + A$$

Using the standard result $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$ with $a = \frac{4}{5}$.

$$= \frac{1}{2}\arctan\left(\frac{5}{4}\left(e^x+\frac{3}{5}\right)\right) + A = \frac{1}{2}\arctan\left(\frac{5e^x+3}{4}\right) + A$$

Remember to return to the original variable, which is x not t.

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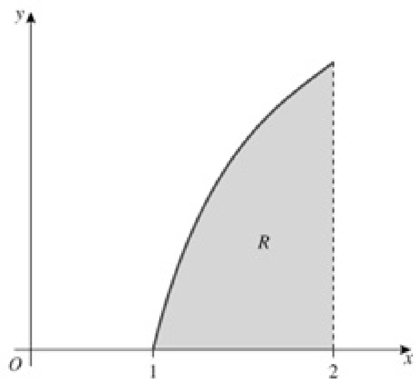
Review Exercise 1
Exercise A, Question 49

Question:



The figure above shows a sketch of the curve with equation $y = x\text{arcosh}x, 1 \le x \le 2$.
The region R, shaded in the figure, is bounded by the curve, the x-axis and the line $x = 2$.

Show that the area of R is $\frac{7}{4}\ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$. **[E]**

Solution:

$$\int x \operatorname{arcosh} x \, dx = \frac{x^2}{2}\operatorname{arcosh} x - \int \frac{x^2}{2\sqrt{(x^2-1)}}\, dx$$

This solution uses integration by parts, $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$, with $u = \operatorname{arcosh} x$ and $\frac{dv}{dx} = x$.
There are other possible approaches to this question, for example, substituting $u = \operatorname{arcosh} x$.

To find the remaining integral, let $x = \cosh\theta$.

$$\frac{dx}{d\theta} = \sinh\theta$$

$$\int \frac{x^2}{2\sqrt{(x^2-1)}}\, dx = \int \frac{\cosh^2\theta}{2\sqrt{(\cosh^2\theta-1)}}\left(\frac{dx}{d\theta}\right) d\theta$$

$$= \int \frac{\cosh^2\theta}{2\sinh\theta}\sinh\theta \, d\theta = \frac{1}{2}\int \cosh^2\theta \, d\theta$$

$$= \frac{1}{4}\int (\cosh 2\theta + 1)\, d\theta$$

Using the identity $\cosh 2\theta = 2\cosh^2\theta - 1$.

$$= \frac{\sinh 2\theta}{8} + \frac{\theta}{4} = \frac{\sinh\theta\cosh\theta}{4} + \frac{\theta}{4}$$

$$= \frac{\left[\sqrt{(x^2-1)}\right]x}{4} + \frac{1}{4}\operatorname{arcosh} x$$

$\sinh\theta = \sqrt{(\cosh^2\theta - 1)} = \sqrt{(x^2-1)}$

Hence the area, A, of R is given by

$$A = \left[\frac{x^2}{2}\operatorname{arcosh} x - \frac{1}{4}x\sqrt{(x^2-1)} - \frac{1}{4}\operatorname{arcosh} x\right]_1^2$$

$$= \left[\left(\frac{x^2}{2} - \frac{1}{4}\right)\operatorname{arcosh} x - \frac{1}{4}x\sqrt{(x^2-1)}\right]_1^2$$

$$= \left[\frac{7}{4}\operatorname{arcosh} 2 - \frac{\sqrt{3}}{2}\right] - [0]$$

As $\operatorname{arcosh} 1 = 0$ and $\sqrt{(1^2-1)} = 0$, both terms are zero at the lower limit.

$$= \frac{7}{4}\ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 50

Question:

$4x^2+4x+5 \equiv (px+q)^2+r$

a Find the values of the constants p, q and r.

b Hence, or otherwise, find $\int \frac{1}{4x^2+4x+5}\,dx$.

c Show that $\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \ln[(2x+1)+\sqrt{(4x^2+4x+5)}]+k$, where k is an arbitrary constant. **[E]**

Solution:

a $4x^2+4x+5 = (px+q)^2+r$

$= p^2x^2+2pqx+q^2+r$

Equating coefficients of x^2

$4 = p^2 \Rightarrow p = 2$

Equating coefficients of x

$4 = 2pq = 4q \Rightarrow q = 1$

Equating constant coefficients

$5 = q^2+r = 1+r \Rightarrow r = 4$

$p = 2, q = 1, r = 4$

The conditions of the question allow $p = -2$ as an answer, but the negative sign would make the integrals following awkward, so choose the positive root.

b $\int \frac{1}{4x^2+4x+5}\,dx = \int \frac{1}{(2x+1)^2+4}\,dx$

If you know a formula of the type $\int \frac{1}{a^2x^2+b^2}\,dx = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right)$, or you are confident at writing down integrals by inspection, you may be able to find this integral without working. It is, however, very easy to make errors with the constant and get, for example, the common error $\frac{1}{2}\arctan\left(\frac{2x+1}{2}\right)+C$.

Let $2x+1 = 2\tan\theta$

$$2\frac{dx}{d\theta} = 2\sec^2\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$$

$$\int \frac{1}{(2x+1)^2+4}\,dx = \int \frac{1}{4\tan^2\theta+4}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{1}{4\cancel{\sec^2\theta}}\left(\cancel{\sec^2\theta}\right)d\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\arctan\left(\frac{2x+1}{2}\right)+C$$

c $\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \int \frac{2}{\sqrt{((2x+1)^2+4)}}\,dx$

As in part **b**, you may be able to write down this integral without working.

Let $2x+1 = 2\sinh\theta$

$$2\frac{dx}{d\theta} = 2\cosh\theta \Rightarrow \frac{dx}{d\theta} = \cosh\theta$$

$$\int \frac{2}{\sqrt{((2x+1)^2+4)}}\,dx = \int \frac{2}{\sqrt{(4\sinh^2\theta+4)}}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{2}{2\cosh\theta}(\cosh\theta)\,d\theta = \int 1\,d\theta$$

$$= \theta + C = \text{arsinh}\left(\frac{2x+1}{2}\right)+C$$

Using $\text{arsinh}\,x = \ln\left(x+\sqrt{(x^2+1)}\right)$

$$\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{(2x+1)^2}{4}+1\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{4x^2+4x+1+4}{4}\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\frac{1}{2}\sqrt{(4x^2+4x+5)}\right]+C$$

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]-\ln 2+C$$

$-\ln 2 +$ an arbitrary constant is another arbitrary constant.

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]+k, \text{ as required.}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 51

Question:

Using the substitution $x = 2\cosh^2 t - \sinh^2 t$, evaluate $\int_2^3 (x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}\,dx$. **[E]**

Solution:

If $x = 2\cosh^2 t - \sinh^2 t$ then

$$x-1 = 2\cosh^2 t - \sinh^2 t - 1$$
$$= 2\cosh^2 t - \left(1+\sinh^2 t\right)$$
$$= 2\cosh^2 t - \cosh^2 t = \cosh^2 t$$

Simplify using $1+\sinh^2 t = \cosh^2 t$.

$$x-2 = 2\cosh^2 t - \sinh^2 t - 2$$
$$= 2\left(\cosh^2 t - 1\right) - \sinh^2 t$$
$$= 2\sinh^2 t - \sinh^2 t = \sinh^2 t$$

Simplify using $\cosh^2 t - 1 = \sinh^2 t$.

$$\frac{dx}{dt} = 4\cosh t \sinh t - 2\cosh t \sinh t = 2\cosh t \sinh t$$

Substituting into the integral

$$\int (x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}\,dx = \int \left(\cosh^2 t\right)^{\frac{1}{2}}\left(\sinh^2 t\right)^{\frac{1}{2}}\frac{dx}{dt}\,dt$$
$$= \int \cosh t \sinh t\left(2\cosh t \sin t\right)dt$$
$$= \int 2\left(\cosh t \sinh t\right)^2\,dt$$
$$= \frac{1}{2}\int \sinh^2 2t\,dt = \frac{1}{4}\int\left(\cosh 4t - 1\right)dt$$
$$= \frac{1}{16}\sinh 4t - \frac{t}{4}$$

To find the integral you need the hyperbolic identities $\sinh 2t = 2\sinh t \cosh t$ and $\cosh 4t = 1 + 2\sinh^2 2t$.

For the limits

At $x = 2$

$$2 = 2\cosh^2 t - \sinh^2 t = \cosh^2 t + \left(\cosh^2 t - \sinh^2 t\right)$$
$$2 = \cosh^2 t + 1 \Rightarrow \cosh t = 1 \Rightarrow t = 0$$

At $x = 3$

$$3 = 2\cosh^2 t - \sinh^2 t = \cosh^2 t + \left(\cosh^2 t - \sinh^2 t\right)$$
$$3 = \cosh^2 t + 1 \Rightarrow \cosh^2 t = 2$$
$$\cosh t = \sqrt{2} \Rightarrow t = \ln\left(\sqrt{2}+1\right)$$
$$\sinh t = \sqrt{\left(\cosh^2 t - 1\right)} = \sqrt{(2-1)} = 1$$

Hence at $x = 3$

Using the formula

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{\left(x^2-1\right)}\right),$$
$$\operatorname{arcosh}\sqrt{2} = \ln\left(\sqrt{2} + \sqrt{(2-1)}\right)$$
$$= \ln\left(\sqrt{2}+1\right)$$

$$\frac{1}{16}\sinh 4t = \frac{1}{8}\sinh 2t \cosh 2t$$
$$= \frac{1}{8}\left(2\sinh t\cosh t\right)\left(1+2\sinh^2 t\right)$$
$$= \frac{1}{8}\left(2\sqrt{2}\right)(1+2) = \frac{3\sqrt{2}}{4}$$

The evaluation of $\frac{1}{16}\sinh 4t$ at the upper limit requires the use of two hyperbolic double angle formulae and it is a good idea to work this out as a separate calculation before attempting the complete integral.

Hence

$$\int_2^3 (x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}\,dx = \left[\frac{1}{16}\sinh 4t - \frac{t}{4}\right]_0^{\ln(\sqrt{2}+1)}$$
$$= \frac{3\sqrt{2}}{4} - \frac{1}{4}\ln\left(\sqrt{2}+1\right)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 52

Question:

$\mathrm{f}(x) = \arcsin x$

a Show that $\mathrm{f}'(x) = \dfrac{1}{\sqrt{(1-x^2)}}$.

b Given that $y = \arcsin 2x$, obtain $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as an algebraic fraction.

c Using the substitution $x = \dfrac{1}{2}\sin\theta$, show that $\displaystyle\int_0^{\frac{1}{4}} \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}}\,\mathrm{d}x = \frac{1}{48}(6-\pi\sqrt{3})$. **[E]**

Solution:

a Let $y = \mathrm{f}(x) = \arcsin x$

$\sin y = x$

Differentiating implicitly with respect to x

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$$

$$\mathrm{f}'(x) = \frac{1}{\sqrt{(1-x^2)}}, \text{ as required}$$

Unless otherwise stated, arcsin x is taken to have the range $-\frac{\pi}{2} < \arcsin x < \frac{\pi}{2}$. These are the principal values of arcsin x. In this range, arcsin x is an increasing function of x, $\frac{dy}{dx}$ is positive and you can take the positive value of the square root.

b $y = \arcsin 2x$

Let $u = 2x, \frac{du}{dx} = 2$

$y = \arcsin u$

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{(1-u^2)}} \times 2 = \frac{2}{\sqrt{(1-4x^2)}}$$

c $x = \frac{1}{2}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\cos\theta$

At $x = \frac{1}{4}, \frac{1}{4} = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

At $x = 0, 0 = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

In this question it is convenient to carry out the substitution without returning to the original variable x. So at some stage you must change the x limits to θ limits.

$$\int \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}} dx = \int \frac{\frac{1}{2}\sin\theta \arcsin(\sin\theta)}{\sqrt{(1-\sin^2\theta)}} \left(\frac{dx}{d\theta}\right) d\theta$$

$$= \int \frac{\frac{1}{2}\sin\theta \times \theta}{\cos\theta}\left(\frac{1}{2}\cos\theta\right) d\theta$$

By definition, $\arcsin(\sin\theta) = \theta$.

$$= \frac{1}{4}\int \theta \sin\theta \, d\theta$$

$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\int \cos\theta \, d\theta$$

$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta$$

You use integration by parts, $\int u\frac{dv}{d\theta} = uv - \int v\frac{du}{d\theta}\, d\theta$, with $u = \theta$ and $\frac{dv}{d\theta} = \sin\theta$.

Hence

$$\int_0^{\frac{1}{4}} \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}} dx = \left[-\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left[-\frac{\pi}{24}\cos\frac{\pi}{6} + \frac{1}{4}\sin\frac{\pi}{6}\right] - [0]$$

$$= -\frac{\pi}{24} \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{48}\left(6 - \pi\sqrt{3}\right), \text{ as required.}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 53

Question:

a Show that $\text{artanh}\left(\sin\frac{\pi}{4}\right) = \ln(1+\sqrt{2})$.

b Given that $y = \text{artanh}(\sin x)$, show that $\frac{dy}{dx} = \sec x$.

c Find the exact value of $\int_0^{\frac{\pi}{4}} \sin x \, \text{artanh}(\sin x) \, dx$. **[E]**

Solution:

a Using $\text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$$\text{artanh}\left(\sin\frac{\pi}{4}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)$$

Multiply the numerator and denominator of this fraction by $\sqrt{2}$.

$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Rationalise the denominator of the fraction by multiplying the numerator and denominator by $(\sqrt{2}+1)$.

$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{1}{2}\ln\left(\frac{(\sqrt{2}+1)^2}{1}\right)$$

$$= 2\times\frac{1}{2}\ln(1+\sqrt{2}) = \ln(1+\sqrt{2}), \text{ as required}$$

Use the property of logarithms $\ln a^2 = 2\ln a$.

b $y = \text{artanh}(\sin x)$

Let $u = \sin x$, then $\frac{du}{dx} = \cos x$

$y = \text{artanh}\,u$

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

$$= \frac{1}{1-u^2}\times\cos x$$

The formula $\frac{d}{dx}(\text{artanh}\,x) = \frac{1}{1-x^2}$ is given in the Edexcel formulae booklet which is provided for use with the paper.

$$= \frac{1}{1-\sin^2 x}\times\cos x = \frac{\cos x}{\cos^2 x}$$

$$= \frac{1}{\cos x} = \sec x, \text{ as required.}$$

c $\int \sin x\,\text{artanh}(\sin x)\,dx$

From part **a**, you know that you can differentiate artanh $(\sin x)$ and this suggests using integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$, with $u = \text{artanh}(\sin x)$ and $\frac{dv}{dx} = \sin x$.

$$= -\text{artanh}(\sin x)\cos x + \int \cos x \sec x\,dx$$

$$= -\text{artanh}(\sin x)\cos x + \int 1\,dx$$

$$= -\text{artanh}(\sin x)\cos x + x$$

Hence

$$\int_0^{\frac{\pi}{4}} \sin x\,\text{artanh}(\sin x)\,dx = \left[-\text{artanh}(\sin x)\cos x + x\right]_0^{\frac{\pi}{4}}$$

$$= \left[-\text{artanh}\left(\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} + \frac{\pi}{4}\right] - [0]$$

You evaluate the upper limit using the result proved in part **a** that $\text{artanh}\left(\sin\frac{\pi}{4}\right) = \ln(1+\sqrt{2})$.

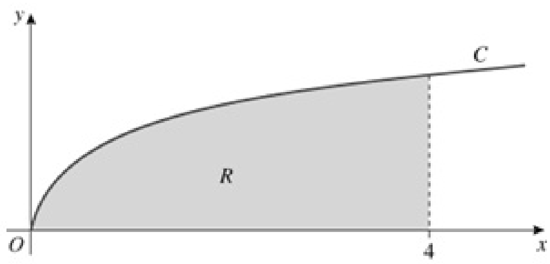
$$= -\ln(1+\sqrt{2})\times\frac{1}{\sqrt{2}} + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{\sqrt{2}}{2}\ln(1+\sqrt{2})$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 54

Question:



The figure shows part of the curve C with equation $y = \text{arsinh}(\sqrt{x}), x \geq 0$.

a Find the gradient of C at the point where $x = 4$.

The region R, shown shaded in the figure, is bounded by C, the x-axis and the line $x = 4$.

b Using the substitution $x = \sinh^2 \theta$, or otherwise, show that the area of R is $k \ln(2 + \sqrt{5}) - \sqrt{5}$, where k is a constant. **[E]**

Solution:

a $y = \text{arsinh}\left(\sqrt{x}\right)$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$y = \text{arsinh}u$

Using the chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{\sqrt{(u^2+1)}} \times \frac{1}{2}x^{-\frac{1}{2}}$

As $u = x^{\frac{1}{2}}$, then $u^2 = x$ and $\sqrt{(u^2+1)} = \sqrt{(x+1)}$.

$= \frac{1}{2\sqrt{x}\sqrt{(x+1)}}$

At $x = 4$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4}\sqrt{(4+1)}} = \frac{1}{4\sqrt{5}} = \frac{\sqrt{5}}{20}$

b If $x = \sinh^2\theta, \frac{dx}{d\theta} = 2\sinh\theta\cosh\theta = \sinh 2\theta$

$\int \text{arsinh}\sqrt{x}\,dx = \int \text{arsinh}\left(\sqrt{\left(\sinh^2\theta\right)}\right) \times \frac{dx}{d\theta}\,d\theta$

$= \int \text{arsinh}(\sinh\theta) \times \sinh 2\theta\,d\theta$

By definition $\text{arsinh}(\sinh\theta) = \theta$.

$= \int \theta \sinh 2\theta\,d\theta$

You use integration by parts, $\int u\frac{dv}{d\theta}\,d\theta = uv - \int v\frac{du}{d\theta}\,d\theta$, with $u = \theta$ and $\frac{dv}{d\theta} = \sinh 2\theta$.

$= \frac{\theta\cosh 2\theta}{2} - \int \frac{\cosh 2\theta}{2}\,d\theta$

$= \frac{\theta\cosh 2\theta}{2} - \frac{\sinh 2\theta}{4}$

$= \frac{\theta\left(1+2\sinh^2\theta\right)}{2} - \frac{2\sinh\theta\cosh\theta}{4}$

This solution uses double angle formulae to transform the expression back to the original variable x before substituting in the limits.

$= \frac{\text{arsinh}\left(\sqrt{x}\right)(1+2x)}{2} - \frac{\sqrt{x}\sqrt{(1+x)}}{2}$

Hence the area, A, of R is given by

$A = \left[\frac{\text{arsinh}\left(\sqrt{x}\right)(1+2x)}{2} - \frac{\sqrt{x}\sqrt{(1+x)}}{2}\right]_0^4$

$= \left[\frac{\text{arsinh}(2)(9)}{2} - \frac{2\sqrt{(5)}}{2}\right] - [0]$

$= \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$

This is the required result with $k = \frac{9}{2}$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 55

Question:

$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx, n \geq 0$.

a Prove that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}, n \geq 2$.

b Find an exact expression for I_6. **[E]**

Solution:

a $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$

$= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x \, dx$

$= \left(\frac{\pi}{2}\right)^n + \left[nx^{n-1} \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \cos x \, dx$

$= \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$, as required.

You use integration by parts, $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = x^n$ and $\frac{dv}{dx} = \cos x$. Then $v = \sin x$.

You repeat integration by parts this time with $u = nx^{n-1}$ and $\frac{dv}{dx} = \sin x$.

This expression is zero at both the lower and upper limit.

b $I_6 = \left(\frac{\pi}{2}\right)^6 - 6 \times 5 I_4$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\left(\frac{\pi}{2}\right)^4 - 4 \times 3 I_2\right)$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360 I_2$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\left(\frac{\pi}{2}\right)^2 - 2 \times 1 I_0\right)$

$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720 I_0$

This is the result of part a with 6 substituted for n. You have now reduced the integral to $n = 4$. You then repeat the procedure until you reach an integral which you can evaluate directly.

$I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} = 1 - 0 = 1$

$I_0 = \int_0^{\frac{\pi}{2}} x^0 \cos x \, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx$ and as the integral of $\cos x$ is $\sin x$ you can work this out without further use of the reduction formula.

Hence

$I_6 = \left(\frac{\pi}{2}\right)^6 - 360\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 56

Question:

Given that $I_n = \int_0^4 x^n \sqrt{(4-x)}\,dx, n \geq 0$.

a show that $I_n = \frac{8n}{2n+3} I_{n-1}, n \geq 1$.

Given that $\int_0^4 \sqrt{(4-x)}\,dx = \frac{16}{3}$,

b use the result in part **a** to find the exact value of $\int_0^4 x^2 \sqrt{(4-x)}\,dx$. **[E]**

Solution:

a $I_n = \int_0^4 x^n \sqrt{(4-x)}\,dx$

$= \left[-\frac{2}{3}(4-x)^{\frac{3}{2}} x^n\right]_0^4 + \frac{2}{3}\int_0^4 nx^{n-1}(4-x)^{\frac{3}{2}}\,dx$

$= \frac{2}{3}\int_0^4 nx^{n-1}(4-x)(4-x)^{\frac{1}{2}}\,dx$

You use integration by parts, $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, with $u = x^n$ and $\frac{dv}{dx} = (4-x)^{\frac{1}{2}}$. Then

$v = \int (4-x)^{\frac{1}{2}}\,dx = \frac{(4-x)^{\frac{3}{2}}}{-1\times\frac{3}{2}} = -\frac{2}{3}(4-x)^{\frac{3}{2}}.$

$= \frac{2}{3}\int_0^4 nx^{n-1}4(4-x)^{\frac{1}{2}}\,dx - \frac{2}{3}\int_0^4 nx^{n-1}x(4-x)^{\frac{1}{2}}\,dx$

$= \frac{8n}{3}\int_0^4 x^{n-1}(4-x)^{\frac{1}{2}}\,dx - \frac{2n}{3}\int_0^4 x^n(4-x)^{\frac{1}{2}}\,dx$

$= \frac{8n}{3}I_{n-1} - \frac{2n}{3}I_n$

You split this integral into two separate integrals using

$$\begin{aligned}(4-x)^{\frac{3}{2}} &= (4-x)^1(4-x)^{\frac{1}{2}}\\ &= (4-x)(4-x)^{\frac{1}{2}}\\ &= 4(4-x)^{\frac{1}{2}} - x(4-x)^{\frac{1}{2}}\end{aligned}$$

Hence

$I_n + \frac{2n}{3}I_n =$ so $\frac{3+2n}{3}I_n = \frac{8n}{3}I_{n-1}$

$I_n = \frac{8n}{2n+3}I_{n-1}$, as required.

Collect the terms in I_n on one side of the equation and solve for I_n in terms of n and I_{n-1}.

b $I_2 = \frac{8\times 2}{2\times 2+3}I_1 = \frac{16}{7}I_1$

$= \frac{16}{7}\times\frac{8\times 1}{2\times 1+3}I_0 = \frac{16}{7}\times\frac{8}{5}I_0$

$= \frac{16}{7}\times\frac{8}{5}\times\frac{16}{3} = \frac{2048}{105}$

This is the result of part a with 2 substituted for n. You have now reduced the integral to $n = 1$. You then repeat the procedure reaching $n = 0$ and, in this question, you have been given I_0.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 57

Question:

Given that $y = \sinh^{n-1} x \cosh x$,

a show that $\frac{dy}{dx} = (n-1)\sinh^{n-2} x + n\sinh^n x$.

The integral I_n is defined by $I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x \, dx, n \geq 0$.

b Using the result in part **a**, or otherwise, show that $nI_n = \sqrt{2} - (n-1)I_{n-2}, n \geq 2$.

c Hence find the value of I_4. **[E]**

Solution:

a $y = \sinh^{n-1} x \cosh x$

$$\frac{dy}{dx} = (n-1)\sinh^{n-2} x \cosh x \times \cosh x + \sinh^{n-1} x \times \sinh x$$

Using the product rule for differentiation.

$$= (n-1)\sinh^{n-2} x \cosh^2 x + \sinh^n x$$
$$= (n-1)\sinh^{n-2} x (1 + \sinh^2 x) + \sinh^n x$$
$$= (n-1)\sinh^{n-2} x + (n-1)\sinh^n x + \sinh^n x$$
$$= (n-1)\sinh^{n-2} x + n\sinh^n x, \text{ as required.}$$

You use the identity $\cosh^2 x - \sinh^2 x = 1$ to write this expression in terms of the powers of $\sinh x$ only.

b Integrating the result of part **a** throughout with respect to x.

$$\int \frac{dy}{dx} dx = \int (n-1)\sinh^{n-2} x \, dx + \int n\sinh^n x \, dx$$
$$y = \int (n-1)\sinh^{n-2} x \, dx + \int n\sinh^n x \, dx$$
$$\sinh^{n-1} x \cosh x = \int (n-1)\sinh^{n-2} x \, dx + \int n\sinh^n x \, dx$$

As integration is the reverse process of differentiation $\int \frac{dy}{dx} dx = y$ and, in this question, $y = \sinh^{n-1} x \cosh x$.

Between the limits 0 and arsinh 1

$$\left[\sinh^{n-1} x \cosh x\right]_0^{\text{arsinh}1} = \int_0^{\text{arsinh}1} (n-1)\sinh^{n-2} x \, dx + \int_0^{\text{arsinh}1} n\sinh^n x \, dx$$
$$1 \times \sqrt{2} - 0 = (n-1)I_{n-2} + nI_n$$
$$nI_n = \sqrt{2} - (n-1)I_{n-2}, \text{ as required.}$$

$x = \text{arsinh}1 \Rightarrow \sinh x = 1$ and, as $\cosh^2 x = 1 + \sinh^2 x$, then $\cosh^2 1 = 1 + \sinh^2 x = 2 \Rightarrow \cosh 1 = \sqrt{2}$

c From part **b** $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$

Hence

$$I_4 = \frac{\sqrt{2}}{4} - \frac{3}{4} I_2$$
$$= \frac{\sqrt{2}}{4} - \frac{3}{4}\left(\frac{\sqrt{2}}{2} - \frac{1}{2} I_0\right) = \frac{3}{8} I_0 - \frac{\sqrt{2}}{8}$$
$$I_0 = \int_0^{\text{arsinh}1} \sinh^0 x \, dx = \int_0^{\text{arsinh}1} 1 \, dx$$
$$= [x]_0^{\text{arsinh}1} = \text{arsinh}1 - 0 = \ln(1 + \sqrt{2})$$

Using $\text{arsinh} x = \ln\left[x + \sqrt{(x^2 + 1)}\right]$ with $x = 1$.

Hence

$$I_4 = \frac{1}{8}\left(3\ln(1+\sqrt{2}) - \sqrt{2}\right)$$

It is usual to give values involving inverse hyperbolic functions in terms of natural logarithms but, as this question specifies no form of the answer, $\frac{1}{8}(3\,\text{arsinh}1 - \sqrt{2})$ would be acceptable.

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Review Exercise 1
Exercise A, Question 58

Question:

Given that $I_n = \int_0^8 x^n (8-x)^{\frac{1}{3}} \mathrm{d}x, n \geq 0$,

a show that $I_n = \frac{24n}{3n+4} I_{n-1}, n \geq 1$.

b Hence find the exact value of $\int_0^8 x(x+5)(8-x)^{\frac{1}{3}} \mathrm{d}x$. **[E]**

Solution:

a $I_n = \int_0^8 x^n(8-x)^{\frac{1}{3}}\,dx$

$= \left[x^n\left(-\frac{3}{4}\right)(8-x)^{\frac{4}{3}}\right]_0^8 - \int_0^8 nx^{n-1}\left(-\frac{3}{4}\right)(8-x)^{\frac{4}{3}}\,dx$

$= \frac{3n}{4}\int_0^8 x^{n-1}(8-x)(8-x)^{\frac{1}{3}}\,dx$

$= \frac{3n}{4}\int_0^8 x^{n-1}(8)(8-x)^{\frac{1}{3}}\,dx - \frac{3n}{4}\int_0^8 x^{n-1}(x)(8-x)^{\frac{1}{3}}\,dx$

$= 6n\int_0^8 x^{n-1}(8-x)^{\frac{1}{3}}\,dx - \frac{3n}{4}\int_0^8 x^n(8-x)^{\frac{1}{3}}\,dx$

$I_n = 6nI_{n-1} - \frac{3n}{4}I_n$

$\left(1+\frac{3n}{4}\right)I_n = \therefore \frac{4+3n}{4}I_n = 6nI_{n-1}$

$I_n = \frac{24n}{3n+4}I_{n-1}$

You use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, with $u = x^n$ and $\frac{dv}{dx} = (8-x)^{\frac{1}{3}}$.

$v = \int (8-x)^{\frac{1}{3}}\,dx = \frac{(8-x)^{\frac{4}{3}}}{-\frac{4}{3}} = -\frac{3}{4}(8-x)^{\frac{4}{3}}$

You split this integral into two separate integrals using

$(8-x)^{\frac{4}{3}} = (8-x)^1(8-x)^{\frac{1}{3}}$

$= (8-x)(8-x)^{\frac{1}{3}}$

$= 8(8-x)^{\frac{1}{3}} - x(8-x)^{\frac{1}{3}}$.

Collect the terms in I_n on one side of the equation and solve for I_n in terms of n and I_{n-1}.

b $\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\,dx = \int_0^8 (x^2+5x)(8-x)^{\frac{1}{3}}\,dx$

$= \int_0^8 x^2(8-x)^{\frac{1}{3}}\,dx + 5\int_0^8 x(8-x)^{\frac{1}{3}}\,dx$

$= I_2 + 5I_1$

$I_0 = \int_0^8 (8-x)^{\frac{1}{3}}\,dx = \left[\frac{(8-x)^{\frac{4}{3}}}{-\frac{4}{3}}\right]_0^8$

$= \left[-\frac{3}{4}(8-x)^{\frac{4}{3}}\right]_0^8 = 0 - \left(-\frac{3}{4}\times 8^{\frac{4}{3}}\right)$

$= \frac{3}{4}\times 16 = 12$

Using the result of part **a**

$I_1 = \frac{24}{7}I_0 = \frac{24}{7}\times 12 = \frac{288}{7}$

$I_2 = \frac{48}{10}I_1 = \frac{48}{10}\times\frac{288}{7} = \frac{6912}{35}$

These fractions are awkward. Use your calculator to manipulate the fractions.

$\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\,dx = I_2 + 5I_1$

$= \frac{6912}{35} + 5\times\frac{288}{7} = \frac{2016}{5}$

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Review Exercise 1
Exercise A, Question 59

Question:

$I_n = \int \frac{\sin nx}{\sin x} \mathrm{d}x \; n > 0, n \in \mathbb{Z}$.

a By considering $I_{n+2} - I_n$, or otherwise, show that $I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n$.

b Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x} \mathrm{d}x$, giving your answer in the form $p\sqrt{2} + q\sqrt{3}$, where p and q are rational numbers to be found. **[E]**

Solution:

a $I_{n+2} - I_n = \int \frac{\sin(n+2)x}{\sin x}\,dx - \int \frac{\sin nx}{\sin x}\,dx$

$= \int \frac{\sin(n+2)x - \sin nx}{\sin x}\,dx$

$= \int \frac{2\cos(n+1)x\sin x}{\sin x}\,dx$

$= \int 2\cos(n+1)x\,dx$

$= \frac{2\sin(n+1)x}{n+1}$

Hence

$I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n$, as required.

You use the trigonometric identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ with $A = (n+2)x$ and $B = nx$. The identity can be found among the formulae for module C3 in the Edexcel formulae booklet which is provided for use in the examination. The specification for FP3 requires knowledge of the specifications for C1, C2, C3, C4 and FP1 and their associated formulae.

b Using the result in part **a**

$I_6 = \frac{2\sin 5x}{5} + I_4$

$= \frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + I_2$

$I_2 = \int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx$

$= \int 2\cos x\,dx = 2\sin x + C$

I_2 can be found directly. You should not reduce the integral to I_0 as the first line of the question specifies $n > 0$.

The constant of integration will disappear when limits are applied.

Hence

$$I_6 = \frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + 2\sin x + C$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\,dx = \left[\frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + 2\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\frac{2}{5}\times -\frac{\sqrt{3}}{2} + \frac{2}{3}\times 0 + 2\times\frac{\sqrt{3}}{2}\right) - \left(\frac{2}{5}\times -\frac{\sqrt{2}}{2} + \frac{2}{3}\times\frac{\sqrt{2}}{2} + 2\times\frac{\sqrt{2}}{2}\right)$$

$$= \frac{4}{5}\sqrt{3} - \frac{17}{15}\sqrt{2}$$

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Review Exercise 1
Exercise A, Question 60

Question:

$I_n = \int_0^1 x^n \, e^x \, dx$ and $J_n = \int_0^1 x^n \, e^{-x} dx, n \geq 0$.

a Show that, for $n \geq 1$, $I_n = e - nI_{n-1}$.

b Find a similar formula for J_n.

c Show that $J_2 = 2 - \frac{5}{e}$.

d Show that $\int_0^1 x^n \cosh x dx = \frac{1}{2}(I_n + J_n)$.

e Hence, or otherwise, evaluate $\int_0^1 x^2 \cosh x dx$, giving your answer in terms of e. **[E]**

Solution:

a $I_n = \int_0^1 x^n \, e^x \, dx$

$= \left[x^n \, e^x\right]_0^1 - \int_0^1 nx^{n-1} \, e^x \, dx$

$= e^1 - 0 - n\int_0^1 x^{n-1} e^x \, dx$

$= e - nI_{n-1}$, as required.

You use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = x^n$ and $\frac{dv}{dx} = e^x$.

b $J_n = \int_0^1 x^n \, e^{-x} \, dx$

$= \left[-x^n \, e^{-x}\right]_0^1 + \int_0^1 nx^{n-1} \, e^{-x} \, dx$

$= -e^{-1} - 0 + n\int_0^1 x^{n-1} \, e^{-x} dx$

$= -e^{-1} + nJ_{n-1}$

As you are asked to find a similar formula, it is sensible to pattern your solution to part **b** on that of part **a**. Here you use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$, with $u = x^n$ and $\frac{dv}{dx} = e^{-x}$.

c $J_2 = -e^{-1} + 2J_1$

$= -e^{-1} + 2\left(-e^{-1} + J_0\right) = -3e^{-1} + 2J_0$

$J_0 = \int_0^1 x^0 e^{-x} dx = \int_0^1 e^{-x} dx$

$= \left[-e^{-x}\right]_0^1 = -e^{-1} - (-1) = 1 - e^{-1}$

You use the result of part **b** twice and evaluate J_0 directly.

Hence

$J_2 = -3e^{-1} + 2\left(1 - e^{-1}\right) = 2 - 5e^{-1}$

$= 2 - \frac{5}{e}$, as required.

d $\int_0^1 x^n \cosh x dx = \int_0^1 x^n \left(\frac{e^x + e^{-x}}{2}\right) dx$

$= \frac{1}{2}\int_0^1 x^n e^x dx + \frac{1}{2}\int_0^1 x^n e^{-x} dx$

$= \frac{1}{2} I_n + \frac{1}{2} J_n = \frac{1}{2}\left(I_n + J_n\right)$, as required.

e $I_2 = \mathrm{e} - 2I_1$

$= \mathrm{e} - 2(\mathrm{e} - I_0) = -\mathrm{e} + 2I_0$

$I_0 = \int_0^1 x^0 \mathrm{e}^x \, dx = \int_0^1 \mathrm{e}^x \, dx$

$= \left[\mathrm{e}^x\right]_0^1 = \mathrm{e}^1 - 1 = \mathrm{e} - 1$

You use the result of part **a** twice and evaluate I_0 directly.

Hence

$I_2 = -\mathrm{e} + 2(\mathrm{e} - 1) = \mathrm{e} - 2$

$\int_0^1 x^2 \cosh x \, dx = \frac{1}{2}(I_2 + J_2)$

$= \frac{1}{2}\left(\mathrm{e} - 2 + 2 - \frac{5}{\mathrm{e}}\right) = \frac{1}{2}\left(\mathrm{e} - \frac{5}{\mathrm{e}}\right)$

This is the result of part **d** with $n = 2$.

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 61

Question:

Given that $I_n = \int \sec^n x \, dx$,

a show that $(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}, n \geq 2$.

b Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^3 x \, dx$, giving your answer in terms of natural logarithms and surds. **[E]**

Solution:

a $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \times \tan x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$

$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$

$I_n + (n-2)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$

$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$, as required.

You use $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$, with $u = \sec^{n-2} x$ and $\frac{dv}{dx} = \sec^2 x$.

Using the chain rule $\frac{d}{dx}\left(\sec^{n-2} x\right)$

$= (n-2)\sec^{n-3} x \frac{d}{dx}(\sec x)$

$= (n-2)\sec^{n-3} x \times \sec x \tan x$.

b From part a $I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}I_{n-2}$

Substituting $n = 3$

$I_3 = \frac{\tan x \sec x}{2} + \frac{1}{2}I_1$

$I_1 = \int \sec x\,dx = \ln(\sec x + \tan x) + C$

The formula for integrating $\sec x$ can be found among the formulae for module C4 in the Edexcel formula booklet, which is provided for use in the examination.

Hence $I_3 = \frac{\tan x \sec x}{2} + \frac{1}{2}\ln(\sec x + \tan x) + C$

With the limits $x = 0$ and $x = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} \sec^3 x\,dx = \left[\frac{\tan x \sec x}{2} + \frac{1}{2}\ln(\sec x + \tan x)\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{2}\times\sqrt{3}\times 2 + \frac{1}{2}\ln\left(2+\sqrt{3}\right)\right) - 0$$

$$= \sqrt{3} + \frac{1}{2}\ln\left(2+\sqrt{3}\right)$$

$\tan\frac{\pi}{3} = \sqrt{3}$ and

$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 62

Question:

$I_n = \int_0^1 (1-x^2)^n \, dx, n \geq 0$.

a Prove that $(2n+1)I_n = 2nI_{n-1}, n \geq 1$.

b Prove by induction that $I_n \geq \left(\frac{2n}{2n+1}\right)^n$ for $n \in \mathbb{Z}^+$. **[E]**

Solution:

a $I_n = \int_0^1 (1-x^2)^n \, dx = \int_0^1 1\times(1-x^2)^n \, dx$

$= \left[x(1-x^2)^n\right]_0^1 - \int_0^1 x \times n(1-x^2)^{n-1}(-2x)\, dx$

$= 2n\int_0^1 x^2(1-x^2)^{n-1}\, dx$

$= 2n\int_0^1 (x^2-1+1)(1-x^2)^{n-1}\, dx$

$= -2n\int_0^1 (1-x^2)^n \, dx + 2n\int_0^1 (1-x^2)^{n-1}\, dx$

$= -2nI_n + 2nI_{n-1}$

$(2n+1)I_n = 2nI_{n-1}$, as required

You use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, with $u=(1-x^2)^n$, $\frac{dv}{dx}=1$ and, so, $v = x$.

You split this integral into two separate integrals using algebra.

$x^2(1-x^2)^{n-1} = (x^2-1+1)(1-x^2)^{n-1}$

$= (x^2-1)(1-x^2)^{n-1} + 1(1-x^2)^{n-1}$

$= -(1-x^2)^n + (1-x^2)^{n-1}$

b To prove $I \le \left(\frac{2n}{2n+1}\right)$ by mathematical induction.

Mathematical induction is a topic in the FP1 specification. The FP3 specification assumes knowledge of the FP1 specification.

Let $n = 1$

$I_1 = \int_0^1 (1-x^2)^1 \, dx = \int_0^1 (1-x^2)\, dx$

$= \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3} = \left(\frac{2\times 1}{2\times 1+1}\right)^1$

Hence for $n = 1$, $I_n \le \left(\frac{2n}{2n+1}\right)^n$

For $n = 1$, equality holds.

Assume the inequality is true for $n = k$, that is $I_k \le \left(\frac{2k}{2k+1}\right)^k$.

From part **a**, $I_n = \frac{2n}{2n+1} I_{n-1}$

With $n = k+1$ and using the induction hypothesis

$I_{k+1} = \frac{2k+2}{2k+3} I_k \le \frac{2k+2}{2k+3}\left(\frac{2k}{2k+1}\right)^k$

To complete the proof it is necessary to show that,

for $k > 0$, $\frac{2k}{2k+1} \le \frac{2k+2}{2k+3}$

To complete the proof the $\frac{2k}{2k+1}$ in the bracket needs to be replaced by $\frac{2k+2}{2k+3}$, which is the expression $\frac{2n}{2n+1}$ with $n = k+1$.

You are also using the property that, for positive numbers, $a \le b \Rightarrow a^k \le b^k$.

$$\frac{2k}{2k+1}-\frac{2k+2}{2k+3}=\frac{2k(2k+3)-(2k+2)(2k+1)}{(2k+1)(2k+3)}$$
$$=\frac{4k^2+6k-(4k^2+6k+2)}{(2k+1)(2k+3)}$$
$$=\frac{-2}{(2k+1)(2k+3)}<0,\text{ for } k>0$$

Hence $\frac{2k}{2k+1}\le\frac{2k+2}{2k+3}$ and $I_{k+1}\le\frac{2k+2}{2k+3}\left(\frac{2k}{2k+1}\right)^k\le\frac{2k+2}{2k+3}\left(\frac{2k+2}{2k+3}\right)^k=\left(\frac{2k+2}{2k+3}\right)^{k+1}$

This is the inequality with $n=k+1$.

The inequality is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the inequality is true for all positive integers n.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 63

Question:

A curve is defined by $x = t + \sin t, y = 1 - \cos t$, where t is a parameter.

Find the length of the curve from $t = 0$ to $t = \frac{\pi}{2}$, giving your answer in surd form. **[E]**

Solution:

$x = t + \sin t \quad y = 1 - \cos t$

$\frac{dx}{dt} = 1 + \cos t \quad \frac{dy}{dt} = \sin t$

$$s = \int \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)} dt$$

It is always a good idea to quote any formula you are going to use in answering a question.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + \cos t)^2 + \sin^2 t$$
$$= 1 + 2\cos t + \cos^2 t + \sin^2 t$$
$$= 2 + 2\cos t$$
$$= 4\cos^2 \frac{t}{2}$$

You simplify this expression using the identity $\sin^2 t + \cos^2 t = 1$ and the double angle formula $\cos 2x = 2\cos^2 x - 1$, with $x = \frac{t}{2}$.

Hence, the length of the curve is given by

$$s = \int_0^{\frac{\pi}{2}} \sqrt{\left(4\cos^2 \frac{t}{2}\right)} dt = \int_0^{\frac{\pi}{2}} 2\cos \frac{t}{2} dt$$

$$\int 2\cos \frac{t}{2} dt = \frac{2\sin \frac{t}{2}}{\frac{1}{2}} = 4\sin \frac{t}{2}$$

$$= \left[4\sin \frac{t}{2}\right]_0^{\frac{\pi}{2}} = 4\sin \frac{\pi}{4}$$

$$= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 64

Question:

Parametric equations for the curve C are $x = \cosh t + t, y = \cosh t - t, t \geq 0$.
Show that the length of the arc of the curve C between points at which $t = 0$ and $t = a$, where a is a positive constant, is $(\sqrt{2}) \sinh a$. **[E]**

Solution:

$x = \cosh t + t \quad y = \cosh t - t$

$\frac{dx}{dt} = 1 + \sinh t \quad \frac{dy}{dt} = \sinh t - 1$

$s = \int \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)} dt$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\sinh t + 1)^2 + (\sinh t - 1)^2$

$= \sinh^2 t + 2\sinh t + 1 + \sinh^2 t - 2\sinh t + 1$

$= 2\sinh^2 t + 2 = 2\cosh^2 t$

To save a lot of writing, it is often sensible to work out complicated expressions like $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, and to transform them into a form where a square root can be easily found, before substituting into the integral.

Using $\cosh^2 t - \sinh^2 t = 1$

Hence, the length of the curve is given by

$s = \int_0^a \sqrt{(2\cosh^2 t)} dt = \sqrt{2} \int_0^a \cosh t \, dt$

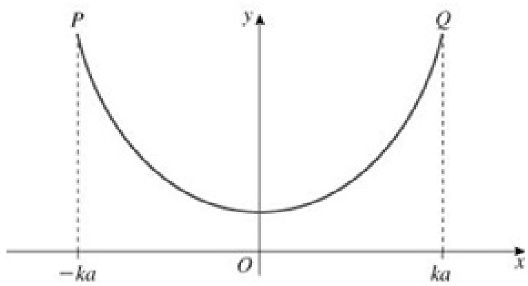
$= \sqrt{2} [\sinh t]_0^a = \sqrt{2} (\sinh a - \sinh 0)$

$= \sqrt{2} \sinh a$, as required.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 65

Question:



A rope is hung from points P and Q on the same horizontal line, as shown in the figure. The curve formed is modelled by the equation $y = a\cosh\left(\frac{x}{a}\right), -ka \leq x \leq ka$.

where a and k are constants.

a Prove that the length of the rope is $2a\sinh k$.

Given that the length of the rope is $8a$,

b find the coordinates of Q, leaving your answer in terms of natural logarithms and surds, where appropriate. **[E]**

Solution:

a $$y = a\cosh\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{1}{a} \times a\sinh\left(\frac{x}{a}\right) = \sinh\left(\frac{x}{a}\right)$$

$$s = \int \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} dx$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$$

The length of the rope is given by

$$s = 2\int_0^{ka} \cosh\left(\frac{x}{a}\right) dx$$

$$= 2\left[a\sinh\left(\frac{x}{a}\right)\right]_0^{ka} = 2a(\sinh k - \sinh 0)$$

$= 2a\sinh k$, as required.

From the symmetry of the diagram, the length of the rope from P to Q is twice the length of the rope from the point where $x = 0$ to Q.

b $2a\sinh k = 8a$

$\sinh k = 4 \Rightarrow k = \text{arsinh}4$

$= \ln(4 + \sqrt{17})$

You use the formula $\text{arsinh}x = \ln\left(x + \sqrt{(x^2 + 1)}\right)$ to find the x-coordinate of Q in terms of a natural logarithm. The question specifies that you should give your answer in this form.

At Q, $x = ka = a\ln(4 + \sqrt{17})$ and

$$y = a\cosh\left(\frac{x}{a}\right) = a\cosh\left(\frac{ka}{a}\right) = a\cosh k$$

$\cosh^2 k = 1 + \sinh^2 k = 1 + 4^2 = 17 \Rightarrow \cosh k = \sqrt{17}$

The coordinates of Q are $\left(a\ln(4 + \sqrt{17}), a\sqrt{17}\right)$.

As you know that $\sinh k = 4$, you can find the value of $\cosh k$ using the identity $\cosh^2 x = 1 + \sinh^2 x$.

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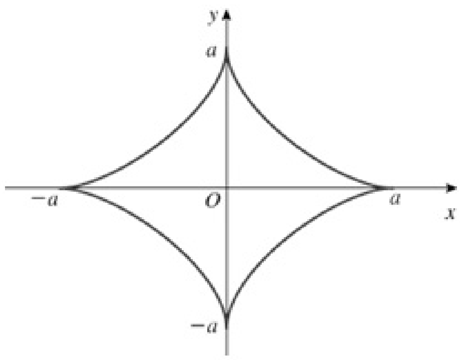
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 66

Question:



The figure shows the curve with parametric equations
$x = a\cos^3\theta, y = a\sin^3\theta, 0 \le \theta < 2\pi$.

a Find the total length of the curve.
The curve is rotated through π radians about the x-axis.
b Find the area of the surface generated. **[E]**

Solution:

a $x = a\cos^3\theta \qquad y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$s = \int\sqrt{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)}d\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2$$
$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\cos^2\theta\sin^4\theta$$
$$= 9a^2\cos^2\theta\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$
$$= 9a^2\cos^2\theta\sin^2\theta$$

Hence the length of the curve is given by

$$s = 4\times\int_0^{\frac{\pi}{2}}\sqrt{\left(9a^2\cos^2\theta\sin^2\theta\right)}d\theta$$
$$= 12a\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta$$
$$= 12a\left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}}$$
$$= 12a\left(\frac{1}{2} - 0\right)$$
$$= 6a$$

The symmetries of the diagram show that the total length of the curve is four times the length in the first quadrant. As $x\left(= a\cos^3\theta\right)$ varies from 0 to a, $\cos\theta$ varies from 0 to 1, and so θ varies from $\frac{\pi}{2}$ to 0 in that order.

There are number of alternative ways of evaluating this integral. You could use a double angle formula.

b $A = 2\pi\int y\sqrt{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)}d\theta$

The area of the surface generated is given by

$$A = 2\times 2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta\times 3a\cos\theta\sin\theta\ d\theta$$
$$= 12\pi a^2\int_0^{\frac{\pi}{2}}\sin^4\theta\cos\theta\ d\theta$$
$$= 12\pi a^2\left[4\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = 12\pi a^2\left(\frac{1}{5} - 0\right)$$
$$= \frac{12}{5}\pi a^2$$

The total area is twice the area formed by rotating the two portions of the curve on the positive side of the x-axis.

You have already worked out $\sqrt{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)}$ in part **a** and there is no need to repeat the working here.

Here the integral is found using the formula $\int\sin^n\theta\cos\theta\ d\theta = \frac{\sin^{n+1}\theta}{n+1}$ with $n = 4$. If you do not know this formula, you can find the integral using the substitution $u = \sin\theta$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 67

Question:

a By using the definition of $\cosh x$ in terms of exponentials, show that $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$.

b The arc of the curve with equation $y = \cosh x$ from $x = 0$ to $x = \ln 2$ is rotated through 2π radians about the x-axis. Determine the area of the curved surface generated, leaving your answer in terms of π. **[E]**

Solution:

a $\cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{4} + \frac{2}{4}$

$= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2}\right) + \frac{1}{2} = \frac{1}{2}\cosh 2x + \frac{1}{2}$

$= \frac{1}{2}(\cosh 2x + 1)$, as required.

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b $y = \cosh x \Rightarrow \frac{dy}{dx} = \sinh x$

$A = 2\pi \int y \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} dx$

$= 2\pi \int_0^{\ln 2} \cosh x \sqrt{(1 + \sinh^2 x)} dx$

$= 2\pi \int_0^{\ln 2} \cosh^2 x dx$

$= 2\pi \int_0^{\ln 2} \frac{1}{2}(\cosh 2x + 1) dx = \pi \int_0^{\ln 2} (\cosh 2x + 1) dx$

You use the identity shown in part **a** to find the integral.

$= \pi \left[\frac{\sinh 2x}{2} + x\right]_0^{\ln 2}$

$= \pi [\sinh x \cosh x + x]_0^{\ln 2}$

Using the identity $\sinh 2x = 2\sinh x \cosh x$.

$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$

As, for any x, $e^{-\ln x} = e^{\ln 1 - \ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$,

then $e^{-\ln 2} = \frac{1}{2}$.

Hence the area is given by $A = \pi\left(\frac{3}{4} \times \frac{5}{4} + \ln 2\right) = \pi\left(\frac{15}{16} + \ln 2\right)$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 1

Question:

Find the magnitude of the vector $(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})$. **[E]**

Solution:

$$(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 1\\ -1 & 1 & -1\end{vmatrix}$$
$$=\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ -1 & 1\end{vmatrix}\mathbf{k}$$
$$=((-1\times-1)-(1\times1))\mathbf{i}-((-1\times-1)-(-1\times1))\mathbf{j}+((-1\times1)-(-1\times-1))\mathbf{k}$$
$$=(1-1)\mathbf{i}-(1-(-1))\mathbf{j}+(-1-1)\mathbf{k}=-2\mathbf{j}-2\mathbf{k}$$

Formulae for finding the vector product are given in the Edexcel formulae booklet which is provided for the examination. One form it gives is $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ and that has been used here.

Hence

$$|(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})|=\sqrt{((-2)^2+(-2)^2)}$$
$$=\sqrt{8}=2\sqrt{2}$$

You use the formula for the magnitude of a vector $|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}|=\sqrt{(x^2+y^2+z^2)}$.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 2

Question:

Given that $\mathbf{p} = 3\mathbf{i} + \mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 3\mathbf{j} + c\mathbf{k}$, find the value of the constant c for which the vector $(\mathbf{p} \times \mathbf{q}) + \mathbf{p}$ is parallel to the vector $\mathbf{k}$. **[E]**

Solution:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 1 \\ 1 & 3 & c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 3 & c \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & c \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 0 \\ 1 & 3 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} - (3c - 1)\mathbf{j} + 9\mathbf{k}$$

$$\mathbf{p} \times \mathbf{q} + \mathbf{p} = (-3\mathbf{i} - (3c-1)\mathbf{j} + 9\mathbf{k}) + (3\mathbf{i} + \mathbf{k})$$

$$= (1 - 3c)\mathbf{j} + 10\mathbf{k}$$

If $\mathbf{p} \times \mathbf{q} + \mathbf{p}$ is parallel to $\mathbf{k}$ then

$$1 - 3c = 0 \Rightarrow c = \frac{1}{3}$$

You work out 2×2 determinants using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which is in the FP1 specification. Knowledge of the FP1 specification is a prerequisite of the FP3 specification.

If a vector is parallel to $\mathbf{k}$ then both its $\mathbf{i}$ and $\mathbf{j}$ components must be 0. The $\mathbf{i}$ component of $\mathbf{p} \times \mathbf{q} + \mathbf{p}$ is 0 and the $\mathbf{j}$ component, $1 - 3c$ must equal 0, which gives you a simple equation to find c.

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 3

Question:

Referred to a fixed origin O, the position vectors of three non-linear points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. By considering $\overrightarrow{AB} \times \overrightarrow{AC}$, prove that the area of ΔABC can be expressed in the form $\frac{1}{2}|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|$. **[E]**

Solution:

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$

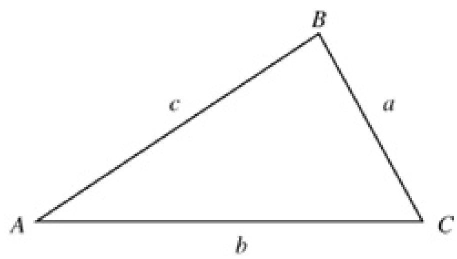
$= \mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} - \mathbf{a}\times\mathbf{c} + \mathbf{a}\times\mathbf{a}$

As $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$ and $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{b}\times\mathbf{c} + \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}$

$= \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$

You multiply out the brackets using the usual rules of algebra. You must take care with the order in which the vectors are multiplied as the vector product is not commutative. For a vector product $\mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}$.



The area of the triangle, Δ, say, is given by

$$\Delta = \frac{1}{2}bc\sin A$$

$$= \frac{1}{2}AC \times AB \sin A$$

$$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|, \text{ as required.}$$

.The magnitude of the vector product $\mathbf{a}\times\mathbf{b}$ is $|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between the vectors. The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 4

Question:

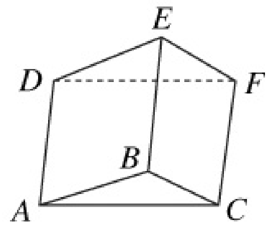
The figure shows a right prism with triangular ends ABC and DEF, and parallel edges AD, BE, CF.

Given that A is $(2,7,-1)$, B is $(5, 8, 2)$, C is $(6, 7, 4)$ and D is $(12,1,-9)$,

a find $\overrightarrow{AB}\times\overrightarrow{AC}$,

b find $\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})$.

c Calculate the volume of the prism.



Solution:

a

$$\overrightarrow{AB}=(5\mathbf{i}+8\mathbf{j}+2\mathbf{k})-(2\mathbf{i}+7\mathbf{j}-\mathbf{k})$$
$$=3\mathbf{i}+\mathbf{j}+3\mathbf{k}$$
$$\overrightarrow{AC}=(6\mathbf{i}+7\mathbf{j}+4\mathbf{k})-(2\mathbf{i}+7\mathbf{j}-\mathbf{k})$$
$$=4\mathbf{i}+5\mathbf{k}$$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$. It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB}=\overrightarrow{OA}-\overrightarrow{OB}$ and obtain the negative of the correct answer.

$$\overrightarrow{AB}\times\overrightarrow{AC}=(3\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(4\mathbf{i}+5\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&1&3\\4&0&5\end{vmatrix}$$
$$=\begin{vmatrix}1&3\\0&5\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&3\\4&5\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&1\\4&0\end{vmatrix}\mathbf{k}$$
$$=5\mathbf{i}-3\mathbf{j}-4\mathbf{k}$$

b

$$\overrightarrow{AD}=(12\mathbf{i}+\mathbf{j}-9\mathbf{k})-(2\mathbf{i}+7\mathbf{j}-\mathbf{k})$$
$$=10\mathbf{i}-6\mathbf{j}-8\mathbf{k}$$
$$\overrightarrow{AD}.(\overrightarrow{AB}\times\overrightarrow{AC})=(10\mathbf{i}-6\mathbf{j}-8\mathbf{k})\cdot(5\mathbf{i}-3\mathbf{j}-4\mathbf{k})$$
$$=10\times5+(-6)\times(-3)+(-8)\times(-4)$$
$$=50+18+32=100$$

$10\mathbf{i}-6\mathbf{j}-9\mathbf{k}=2(5\mathbf{i}-3\mathbf{j}-4\mathbf{k})$ so $\overrightarrow{AD}$ and $\overrightarrow{AB}\times\overrightarrow{AC}$ are parallel. As the vector product is perpendicular to AB and AC, it follows that the line AD is perpendicular to the plane of the triangle ABC.

c The volume of the prism, P say, is given by

$$P=\frac{1}{2}\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=\frac{1}{2}\times100=50$$

In this case, the volume of the prism is the area of the triangle ABC, which is half the magnitude of $\overrightarrow{AB}\times\overrightarrow{AC}$, multiplied by the distance AD. (Even if the line AD is not perpendicular to the plane of the triangle ABC, the triple scalar product is still twice the volume of the prism.)

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Review Exercise 2
Exercise A, Question 5

Question:

The plane Π_1, has vector equation $\mathbf{r} = (5\mathbf{i} + \mathbf{j}) + u(-4\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + v(\mathbf{j} + 2\mathbf{k})$, where u and v are parameters.

a Find a vector $\mathbf{n}_1$ normal to Π_1.

The plane Π_2 has equation $3x + y - z = 3$.

b Write down a vector $\mathbf{n}_2$ normal to Π_2.

c Show that $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Given that the point (1, 1, 1) lies on both Π_1 and Π_2,

d write down an equation of the line of intersection of Π_1 and Π_2 in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is a parameter. **[E]**

Solution:

a $\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$

If the equation of a plane is given to you in the form $\mathbf{r} = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$, then you can find a normal to the plane by finding $\mathbf{b} \times \mathbf{c}$.

b $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

The Cartesian equation $3x + y - z = 3$ can be written in the vector form $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comparison with the standard form, $\mathbf{r} \cdot \mathbf{n} = p$, gives you that $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to Π_2.

c $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 8 & -4 \\ 3 & 1 & -1 \end{vmatrix}$

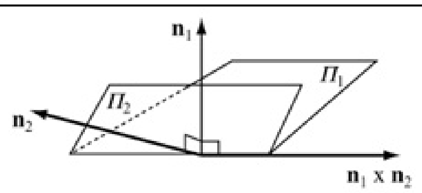
$= \begin{vmatrix} 8 & -4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -4 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 8 \\ 3 & 1 \end{vmatrix}\mathbf{k}$

$= -4\mathbf{i} - 13\mathbf{j} - 25\mathbf{k} = -1(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$

The scalar product $\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. So to show that a vector, $\mathbf{r}$ say, is perpendicular to two other vectors, you can show that $\mathbf{r}$ is parallel to the vector product of the two other vectors. An alternative method is to show that the scalar product of $\mathbf{r}$ with each of the other two vectors is zero.

$\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to the plane containing $\mathbf{n}_1$ and $\mathbf{n}_2$, and $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is a multiple of $\mathbf{n}_1 \times \mathbf{n}_2$. Hence $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$.

d $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$



This diagram illustrates that the line of intersection of the planes Π_1 and Π_2 lies in the direction of $\mathbf{n}_1 \times \mathbf{n}_2$. In this case, $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k} = -\mathbf{n}_1 \times \mathbf{n}_2$ and can be used as the direction of the line, as can any other multiple of this vector.

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Review Exercise 2
Exercise A, Question 6

Question:

The points A, B and C lie on the plane Π and, relative to a fixed origin O, they have position vectors

$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \mathbf{b} = -\mathbf{i} + 2\mathbf{j},$

$\mathbf{c} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

respectively.

a Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

b Obtain the equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$.

The point D has position vector $5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

c Calculate the volume of the tetrahedron $ABCD$. **[E]**

Solution:

a

$$\overrightarrow{AB} = -\mathbf{i}+2\mathbf{j}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$$
$$= -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}$$
$$\overrightarrow{AC} = 5\mathbf{i}-3\mathbf{j}+7\mathbf{k}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$$
$$= 2\mathbf{i}-2\mathbf{j}+3\mathbf{k}$$
$$\overrightarrow{AB}\times\overrightarrow{AC} = (-4\mathbf{i}+3\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix}$$
$$= \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -4 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 3 \\ 2 & -2 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{i}+4\mathbf{j}+2\mathbf{k}$$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$ and obtain the negative of the correct answer.

b An equation of Π is

$$\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}+2\mathbf{k}) = (3\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+4\mathbf{j}+2\mathbf{k})$$
$$= 3\times 1+(-1)\times 4+4\times 2$$
$$= 3-4+8=7$$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of A has been used but the position vectors of B and C would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here 7.

c

$$\overrightarrow{AD} = 5\mathbf{i}+2\mathbf{j}+3\mathbf{k}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$$
$$= 2\mathbf{i}+3\mathbf{j}-\mathbf{k}$$
$$\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = (2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+4\mathbf{j}+2\mathbf{k})$$
$$= 2\times 1+3\times 4+(-1)\times 2$$
$$= 2+12-2=12$$

The volume, V say, of the tetrahedron is given by

$$V = \frac{1}{6}\left|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\right| = \frac{1}{6}\times 12 = 2$$

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Review Exercise 2
Exercise A, Question 7

Question:

The points A and B have position vectors $4\mathbf{i}+\mathbf{j}-7\mathbf{k}$ and $2\mathbf{i}+6\mathbf{j}+2\mathbf{k}$ respectively relative to a fixed origin O.

a Show that angle AOB is a right angle.

b Find a vector equation for the median AM of the triangle OAB.

c Find a vector equation of the plane OAB, giving your answer in the form $\mathbf{r}\cdot\mathbf{n}=p$.

[E]

Solution:

a $\overrightarrow{OA}\cdot\overrightarrow{OB}=(4\mathbf{i}+\mathbf{j}-7\mathbf{k})\cdot(2\mathbf{i}+6\mathbf{j}+2\mathbf{k})$

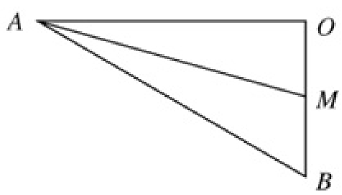
$=4\times 2+1\times 6+(-7)\times 2$

$=8+6-14=0$

Hence $\angle AOB=90°$, as required.

As the scalar product of two vectors $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors, if the scalar product of two non-zero vectors is zero, then $\cos\theta=0$ and the angle between the vectors is a right angle.

b



The median AM of a triangle is the line joining the vertex A of the triangle to the mid-point M of the side of the triangle which is opposite to A.

The coordinates of M, the mid-point of O (0, 0, 0) and B (2, 6, 2) are

$\left(\frac{0+2}{2},\frac{0+6}{2},\frac{0+2}{2}\right)=(1,3,1)$

$\overrightarrow{AM}=\mathbf{i}+3\mathbf{j}+\mathbf{k}-(4\mathbf{i}+\mathbf{j}-7\mathbf{k})$

$=-3\mathbf{i}+2\mathbf{j}+8\mathbf{k}$

There are many possible alternative forms for this answer. For example, you could use M as the 'starting point' of the line and obtain an answer such as $\mathbf{r}=\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(3\mathbf{i}-2\mathbf{j}+8\mathbf{k})$.

An equation of AM is $\mathbf{r}=4\mathbf{i}+\mathbf{j}-7\mathbf{k}+\lambda(-3\mathbf{i}+2\mathbf{j}+8\mathbf{k})$

c $\overrightarrow{OA}\times\overrightarrow{OB}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 1 & -7\\ 2 & 6 & 2\end{vmatrix}$

$=\begin{vmatrix}1 & -7\\ 6 & 2\end{vmatrix}\mathbf{i}-\begin{vmatrix}4 & -7\\ 2 & 2\end{vmatrix}\mathbf{j}+\begin{vmatrix}4 & 1\\ 2 & 6\end{vmatrix}\mathbf{k}$

$=44\mathbf{i}-22\mathbf{j}+22\mathbf{k}$

You can use $44\mathbf{i}-22\mathbf{j}+22\mathbf{k}$ or any multiple of this vector as $\mathbf{n}$ in $\mathbf{r}\cdot\mathbf{n}=p$.
$2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is used here as it gives a simpler answer.

A vector parallel to $44\mathbf{i}-22\mathbf{j}+22\mathbf{k}$ is $2\mathbf{i}-\mathbf{j}+\mathbf{k}$

An equation of Π is $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=0$

As the plane goes through the origin, the p in $\mathbf{r}.\mathbf{n}=p$ must be zero.

Solutionbank FP3

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Review Exercise 2
Exercise A, Question 8

Question:

Referred to a fixed origin O, the point A has position vector $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and the plane Π has equation $\mathbf{r}\cdot(\mathbf{i}-5\mathbf{j}+3\mathbf{k})=5a$, where a is a scalar constant.

a Show that A lies in the plane Π.

The point B has position vector $a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$.

b Show that $\overrightarrow{BA}$ is perpendicular to the plane Π.

c Calculate, to the nearest one tenth of a degree, $\angle OBA$. **[E]**

Solution:

a $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k}).(\mathbf{i}-5\mathbf{j}+3\mathbf{k})=a(4\times1+1\times(-5)+2\times3)$

$=a(4-5+6)=5a$

Hence A lies in the plane Π, as required.

For A to lie on the plane with equation $\mathbf{r}.\mathbf{n}=5a$, when $\mathbf{r}$ is replaced by the position vector of A, $\mathbf{r}.\mathbf{n}$ must give $5a$.

b $\overrightarrow{BA}=a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})-a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$

$=a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

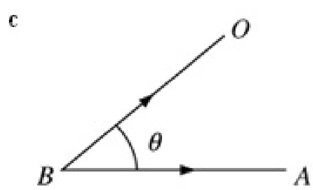
$\overrightarrow{BA}=2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$

$\overrightarrow{BA}$ is parallel to the vector $\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, which is perpendicular to the plane Π.

Hence $\overrightarrow{BA}$ is perpendicular to the plane Π, as required.

When a plane has an equation of the form $\mathbf{r}.\mathbf{n}=p$, the vector $\mathbf{n}$ is perpendicular to the plane.

c



The angle OBA is the angle between BO and BA. Both these line segments have a definite sense and so you must use the scalar product $\overrightarrow{BO}.\overrightarrow{BA}$ to find θ. If you used $\overrightarrow{OB}.\overrightarrow{BA}$, you would obtain the supplementary angle $(180^\circ-\theta)$, which is not the correct answer.

Let $\angle OAB=\theta$

$|\overrightarrow{BO}|=a\sqrt{((-2^2)+(-11)^2+4^2)}=a\sqrt{(141)}$

$|\overrightarrow{BA}|=a\sqrt{(2^2+(-10)^2+6^2)}=a\sqrt{(140)}$

$\overrightarrow{BO}.\overrightarrow{BA}=a(-2\mathbf{i}-11\mathbf{j}+4\mathbf{k}).a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

$|\overrightarrow{BO}|.|\overrightarrow{BA}|\cos\theta=a^2((-2)\times2+(-11)\times(-10)+4\times6)$

$a\sqrt{(141)}\times a\sqrt{(140)}\cos\theta=a^2(-4+110+24)$

$\cos\theta=\dfrac{130}{\sqrt{(141)}\sqrt{(140)}}=0.925272\ldots$

$\theta=22.3^\circ$ (to the nearest one tenth of a degree)

Finding the angle between two vectors using the scalar product is part of the C4 specification. Knowledge of the C4 specification is a pre-requisite of the FP3 specification.

Solutionbank FP3
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Review Exercise 2
Exercise A, Question 9

Question:

The points A, B, C and D have coordinates $(3, 1, 2)$, $(5, 2, -1)$, $(6, 4, 5)$ and $(-7, 6, -3)$ respectively.

a Find $\overrightarrow{AC} \times \overrightarrow{AD}$.

b Find a vector equation of the line through A which is perpendicular to $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

c Verify that B lies on this line.

d Find the volume of the tetrahedron $ABCD$. **[E]**

Solution:

a $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix} -7 \\ 6 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix}$

$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 3\times(-5) - 3\times 5 \\ 3\times(-10) - 3\times(-5) \\ 3\times 5 - 3\times(-10) \end{pmatrix}$

$= \begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix}$

For writing vectors, you can use either the form with **i**s, **j**s and **k**s, or column vectors, which are used in this solution. Sometimes it may even be appropriate to use a mixture of the two. The form using **i**, **j** and **k** usually gives a more compact solution but many find column vectors quicker to write. The choice is entirely up to you and you may choose to vary it from question to question.

b $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$

The vector $\begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix}$ is perpendicular to both $\overrightarrow{AC}$ and $\overrightarrow{AD}$. This vector or any multiple of it may be used for the equation of the line.

c For B to lie on the line there must be a value of λ for which

$\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$

Equating the x components of the vectors

$5 = 3 - 2\lambda \Rightarrow \lambda = -1$

Checking this value of λ for the other components

y component:

$1 + \lambda \times (-1) = 1 + (-1) \times (-1) = 2$, as required

z component:

$2 + \lambda \times 3 = 2 + (-1) \times 3 = -1$, as required

Hence, B lies on the line.

d $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix} = 2\times(-30) + 1\times(-15) + (-3)\times 45$

$= -60 - 15 - 135 = -210$

The volume of the tetrahedron, V say, is given by

$V = \frac{1}{6}\left|\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD})\right| = \frac{1}{6}|-210| = \frac{1}{6} \times 210 = 35$

The volume of the tetrahedron is one sixth of the triple scalar product.

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Review Exercise 2
Exercise A, Question 10

Question:

The line l_1 has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k})$ and the line l_2 has equation $\mathbf{r} = 3\mathbf{i} + p\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, where p is a constant.

The plane Π_1 contains l_1 and l_2.

a Find a vector which is normal to Π_1.

b Show that an equation for Π_1 is $6x + y - 4z = 16$.

c Find the value of p.

The plane Π_2 has equation $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2$.

d Find an equation for the line of intersection of Π_1 and Π_2, giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$. **[E]**

Solution:

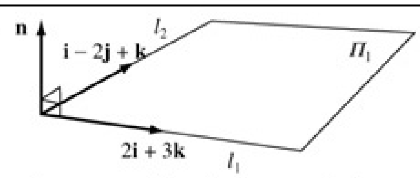
a A vector $\mathbf{n}$ perpendicular to l_1 and l_2 is given by

$$\mathbf{n} = (2\mathbf{i} + 3\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$



The vector $2\mathbf{i} + 3\mathbf{k}$ is parallel to l_1 and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is parallel to l_2. The vector product of two vectors is perpendicular to both vectors and so is perpendicular to the plane containing both lines

b An equation for Π_1 has the form

$\mathbf{r}.(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = p$

$p = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}).(6\mathbf{i} + \mathbf{j} - 4\mathbf{k})$

$= 6 + 6 + 4 = 16$

A vector equation of Π_1 is

$\mathbf{r}.(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 16$

A Cartesian equation of Π_1 is given by

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 16$

$6x + y - 4z = 16$, as required.

To obtain a Cartesian equation of a plane when you have a vector equation in the form $\mathbf{r}.\mathbf{n} = p$, replace $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and work out the scalar product.

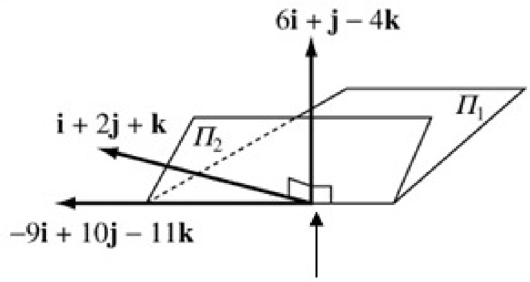
c The point with coordinates $(3, p, 0)$ lies on l_1 and, hence, must lie on Π_1.
Substituting $(3, p, 0)$ into the result of part **b**
$18 + p = 16 \Rightarrow p = -2$

d The line of intersection lies in the direction given by

$$(\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 6 & 1 & -4\end{vmatrix}$$

$$=\begin{vmatrix}2 & 1\\ 1 & -4\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 1\\ 6 & -4\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ 6 & 1\end{vmatrix}\mathbf{k}$$

$$=-9\mathbf{i}+10\mathbf{j}-11\mathbf{k}$$



To find one point that lies on both Π_1 and Π_2

Π_1: $6x+y-4z=16$ ①

Π_2: $x+2y+z=2$ ②

① $+4\times$ ② gives $10x+9y=24$

Choose $x=-3, y=6$

Substitute into ②

$-3+12+z=2 \Rightarrow z=-7$

One point on the line is $(-3, 6, -7)$

An equation of the line is

$(\mathbf{r}-(-3\mathbf{i}+6\mathbf{j}-7\mathbf{k}))\times(-9\mathbf{i}+10\mathbf{j}-11\mathbf{k})=0$

You need to find just one point that is on both planes and there are infinitely many possibilities. Here you can choose any pair of values of x and y which fit this equation. A choice here has been made which gives whole numbers but you may find, for example, $y=0$, $x=2.4$ easier to see.

The form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, for the equation of a straight line, represents a line that passes through the point with position vector $\mathbf{a}$ and is parallel to the vector $\mathbf{b}$.

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Review Exercise 2
Exercise A, Question 11

Question:

The plane Π passes through the points $A(-2,3,5), B(1,-3,1)$ and $C(4,-6,-7)$.

a Find $\overrightarrow{AC} \times \overrightarrow{BC}$.

b Hence, or otherwise, find the equation of the plane Π in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{p}$.

The perpendicular from the point (25, 5, 7) to Π meets the plane at the point F.

c Find the coordinates of F. **[E]**

Solution:

a

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ -6 \\ -7 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 4 \\ -6 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix}$$

$$\overrightarrow{AC} \times \overrightarrow{BC} = \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix} = \begin{pmatrix} 72-36 \\ -36+48 \\ -18+27 \end{pmatrix}$$

$$= \begin{pmatrix} 36 \\ 12 \\ 9 \end{pmatrix}$$

This vector, or any multiple of this vector, can be used for the vector perpendicular to Π in part **b**. The working in later parts of the question will usually be simplest if you take the multiple which gives the smallest possible integers. In this case one third of the vector has been used in part **b**.

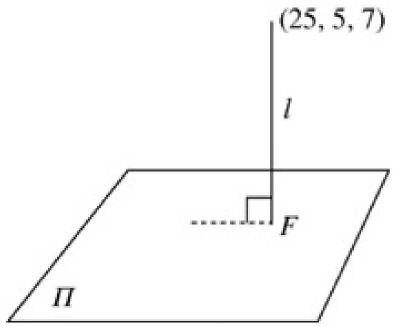
b An equation of Π is

$$\mathbf{r}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = -24+12+15$$

$$\mathbf{r}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = 3$$

The position vector of the point A has been used to evaluate p in $\mathbf{r}.\mathbf{n} = p$. You could use the position vector of any of the points, A, B and C.

c



An equation of the line, l say, which passes through (25, 5, 7) and is perpendicular to Π is

$$\mathbf{r} = \begin{pmatrix} 25 \\ 5 \\ 7 \end{pmatrix} + t\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix}$$

The equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ represents a line passing through the point with position vector $\mathbf{a}$, in this case $\begin{pmatrix} 25 \\ 5 \\ 7 \end{pmatrix}$, which is parallel to the vector $\mathbf{b}$. In this case, l is parallel to the normal to the plane, $\begin{pmatrix} 12 \\ 4 \\ 7 \end{pmatrix}$.

Parametric equations of l are
$x = 25 + 12t, y = 5 + 4t, z = 7 + 3t$

A Cartesian equation of Π is
$12x + 4y + 3z = 3$

Writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, the vector equation $\mathbf{r}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = 3$ becomes the Cartesian equation $12x + 4y + 3z = 3$.

Substituting $(25 + 12t, 5 + 4t, 7 + 3t)$ into the Cartesian equation of Π

$$12(25 + 12t) + 4(5 + 4t) + 3(7 + 3t) = 3$$
$$300 + 144t + 20 + 16t + 21 + 9t = 3$$
$$169t = -338$$
$$t = -2$$

The coordinates of F are given by
$(25 + 12t, 5 + 4t, 7 + 3t)$
$= (25 - 24, 5 - 8, 7 - 6)$
$= (1, -3, 1)$

$t = -2$ is the value of the parameter t at the point where the line intersects the plane. Substituting $t = -2$ into the parametric form of the line then gives you the coordinates of F.

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Review Exercise 2
Exercise A, Question 12

Question:

The plane Π passes through the points $P(-1,3,-2)$, $Q(4,-1,-1)$ and $R(3, 0, c)$, where c is a constant.

a Find, in terms of c, $\overrightarrow{RP} \times \overrightarrow{RQ}$.

Given that $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$, where d is a constant,

b find the value of c and show that $d = 4$.

c Find an equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$, where p is a constant.

The point S has position vector $\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point S' is the image of S under reflection in Π.

d Find the position vector of S'. **[E]**

Solution:

a $\overrightarrow{RP} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix}$

$\overrightarrow{RQ} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$

$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$

$= \begin{pmatrix} 3(-1-c)-(2+c) \\ -2-c-4(1+c) \\ 4-3 \end{pmatrix} = \begin{pmatrix} -5-4c \\ -6-5c \\ 1 \end{pmatrix}$

In this solution, the vector product has be found using the formula $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$. This formula can be found in the Edexcel formulae booklet which is provided for the examination.

b $\begin{pmatrix} -5-4c \\ -6-5c \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ d \\ 1 \end{pmatrix}$

Equating the x components

$-5-4c = 3 \Rightarrow 4c = -8 \Rightarrow c = -2$

Equating the y components

$d = -6-5c = -6-5\times(-2) = -6+10$

$= 4$, as required.

c When $c = -2$

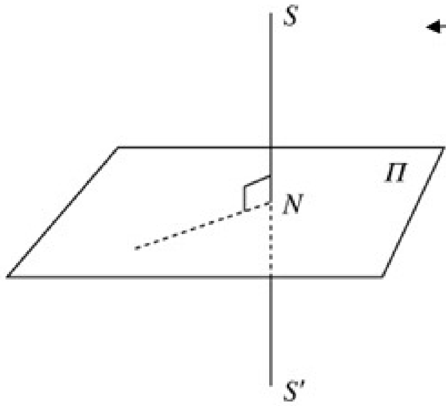
$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{pmatrix} -5-4c \\ -6-5c \\ 1 \end{pmatrix} = \begin{pmatrix} -5+8 \\ -6+10 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

An equation of Π is

$\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = -3+12-2$

$\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7$

d



In this diagram, the point N is the intersection of SS' and the plane. As S' is the reflection of S in Π, SS' is perpendicular to Π and N is the mid-point of SS'. Hence the translation (or displacement) from S to N is the same as the translation (or displacement) from N to S'.
The method used in this solution is to find the position vector of N and, then, to find the translation which maps S to N. This translation can then be used to find the position vector of S' from the position vector of N.

A vector equation of SS' is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Parametric equations for SS' are
$x = 1 + 3t, y = 5 + 4t, z = 10 + t$ ①
A Cartesian equation of Π is
$3x + 4y + z = 7$ ②
To find the position vector of N, the point of intersection of SS' and Π, substitute ① into ②

$$3(1+3t) + 4(5+4t) + 10 + t = 7$$
$$3 + 9t + 20 + 16t + 10 + t = 7$$
$$26t = -26 \Rightarrow t = -1$$

The position vector of N is $\begin{pmatrix} 1+3t \\ 5+4t \\ 10+t \end{pmatrix} = \begin{pmatrix} 1-3 \\ 5-4 \\ 10-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix}$

The translation which maps S to N is represented by the vector

$$\overrightarrow{SN} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$$

The translation which maps S to N will also map N to S'.

The position vector of S' is given by

$$\begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ 8 \end{pmatrix}$$

The position vector of S' is the position vector of N added to the vector representing the translation.

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Review Exercise 2
Exercise A, Question 13

Question:

The points A, B and C lie on the plane Π_1 and, relative to a fixed origin O, they have position vectors
$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$,
$\mathbf{c} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
respectively.

a Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ ..

b Find an equation of Π_1, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = p$.

The plane Π_2 has Cartesian equation $x + z = 3$ and Π_1 and Π_2 intersect in the line l.

c Find an equation of l in the form $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = 0$.

The point P is the point on l that is nearest to the origin O.

d Find the coordinates of P. **[E]**

Solution:

a $\mathbf{b}-\mathbf{a}=3\mathbf{i}+3\mathbf{j}-4\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k})=2\mathbf{i}-3\mathbf{k}$

$\mathbf{c}-\mathbf{a}=5\mathbf{i}-2\mathbf{j}-2\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k})=4\mathbf{i}-5\mathbf{j}-\mathbf{k}$

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-3\\4&-5&-1\end{vmatrix}$$

$$=\begin{vmatrix}0&-3\\-5&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-3\\4&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&0\\4&-5\end{vmatrix}\mathbf{k}$$

$$=-15\mathbf{i}-10\mathbf{j}-10\mathbf{k}$$

The vector $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ is perpendicular to both AB and AC and, so, is perpendicular to the plane Π_1. You can use this vector, or any parallel vector, as the $\mathbf{n}$ in the equation $\mathbf{r}.\mathbf{n}=p$ in part **b**. Here each coefficient has been divided by -5. This eases later working and avoids negative signs

b A vector perpendicular to Π_1 is $3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

A vector equation of Π_1 is

$\mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=(\mathbf{i}+3\mathbf{j}-\mathbf{k}).(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

$=3+6-2=7$

c The line l is parallel to the vector

$$(\mathbf{i}+\mathbf{k})\times(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\3&2&2\end{vmatrix}$$

$$=-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

To find one point on both Π_1 and Π_2

For Π_1 $\quad x+z=3$

Let $z=0$, then $x=3$

The form $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=0$ is that of a line passing through a point with position vector $\mathbf{p}$, parallel to the vector $\mathbf{q}$. So you need to find one point on the line; that is any point which is on both Π_1 and Π_2. As there are infinitely many such points, there are many possible answers to this question. The choice of $z=0$ here is an arbitrary one.

Substituting $z=0, x=3$ into a Cartesian equation of Π_2

$3x+2y+z=7$

$9+2y+0 \;=7\Rightarrow y=-1$

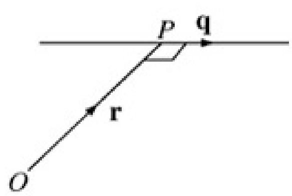
One point on Π_1 and Π_2 and, hence on l is $(3,-1,0)$

Hence, a vector equation of l is $(\mathbf{r}-(3\mathbf{i}-\mathbf{j}))\times(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$

d A vector equation of l is

$\mathbf{r}=(3\mathbf{i}-\mathbf{j})+t(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})$

$=(3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}$



At P, $\mathbf{r}$ is perpendicular to l

$((3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}).(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$

$$-6+4t-1+t+4t=0\Rightarrow 9t=7\Rightarrow t=\frac{7}{9}$$

The coordinates of P are

$$(3-2t,-1+t,2t)=\left(\frac{13}{9},-\frac{2}{9},\frac{14}{9}\right)$$

At the point P which is nearest to the origin O, the position vector of P, $\mathbf{r}$, is perpendicular to the direction of the line, $\mathbf{q}$. Forming the scalar product $\mathbf{r}.\mathbf{q}$ and equating to zero gives you an equation in t.

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Review Exercise 2
Exercise A, Question 14

Question:

The points $A(2, 0, -1)$ and $B(4, 3, 1)$ have position vectors **a** and **b** respectively with respect to a fixed origin O.

a Find $\mathbf{a} \times \mathbf{b}$.

The plane Π_1 contains the points O, A and B.

b Verify that an equation of Π_1 is $x - 2y + 2z = 0$.

The plane Π_2 has equation $\mathbf{r} \cdot \mathbf{n} = d$ where $n = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and d is a constant. Given that B lies on Π_2,

c find the value of d.

The planes Π_1 and Π_2 intersect in the line L.

d Find an equation of L in the form $\mathbf{r} = \mathbf{p} + t\mathbf{q}$, where t is a parameter.

e Find the position vector of the point X on L where OX is perpendicular to L. **[E]**

Solution:

a $\mathbf{a}\times\mathbf{b} = (2\mathbf{i}-\mathbf{k})\times(4\mathbf{i}+3\mathbf{j}+\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 4 & 3 & 1 \end{vmatrix} = 3\mathbf{i}-6\mathbf{j}+6\mathbf{k}$$

b Substituting (0, 0, 0) into $x-2y+2z$

$0-2\times 0+2\times 0=0$

So the plane with equation $x-2y+2z=0$ contains O.

Similarly as

$2-2\times 0+2\times(-1)=2-2=0$

and $4-2\times 3+2\times 1=4-6+2=0$,

the plane with equation $x-2y+2z=0$

contains $A(2,0,-1)$ and $B(4,3,1)$.

'Verify' means check that the equation is satisfied by the data of this particular question. To do this you can just show that the coordinates of O, A and B satisfy $x-2y+2z=0$. You do not have to show any general methods.

c For B to lie on the plane with equation

$$\mathbf{r}.\mathbf{n}=d$$

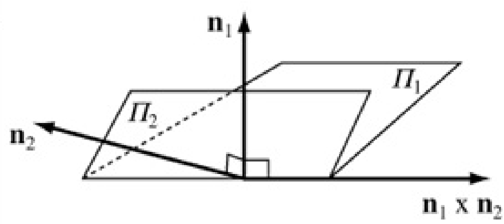
$(4\mathbf{i}+3\mathbf{j}+\mathbf{k}).(3\mathbf{i}+\mathbf{j}-\mathbf{k})=d$

$$d=4\times 3+3\times 1+1\times(-1)=12+3-1=14$$

d The line of intersection L lies in the direction given by

$$(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= 0\mathbf{i}+7\mathbf{j}+7\mathbf{k}$$



The vector $\mathbf{n}_1=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ is perpendicular to Π_1 and the vector $\mathbf{n}_2=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ is perpendicular to Π_2. This diagram illustrates the line of intersection of the planes is parallel to $\mathbf{n}_1\times\mathbf{n}_2$. This gives you the direction of L. To find the equation of L, you also need one point on L. In this case, the information given in the question shows you that you already have such a point, B.

A vector parallel to $7\mathbf{j}+7\mathbf{k}$ is $\mathbf{j}+\mathbf{k}$ and this is parallel to the line L.

The point B, which has position vector $4\mathbf{i}+3\mathbf{j}+\mathbf{k}$, lies on both Π_1 and Π_2 and, hence, on L.

A vector equation of L is

$\mathbf{r}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}+t(\mathbf{j}+\mathbf{k})$

e Rearranging the answer to part **d**

$\mathbf{r}=4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}$

At the point X on L where OX is perpendicular to L

$$\mathbf{r}.(\mathbf{j}+\mathbf{k})=0$$

$(4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}).(\mathbf{j}+\mathbf{k})=3+t+1+t=0$

$$2t=-4\Rightarrow t=-2$$

The position vector of X is

$4\mathbf{i}+(3-2)\mathbf{j}+(1-2)\mathbf{k}=4\mathbf{i}+\mathbf{j}-\mathbf{k}$

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Review Exercise 2
Exercise A, Question 15

Question:

The points A, B and C have position vectors, relative to a fixed origin O,
$\mathbf{a} = 2\mathbf{i} - \mathbf{j}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,
$\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$,
respectively. The plane Π passes through A, B and C.

a Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

b Show that a Cartesian equation of Π is $3x - y + 2z = 7$.

The line l has equation $(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0$.
The line l and the plane Π intersect at the point T.

c Find the coordinates of T.

d Show that A, B and T lie on the same straight line. **[E]**

Solution:

a $\overrightarrow{AB} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\3\\3\end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix}2\\3\\2\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\4\\2\end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-1\\3\\3\end{pmatrix} \times \begin{pmatrix}0\\4\\2\end{pmatrix} = \begin{pmatrix}-6\\2\\-4\end{pmatrix}$

b A vector equation of Π is $\mathbf{r}.\begin{pmatrix}-6\\2\\-4\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}.\begin{pmatrix}-6\\2\\-4\end{pmatrix} = -12-2 = -14$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r.n} = \mathbf{a.n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of A has been used but the position vectors of B and C would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here -14.

Let $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\mathbf{r}.\begin{pmatrix}-6\\2\\-4\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-6\\2\\-4\end{pmatrix} = -6x+2y-4z = -14$

A Cartesian equation of Π is

$-6x+2y-4z = -14$

Dividing throughout by -2

$3x-y+2z = 7$, as required

c A vector equation of the line l is

$\mathbf{r} = \begin{pmatrix}5\\5\\3\end{pmatrix} + t\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

The two vector forms of a straight line $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = 0$ and $\mathbf{r} = \mathbf{a}+t\mathbf{b}$ are equivalent and you can always interchange one with the other, Here $\mathbf{a} = \begin{pmatrix}5\\5\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$.

Parametric equations of l are

$x = 5+2t, y = 5-t, z = 3-2t$

Substituting the parametric equations into

$$\begin{aligned} 3x-y+2z &= 7 \\ 3(5+2t)-(5-t)+2(3-2t) &= 7 \\ 15+6t-5+t+6-4t &= 7 \\ 3t &= -9 \Rightarrow t = -3 \end{aligned}$$

The coordinates of T are

$$\begin{aligned} (5+2t, 5-t, 3-2t) &= (5-6, 5+3, 3+6) \\ &= (-1, 8, 9) \end{aligned}$$

d $\overrightarrow{BT} = \overrightarrow{OT} - \overrightarrow{OB} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix}$

When A, B and T lie on the same straight line, AB and BT are in the same direction. So you show that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BT}$ are parallel.

From part **a**

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

Hence

$\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BT}$ and AB is parallel to BT.

Hence A, B and T lie in the same straight line.

Points which lie on the same straight line are called **collinear** points.

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Review Exercise 2
Exercise A, Question 16

Question:

The plane Π passes through the points $A(-1,-1,1)$, $B(4,2,1)$ and $C(2, 1, 0)$.

a Find a vector equation of the line perpendicular to Π which passes through the point $D(1, 2, 3)$.

b Find the volume of the tetrahedron $ABCD$.

c Obtain the equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$.

The perpendicular from D to the plane Π meets Π at the point E.

d Find the coordinates of E.

e Show that $DE = \dfrac{11\sqrt{35}}{35}$.

The point D' is the reflection of D in Π.

f Find the coordinates of D'. **[E]**

Solution:

a $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$

> The vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ is, by definition, perpendicular to both AB and AC. So it will also be perpendicular to the plane containing AB and AC.

An equation of the line, l say, which passes through D and is perpendicular to Π is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$

b $\overrightarrow{AD} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

$\overrightarrow{AD}.\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} . \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = -6 + 15 + 2 = 11$

The volume of the tetrahedron, V say, is given by

$V = \frac{1}{6}\left|\overrightarrow{AD}.\left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right| = \frac{1}{6}|11| = \frac{11}{6}$

c An equation for Π is

$\mathbf{r}.\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = 3 - 5 + 1$

$\mathbf{r}.\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = -1$

> The vector equation $\mathbf{r}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = p$ and the Cartesian equation $ax + by + cz = p$ are equivalents and one can always be replaced by the other.

d A Cartesian equation for Π is

$-3x + 5y + z = -1$

Parametric equations corresponding to the equation of l found in part **a** are

$x = 1 - 3t, y = 2 + 5t, z = 3 + t$

Substituting these parametric equations into the Cartesian equation for Π

$-3(1 - 3t) + 5(2 + 5t) + 3 + t = -1$

$-3 + 9t + 10 + 25t + 3 + t = -1$

$$35t = -11 \Rightarrow t = -\frac{11}{35}$$

The coordinates of E are given by

$$(1-3t, 2+5t, 3+t)$$

$$= \left(1+3\times\frac{11}{35}, 2-5\times\frac{11}{35}, 3-\frac{11}{35}\right)$$

$$= \left(\frac{68}{35}, \frac{15}{35}, \frac{94}{35}\right)$$

Use your calculator to help you work out these awkward fractions.

Of course, $\frac{15}{35} = \frac{5}{7}$ and this is acceptable as part of the answer. However, the subsequent working is easier if all the coordinates have the same denominator.

e $$DE^2 = \left(1-\frac{68}{35}\right)^2 + \left(2-\frac{15}{35}\right)^2 + \left(3-\frac{94}{35}\right)^2$$

$$= \left(\frac{33}{35}\right)^2 + \left(\frac{55}{35}\right)^2 + \left(\frac{11}{35}\right)^2$$

$$= \frac{33^2+55^2+11^2}{35^2} = \frac{4235}{1225} = \frac{121}{35}$$

Hence $DE = \sqrt{\left(\frac{121}{35}\right)} = \frac{11}{\sqrt{35}} = \frac{11\sqrt{35}}{35}$, as required.

The distance d between points with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) is given by

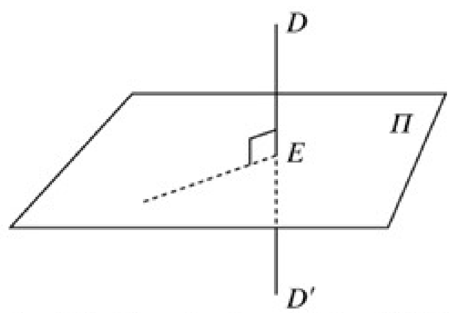
$d^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2$.

f The translation mapping D to E is represented by the vector

$$\overrightarrow{DE} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix}$$

The position vector of D' is given by

$$\overrightarrow{OD'} = \overrightarrow{OE} + \overrightarrow{DE} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix} + \begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix} = \begin{pmatrix} \frac{101}{35} \\ -\frac{40}{35} \\ \frac{83}{35} \end{pmatrix}$$



As D' is the reflection of D in Π, E is the mid-point of DD' and the translation which maps D to E also maps E to D'. So you can find the position vector of D' by adding $\begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix}$ to the position vector of E.

The coordinates of D' are $\left(\frac{101}{35}, -\frac{40}{35}, \frac{83}{35}\right)$

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Review Exercise 2
Exercise A, Question 17

Question:

The points A, B and C have position vectors $(\mathbf{j}+2\mathbf{k}), (2\mathbf{i}+3\mathbf{j}+\mathbf{k})$ and $(\mathbf{i}+\mathbf{j}+3\mathbf{k})$, respectively, relative to the origin O. The plane Π contains the points A, B and C.

a Find a vector which is perpendicular to Π.

b Find the area of ΔABC.

c Find a vector equation of Π in the form $\mathbf{r}\cdot\mathbf{n}=p$.

d Hence, or otherwise, obtain a Cartesian equation of Π.

e Find the distance of the origin O from Π.

The point D has position vector $(3\mathbf{i}+4\mathbf{j}+\mathbf{k})$. The distance of D from Π is $\frac{1}{\sqrt{17}}$.

f Using this distance, or otherwise, calculate the acute angle between the line AD and Π, giving your answer in degrees to one decimal place. **[E]**

Solution:

a Let $\mathbf{a} = \mathbf{j} + 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\mathbf{b} - \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\mathbf{c} - \mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + \mathbf{k}$

A vector which is perpendicular to Π is

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

> The vector product $(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})$ is, by definition, perpendicular to both $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ and, so, it is perpendicular to both AB and AC. It will also be perpendicular to the plane containing AB and AC.

b

$$\Delta ABC = \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$

$$= \frac{1}{2}|2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}|$$

$$= \frac{1}{2}\sqrt{\left(2^2 + (-3)^2 + (-2)^2\right)}$$

$$= \frac{\sqrt{17}}{2}$$

> The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

c A vector equation of Π is

$$\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 0 - 3 - 4$$

$$= -7$$

d A Cartesian equation of Π is $2x - 3y - 2z = -7$

> The vector equation $\mathbf{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = p$ and the Cartesian equation $ax + by + cz = p$ are equivalents.

e The distance from a point (α, β, γ) to a plane

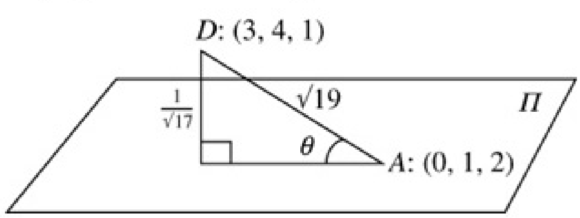
$n_1x + n_2y + n_2z + d = 0$ is $\left| \frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{\left(n_1^2 + n_2^3 + n_3^2\right)}} \right|$

Hence the distance from (0, 0, 0) to $2x - 3y - 2z = -7$

is $\left| \frac{7}{\sqrt{\left(2^2 + (-3)^2 + (-2)^2\right)}} \right| = \frac{7}{\sqrt{17}}$

> This formula is given in the Edexcel formulae booklet. If you use a formula from the booklet, it is sensible to quote it in your solution. The distance of a point from a plane is defined to be the shortest distance from the point to the plane; that is the perpendicular distance from the point to the plane.

f



Let the angle between AD and Π be θ

$$AD^2 = (3-0)^2 + (4-1)^2 + (1-2)^2 = 9 + 9 + 1 = 19$$

$$AD = \sqrt{19}$$

$$\sin\theta = \frac{\left(\frac{1}{\sqrt{17}}\right)}{\sqrt{19}} = 0.055641\ldots$$

$$\theta = 3.2^\circ \text{ (1 d.p.)}$$

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Review Exercise 2
Exercise A, Question 18

Question:

Relative to a fixed origin O the lines l_1 and l_2 have equations

$l_1 : \mathbf{r} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + s(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, $l_2 : \mathbf{r} = -\mathbf{j} + 7\mathbf{k} + t(-\mathbf{i} + \mathbf{j} - \mathbf{k})$,

where s and t are variable parameters.

a Show that the lines intersect and are perpendicular to each other.

b Find a vector equation of the straight line l_3 which passes through the point of intersection of l_1 and l_2 and the point with position vector $4\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k}$, where λ is a real number.

The line l_3 makes an angle θ with the plane containing l_1 and l_2.

c Find $\sin\theta$ in terms of λ.

Given that l_1, l_2 and l_3 are coplanar,

d find the value of λ. **[E]**

Solution:

a Equating the x components
$-1-2s=-t$ ①
Equating the y components
$2+s=-1+t$ ②
① + ② $1-s=-1 \Rightarrow s=2$
Substitute $s=2$ into ② $4=-1+t \Rightarrow t=5$
Checking the z components
For $l_1: -4+3s=-4+6=2$
For $l_2: 7-t=7-5=2$
These are the same, so l_1 and l_2 intersect.
The lines l_1 and l_2 are parallel to
$-2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}-\mathbf{k}$ respectively.
$(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}-\mathbf{k})=2+1-3=0$
Hence l_1 is perpendicular to l_2.

> To show that two lines intersect, you find the values of the two parameters, here s and t, which make two of the components equal and then you show that these values make the third components equal.

> As the scalar product $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors, if, for non-zero vectors, the scalar produce is zero then $\cos\theta=0$ and $\theta=90^\circ$

b Substituting $s=2$ into the equation for l_1, the common point has position vector
$-\mathbf{i}+2\mathbf{j}-4\mathbf{k}+2(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}$
Using $\mathbf{r}=\mathbf{a}+u(\mathbf{b}-\mathbf{a})$, an equation of l_3 is
$$\mathbf{r}=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}+u(4\mathbf{i}+\lambda\mathbf{j}-3\mathbf{k}-(-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}))$$
$$=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}+u(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k})$$

> $\mathbf{r}=\mathbf{a}+u(\mathbf{b}-\mathbf{a})$ is the appropriate form of the equation of a straight line going through two points with position vectors $\mathbf{a}$ and $\mathbf{b}$.
> Here
> $\mathbf{a}=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}$
> $\mathbf{b}=4\mathbf{i}+\lambda\mathbf{j}-3\mathbf{k}$.

c A vector $\mathbf{n}$ perpenticular to the plane, Π say, containing l_1 and l_2 is
$$\mathbf{n}=(-\mathbf{i}+\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & -1\\ -2 & 1 & 3\end{vmatrix}=4\mathbf{i}+5\mathbf{j}+\mathbf{k}$$
Let the angle between l_3 and Π be θ

$|\mathbf{n}|^2 = 4^2 + 5^2 + 1^2 = 42$

$|9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k}| = 9^2 + (\lambda - 4)^2 + (-5)^2$

$= 81 + \lambda^2 - 8\lambda + 16 + 25 = \lambda^2 - 8\lambda + 122$

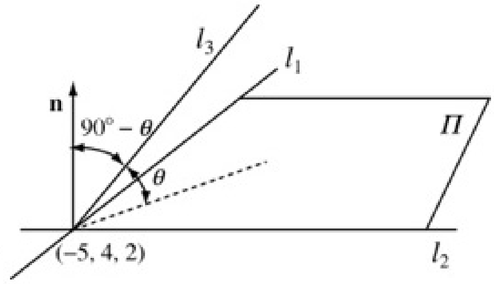
$\mathbf{n} \cdot (9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k}) = |\mathbf{n}||(9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k})| \cos(90^\circ - \theta)$

$(4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) \cdot (9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k})$

$= \sqrt{42} \times \sqrt{(\lambda^2 - 8\lambda + 122)} \sin\theta$

$\sin\theta = \dfrac{4 \times 9 + 5(\lambda - 4) + 1 \times (-5)}{\sqrt{42}\sqrt{(\lambda^2 - 8\lambda + 122)}} =$

$= \dfrac{5\lambda + 11}{\sqrt{42}\sqrt{(\lambda^2 - 8\lambda + 122)}}$



The cosine of the angle between **n** and l_3 can be found using the scalar product of **n** and a vector parallel to l_3. This cosine is $\sin\theta$.

d If l_1, l_2 and l_3 are coplanar then $\theta = 0$ and $\sin\theta = 0$

Hence $5\lambda + 11 = 0 \Rightarrow \lambda = \dfrac{-11}{5}$

Looking at the diagram in part **b** above, if l_3 lies in the plane Π, then $\theta = 0$.

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Review Exercise 2
Exercise A, Question 19

Question:

Referred to a fixed origin O, the planes Π_1 and Π_2 have equations $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 9$ and $\mathbf{r} \cdot (4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 8$ respectively.

a Determine the shortest distance from O to the line of intersection of Π_1 and Π_2.

b Find, in vector form, an equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the point with position vector $2\mathbf{j} + \mathbf{k}$.

c Find the position vector of the point that lies in Π_1, Π_2 and Π_3. **[E]**

Solution:

a The Cartesian equations of the planes are

$\Pi_1 : 2x - y + 2z = 9$ ①

$\Pi_2 : 4x + 3y - z = 8$ ②

① + 2× ②

$10x + 5y = 25$

$2x + y \quad = 5$

Let $x = t$, then $y = 5 - 2x = 5 - 2t$

From ②

> Points on the line of intersection of the two planes can be found by solving simultaneously the Cartesian equations of the two planes. As there are 2 equations in 3 unknowns, there are infinitely many solutions. A free choice can be made for one variable, here x is given the parameter t, and the other variables can then be found in terms of t.

$z = 4x + 3y - 8$

$\quad = 4t + 3(5 - 2t) - 8 = 7 - 2t$

The general point on the line of intersection of the planes has coordinates $(t, 5 - 2t, 7 - 2t)$

> This is the equivalent of the parametric equations of the common line $x = t, y = 5 - 2t, z = 7 - 2t$.
> The equivalent vector equation of this line is $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$.

The distance, y say, from O to this general point is given by

$y^2 = t^2 + (5 - 2t)^2 + (7 - 2t)^2$

$\quad = t^2 + 25 - 20t + 4t^2 + 49 - 28t + 4t^2$

$\quad = 9t^2 - 48t + 74$ ③

Differentiating both sides of ③ with respect to t

$2y\frac{dy}{dt} = 18t - 48$

> A calculus method of finding the minimum distance is shown here. You could instead use the property that, at the shortest distance, the position vector of the point is perpendicular to the common line. This method is illustrated in Question **13**.

At a minimum distance $\frac{dy}{dt} = 0$

$18t - 48 = 0 \Rightarrow t = \frac{48}{18} = \frac{8}{3}$

Substituting into ③

$y^2 = 9 \times \left(\frac{8}{3}\right)^2 - 48 \times \frac{8}{3} + 74$

$\quad = 64 - 128 + 74 = 10$

The shortest distance from O to the line of intersection of the planes is $\sqrt{10}$.

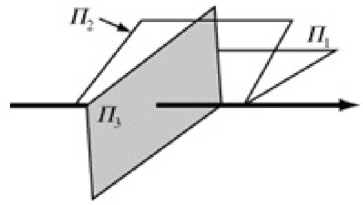
b The line of intersection of Π_1 and Π_2 has vector equation $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

Hence the vector $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is perpendicular to Π_3.

An equation of Π_3 is

$$\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = (2\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$
$$= -4 - 2 = -6$$

The common line of Π_1 and Π_2 is a normal to the plane Π_3 which is perpendicular to Π_1 and Π_2.



c Substituting $(t, 5 - 2t, 7 - 2t)$ into $x - 2y - 2z = -6$

$t - 2(5 - 2t) - 2(7 - 2t) = -6$

$t - 10 + 4t - 14 + 4t = -6 \Rightarrow 9t = 18 \Rightarrow t = 2$

The position vector of the common point is

$t\mathbf{i} + (5 - 2t)\mathbf{j} + (7 - 2t)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

The point that lies on the three planes is given by substituting the general point on the line of intersection of Π_1 and Π_2 into the Cartesian equation of Π_3.

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Review Exercise 2
Exercise A, Question 20

Question:

Vector equations of the two straight lines l and m are respectively

$\mathbf{r} = \mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + \mathbf{j} - \mathbf{k})$,

$\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + u(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$.

a Show that these lines do not intersect.

The point A with parameter t_1 lies on l and the point B with parameter u_1 lies on m.

b Write down the vector $\overrightarrow{AB}$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}, t_1$ and u_1.

Given that the line AB is perpendicular to both l and m,

c find the values of t_1 and u_1 and show that, in this case, the length of AB is $\frac{7}{\sqrt{5}}$. **[E]**

Solution:

a Equating the x components

$2t = 1 - 2u$ ①

Equating the y components

$1 + t = 1 + u \Rightarrow t = u$ ②

Substituting ② into ①

$2u = 1 - 2u \Rightarrow u = \frac{1}{4}$

As $t = u, t = \frac{1}{4}$

Checking the z components

For l: $3 - t = 3 - \frac{1}{4} = \frac{11}{4}$

For m: $-1 + u = -1 + \frac{1}{4} = -\frac{3}{4}$

$\frac{11}{4} \neq -\frac{3}{4}$, so the lines do not intersect.

To show that two lines do not intersect, you find the values of the two parameters, here t and u, which make two of the components equal and then you show that, with these values, the third components are not equal.

b $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - t_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 - 2t_1 - 2u_1 \\ -t_1 + u_1 \\ -4 + t_1 + u_1 \end{pmatrix}$

$= (1 - 2t_1 - 2u_1)\mathbf{i} + (-t_1 + u_1)\mathbf{j} + (-4 + t_1 + u_1)\mathbf{k}$

c If $\overrightarrow{AB}$ is perpendicular to l

$$\begin{pmatrix}1-2t_1-2u_1\\-t_1+u_1\\-4+t_1+u_1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=0$$

$$2-4t_1-4u_1-t_1+u_1+4-t_1-u_1=0$$

$$6t_1+4u_1=6 \quad ③$$

If $\overrightarrow{AB}$ is perpendicular to m

$$\begin{pmatrix}1-2t_1-2u_1\\-t_1+u_1\\-4+t_1+u_1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=0$$

$$-2-4t_1+4u_1-t_1+u_1-4+t_1+u_1=0$$

$$4t_1+6u_1=6 \quad ④$$

$3\times ④ - 2\times ③$

$$10u_1=6 \Rightarrow u_1=\frac{3}{5}$$

Substituting $u_1=\frac{3}{5}$ into ④

$$4t_1+\frac{18}{5}=6 \Rightarrow t_1=\frac{6-\frac{18}{5}}{4}=\frac{3}{5}$$

$$\overrightarrow{AB}=\begin{pmatrix}1-2t_1-2u_1\\-t_1+u_1\\-4+t_1+u_1\end{pmatrix}=\begin{pmatrix}1-\frac{6}{5}-\frac{6}{5}\\-\frac{3}{5}+\frac{3}{5}\\-4+\frac{3}{5}+\frac{3}{5}\end{pmatrix}=\begin{pmatrix}-\frac{7}{5}\\0\\-\frac{14}{5}\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right|^2=\left(-\frac{7}{5}\right)^2+\left(-\frac{14}{5}\right)^2=\frac{245}{25}=\frac{49}{5}$$

The length of AB is given by

$$\left|\overrightarrow{AB}\right|=\sqrt{\left(\frac{49}{5}\right)}=\frac{7}{\sqrt{5}}, \text{ as required.}$$

As $\overrightarrow{AB}$ is perpendicular to l, the scalar product of $\overrightarrow{AB}$ with the direction of l, which is $\begin{pmatrix}2\\1\\-1\end{pmatrix}$, is zero. This gives one equation in t_1 and u_1. Carrying out the same process with the direction of m, gives you a second equation in t_1 and u_1. You solve these simultaneous equations for t_1 and u_1 and use these values to find $\overrightarrow{AB}$. The magnitude of this vector is the length you have been asked to find.

This length is the shortest distance between the two skew lines. This question illustrates one of the methods by which this shortest distance can be found.

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Review Exercise 2
Exercise A, Question 21

Question:

$$A=\begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix}$$

Prove by induction, that for all positive integers n, $A^n=\begin{pmatrix}1&n&\frac{1}{2}(n^2+3n)\\0&1&n\\1&0&1\end{pmatrix}$. **[E]**

Solution:

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Let $n = 1$

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2+3\times 1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}$$

You start all inductions by showing that the formula you are asked to prove is true for a small number, usually 1.

The formula is true for $n = 1$.
Assume the formula is true for $n = k$.
That is

$$\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

This is the **induction hypothesis**. You assume that the formula is true for $n = k$ and show that this implies that the formula is true for $n = k+1$.

$$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$$

$$= \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1+k & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{pmatrix}$$

This term is obtained by multiplying the first row of the first matrix by the third row of the second matrix, that is

$$\begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

$$= 1\times 2 + k\times 1 + \frac{1}{2}(k^2+3k)\times 1$$

$$2+k+\frac{1}{2}(k^2+3k) = \frac{1}{2}k^2 + \frac{3k}{2} + k + 2$$

$$= \frac{1}{2}(k^2+5k+4)$$

$$= \frac{1}{2}(k^2+2k+1+3k+3)$$

$$= \frac{1}{2}\left((k+1)^2 + 3(k+1)\right)$$

Keep in mind what you are aiming for as you work out the algebra. You are looking to prove that the formula is true for $n = k+1$, so you are trying to reach $\frac{1}{2}(n^2+3n)$ with the n replaced by $k+1$.

$$\mathbf{A}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}\left((k+1)^2 + 3(k+1)\right) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the formula with $k+1$ substituted for n.
Hence, the formula is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.
By mathematical induction the formula is true for all positive integers n.

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Review Exercise 2
Exercise A, Question 22

Question:

$$A = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$$, where k is a real constant.

a Find the values of k for which A is singular.
Given that A is non-singular,
b find, in terms of k, A^{-1}. **[E]**

Solution:

a $\det \mathbf{A} = k\begin{vmatrix}-1 & k\\ 1 & 0\end{vmatrix} - 1\begin{vmatrix}0 & k\\ 9 & 0\end{vmatrix} + (-2)\begin{vmatrix}0 & -1\\ 9 & 1\end{vmatrix}$

$= k\times(-k) - 1\times(-9k) + (-2)\times 9$

$= -k^2 + 9k - 18 = 0$

$k^2 - 9k + 18 = (k-3)(k-6) = 0$

$k = 3, 6$

The 2×2 determinants are worked out using the formula $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$, which you learnt in book FP1.

A singular matrix is a matrix without an inverse. The determinant of a singular matrix is 0.

b The matrix of the minors, **M** say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}-1 & k\\ 1 & 0\end{vmatrix} & \begin{vmatrix}0 & k\\ 9 & 0\end{vmatrix} & \begin{vmatrix}0 & -1\\ 9 & 1\end{vmatrix} \\ \begin{vmatrix}1 & -2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}k & -2\\ 9 & 0\end{vmatrix} & \begin{vmatrix}k & 1\\ 9 & 1\end{vmatrix} \\ \begin{vmatrix}1 & -2\\ -1 & k\end{vmatrix} & \begin{vmatrix}k & -2\\ 0 & k\end{vmatrix} & \begin{vmatrix}k & 1\\ 0 & -1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -k & -9k & 9 \\ 2 & 18 & k-9 \\ k-2 & k^2 & -k \end{pmatrix}$$

The matrix of the cofactors, **C** say, is given by

$$\mathbf{C} = \begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & -k+9 \\ k-2 & -k^2 & -k \end{pmatrix}$$

As you have worked out the determinant of **A** in part **a**, the remaining steps for working out an inverse of a 3×3 matrix are:

1 Work out the matrix of the minors.

2 Obtain the matrix of cofactors by adjusting the signs of the minors using the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

3 Transpose the matrix of the cofactors.

4 Divide the transpose of the matrix of cofactors by the determinant of the matrix.

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & -k+9 & -k \end{pmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{-k^2 + 9k - 18}\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & -k+9 & -k \end{pmatrix}$$

You have worked out the determinant of **A** in part **a**. It is perfectly acceptable to leave your answer in this form. You do not have to divide every individual term in the matrix by $-k^2 + 9k - 18$.

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Review Exercise 2
Exercise A, Question 23

Question:

The matrix $\mathbf{M}$ is given by

$\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}$, where p, a, b and c are constants and $a > 0$. Given that $\mathbf{MM}^{\mathrm{T}} = k\mathbf{I}$

for some constant k, find

a the value of p,
b the value of k,
c the values of a, b and c,
d det $\mathbf{M}$. **[E]**

Solution:

a $\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix}$

$$\mathbf{M}\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}\begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix}$$

$$= \begin{pmatrix} 1+16+1 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+pc \\ a+4b-c & 3a+pc & a^2+b^2+c^2 \end{pmatrix}$$

$$= k\mathbf{I} = k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$\begin{pmatrix} 18 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+pc \\ a+4b-c & 3a+pc & a^2+b^2+c^2 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} *$$

If two matrices are equal, then all of the corresponding elements in the matrices must be equal. Potentially, there are 9 equations here. This equation has 5 unknowns and you pick out 5 equations which you can solve to find the unknowns.

Equating the second elements in the top row of *

$3-p=0 \Rightarrow p=3$

b Equating the first elements in the top row of *

$k=18$

c Equating each of the terms in the third row of * and using $p=3$ and $k=18$

$a+4b-c=0$ ①

$3a+3c=0$ ②

$a^2+b^2+c^2=18$ ③

You solve these 3 simultaneous equations by finding b and c in terms of a and, then, eliminating b and c. It is sensible to find a first as that is the unknown for which you are given the additional information that $a>0$.

From ② $c=-a$

Substituting $c=-a$ into ①

$a+4b+a=0 \Rightarrow 4b=-2a \Rightarrow b=-\frac{1}{2}a$

Substituting $c=-a$ and $b=-\frac{1}{2}a$ into ③

$a^2+\frac{1}{4}a^2+a^2=18$

$\frac{9a^2}{4}=18 \Rightarrow a^2=\frac{18\times 4}{9}=8$

As $a>0$

$a=\sqrt{8}=2\sqrt{2}$

$b=-\frac{1}{2}a=-\sqrt{2}$

$c=-a=-2\sqrt{2}$

d $\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & 3 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$

$$\det \mathbf{M} = 1\begin{vmatrix} 0 & 3 \\ -\sqrt{2} & -2\sqrt{2} \end{vmatrix} - 4\begin{vmatrix} 3 & 3 \\ 2\sqrt{2} & -2\sqrt{2} \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 2\sqrt{2} & -\sqrt{2} \end{vmatrix}$$

$$= 1 \times 3\sqrt{2} - 4 \times \left(-6\sqrt{2} - 6\sqrt{2}\right) + (-1) \times \left(-3\sqrt{2}\right)$$

$$= 3\sqrt{2} + 48\sqrt{2} + 3\sqrt{2} = 54\sqrt{2}$$

The 2×2 determinants are worked out using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which you learnt in book FP1.

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Review Exercise 2
Exercise A, Question 24

Question:

a Given that $\mathbf{A}=\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}$, find $\mathbf{A}^2$.

b Using $\mathbf{A}^3=\begin{pmatrix}10 & 9 & 23\\ 5 & 9 & 14\\ 9 & 5 & 19\end{pmatrix}$, show that $\mathbf{A}^3-5\mathbf{A}^2+6\mathbf{A}-\mathbf{I}=0$.

c Deduce that $\mathbf{A}(\mathbf{A}-2\mathbf{I})(\mathbf{A}-3\mathbf{I})=\mathbf{I}$.

d Hence find $\mathbf{A}^{-1}$. **[E]**

Solution:

a $A^2 = A \cdot A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+0+2 & 1+2+0 & 2+1+4 \\ 0+0+1 & 0+4+0 & 0+2+2 \\ 1+0+2 & 1+0+0 & 2+0+4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix}$

As an example, the third element in the third row is found by multiplying the third row of the first matrix by the third column of the second matrix. That is

$(1\ 0\ 2)\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 1\times 2 + 0\times 1 + 2\times 2$

$= 2+0+4 = 6$

b $A^3 - 5A^2 + 6A - I$

$= \begin{pmatrix} 10 & 9 & 23 \\ 5 & 9 & 14 \\ 9 & 5 & 19 \end{pmatrix} - 5\begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix} + 6\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 10-15+6-1 & 9-15+6-0 & 23-35+12-0 \\ 5-5+0-0 & 9-20+12-1 & 14-20+6-0 \\ 9-15+6-0 & 5-5+0-0 & 19-30+12-1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{O}$, as required.

When a matrix is multiplied by a scalar, each element in the matrix is multiplied by the scalar so

$6\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 12 \\ 0 & 12 & 6 \\ 6 & 0 & 12 \end{pmatrix}$

c $A^3 - 5A^2 + 6A - I = 0$

$A^3 - 5A^2 + 6A = I$

$A(A^2 - 5A + 6I) = I$

$A(A - 2I)(A - 3I) = I$, as required.

The rules for factorising expressions with matrices are essentially the same as those for factorising ordinary polynomials, so if $x^2 - 5x + 6 = (x-2)(x-3)$, then $A^2 - 5A + 6I = (A - 2I)(A - 3I)$. The **I**s are needed to preserve the dimensions of the matrices.

d Comparing $\mathbf{A}(\mathbf{A}-2\mathbf{I})(\mathbf{A}-3\mathbf{I})=\mathbf{I}$ with the definition of the inverse matrix $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$

$$\mathbf{A}^{-1}=(\mathbf{A}-2\mathbf{I})(\mathbf{A}-3\mathbf{I})$$
$$=\mathbf{A}^2-5\mathbf{A}+6\mathbf{I}$$

An alternative method is to work out the matrices $(\mathbf{A}-2\mathbf{I})$ and $(\mathbf{A}-3\mathbf{I})$ and multiply them together. The method shown here is a little quicker unless you have a calculator which multiplies matrices.

Hence

$$\mathbf{A}^{-1}=\begin{pmatrix}3&3&7\\1&4&4\\3&1&6\end{pmatrix}-5\begin{pmatrix}1&1&2\\0&2&1\\1&0&2\end{pmatrix}+6\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}3-5+6&3-5+0&7-10+0\\1-0+0&4-10+6&4-5+0\\3-5+0&1-0+0&6-10+6\end{pmatrix}$$
$$=\begin{pmatrix}4&-2&-3\\1&0&-1\\-2&1&2\end{pmatrix}$$

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Review Exercise 2
Exercise A, Question 25

Question:

Given that $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, use matrix multiplication to find

a $\mathbf{A}^2$,
b $\mathbf{A}^3$.

c Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n - 1 \\ 0 & 0 & 1 \end{pmatrix}, n \geq 1$.

d Find the inverse of $\mathbf{A}^n$. **[E]**

Solution:

a $A^2 = A \cdot A = \begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&4&2+1\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&4&3\\0&0&1\end{pmatrix}$

b $A^3 = A^2 \cdot A = \begin{pmatrix}1&0&0\\0&4&3\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&8&4+3\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&8&7\\0&0&1\end{pmatrix}$

> Using $A^3 = A \cdot A^2$ will give you the same result.

c $A^n = \begin{pmatrix}1&0&0\\0&2^n&2^n-1\\0&0&1\end{pmatrix}$

Let $n = 1$

> You start all inductions by showing that the formula you are asked to prove is true for a small number, usually 1.

$A^1 = \begin{pmatrix}1&0&0\\0&2^1&2^1-1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix} = A$

The formula is true for $n = 1$.
Assume the formula is true for $n = k$.
That is

$A^k = \begin{pmatrix}1&0&0\\0&2^k&2^k-1\\0&0&1\end{pmatrix}$

$A^{k+1} = A^k \cdot A$

$= \begin{pmatrix}1&0&0\\0&2^k&2^k-1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&2^k\times 2&2^k+2^k-1\\0&0&1\end{pmatrix}$

$2^k \times 2 = 2^k \times 2^1 = 2^{k+1}$

$2^k + 2^k - 1 = 2\times 2^k - 1 = 2^{k+1} - 1$

> The second element in the second row is found by multiplying the second row of the first matrix by the second column of the second matrix. That is
>
> $\begin{pmatrix}0&2^k&2^k-1\end{pmatrix}\begin{pmatrix}0\\2\\0\end{pmatrix} = 0\times 0 + 2^k \times 2 + (2^k - 1)\times 0$
>
> $= 2^k \times 2 = 2^{k+1}$.
>
> The third element in the second row is found by multiplying the second row of the first matrix by the third column of the second matrix. That is
>
> $\begin{pmatrix}0&2^k&2^k-1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} = 0\times 0 + 2^k \times 1 + (2^k - 1)\times 1$
>
> $= 2^k + 2^k - 1 = 2^{k+1} - 1$.

Hence $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{k+1} & 2^{k+1}-1 \\ 0 & 0 & 1 \end{pmatrix}$

This is the formula with $k+1$ substituted for n.
Hence, the formula is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.
By mathematical induction the formula is true for all positive integers n.

d $\det(\mathbf{A}^n) = 2^n$

The matrix of the minors of $\mathbf{A}^n$, $\mathbf{M}$ say, is given by

$$\mathbf{M} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2^n-1 & 2^n \end{pmatrix}$$

The matrix of the cofactors, $\mathbf{C}$ say, is given by

$$\mathbf{C} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1-2^n & 2^n \end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 1-2^n \\ 0 & 0 & 2^n \end{pmatrix}$$

The inverse of $\mathbf{A}^n$ is given by

$$(\mathbf{A}^n)^{-1} = \frac{1}{\det(\mathbf{A}^n)}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{2^n}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 1-2^n \\ 0 & 0 & 2^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & \frac{1-2^n}{2^n} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$$

A possible alternative approach is to note that the form of $\mathbf{A}^n$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n-1 \\ 0 & 0 & 1 \end{pmatrix}$, suggests that the inverse of $\mathbf{A}^n$, which is $(\mathbf{A}^n)^{-1} = \mathbf{A}^{-n}$, using the laws of indices, might be found by changing the n to $-n$, giving $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$, which is the correct answer.
Relations of this kind are commonly true for the powers of matrices but, in itself, this is not a sufficient argument. However, if you now verified that this was the inverse by multiplying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n-1 \\ 0 & 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$ and obtaining the identity matrix, this would be acceptable.

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Review Exercise 2
Exercise A, Question 26

Question:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & u \end{pmatrix}, u \neq 1$$

a Show that $\det \mathbf{A} = 2(u-1)$.

b Find the inverse of $\mathbf{A}$.

The image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & 6 \end{pmatrix}$ is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

c Find the values of a, b and c. **[E]**

Solution:

a $\det \mathbf{A} = 3\begin{vmatrix}1 & 1\\ 3 & u\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 5 & u\end{vmatrix} + (-1)\begin{vmatrix}1 & 1\\ 5 & 3\end{vmatrix}$

$= 3(u-3) - 1(u-5) - 1\times(-2)$

$= 3u - 9 - u + 5 + 2 = 2u - 2$

$= 2(u-1)$, as required

> Each 2×2 determinant is evaluated using the formula $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$.

b The matrix of the minors, **M** say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}1 & 1\\ 3 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\ 5 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\ 5 & 3\end{vmatrix} \\ \begin{vmatrix}1 & -1\\ 3 & u\end{vmatrix} & \begin{vmatrix}3 & -1\\ 5 & u\end{vmatrix} & \begin{vmatrix}3 & 1\\ 5 & 3\end{vmatrix} \\ \begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}3 & 1\\ 1 & 1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} u-3 & u-5 & -2 \\ u+3 & 3u+5 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

> The minor of an element is found by deleting the row and the column in which the element lies.
> For example, to find the minor of b in $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, delete the row and column through b $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.
> The minor is the determinant of the elements left, $\begin{vmatrix}d & f\\ g & i\end{vmatrix}$.

The matrix of the cofactors, **C** say, is given by

$$\mathbf{C} = \begin{pmatrix} u-3 & -u+5 & -2 \\ -u-3 & 3u+5 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{2(u-1)}\begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

c With $u = 6$

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The matrix in part **c** $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & 6 \end{pmatrix}$ is the matrix **A** of parts **a** and **b** with $u = 6$. To find the object vector when you are given the image vector, you will need the inverse matrix with $u = 6$.

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

This equation expresses the information that the image of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, under the transformation represented by the matrix **A**, is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

Multiplying both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

Hence, as $\mathbf{AA}^{-1} = \mathbf{I}$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 9-9+12 \\ -3+23-24 \\ -6-4+12 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -0.4 \\ 0.2 \end{pmatrix}$$

$a = 1.2, b = -0.4, c = 0.2$

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Review Exercise 2
Exercise A, Question 27

Question:

The transformation R is represented by the matrix $\mathbf{M}$, where $\mathbf{M}=\begin{pmatrix}3 & a & 0\\ 2 & b & 0\\ c & 0 & 1\end{pmatrix}$, and

where a, b and c are constants. Given that $\mathbf{M}=\mathbf{M}^{-1}$,

a find the values of a, b and c,

b evaluate the determinant of $\mathbf{M}$,

c find an equation satisfied by all the points which remain invariant under R. **[E]**

Solution:

a $\mathbf{MM} = \mathbf{I}$

$$\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By definition, $\mathbf{MM}^{-1} = \mathbf{I}$. As you have been given that $\mathbf{M} = \mathbf{M}^{-1}$, it follows that $\mathbf{MM} = \mathbf{I}$. This matrix is self-inverse.

$$\begin{pmatrix} 9+2a & 3a+ab & 0 \\ 6+2b & 2a+b^2 & 0 \\ 4c & ac & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If two matrices are equal, then all of the corresponding elements in the matrices must be equal. Potentially, there are 9 equations here. This question has 3 unknowns and you pick out 3 equations which you can solve to find the unknowns.

Equating the first elements in the first row
$9 + 2a = 1 \Rightarrow a = -4$
Equating the first elements in the second row
$6 + 2b = 0 \Rightarrow b = -3$
Equating the first elements in the third row
$4c = 0 \Rightarrow c = 0$

b Using the values of a, b and c found in part **a**

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \mathbf{M} = 3\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix}$$

$$= 3 \times (-3) + 4 \times 2 = -1$$

c Let the point which is invariant under R have position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The vector of an invariant point is unchanged when multiplied by the matrix representing the transformation.

$$\begin{pmatrix} 3x-4y \\ 2x-3y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The top and middle elements give the same equation and this is the equation satisfied by the invariant points. Equating the lowest elements gives $z = z$. This is an identity, always satisfied, and gives you no further information.

Equating the top elements
$3x - 4y = x \Rightarrow 2x - 4y = 0 \Rightarrow x = 2y$

Equating the middle elements
$2x - 3y = y \Rightarrow 2x - 4y = 0 \Rightarrow x = 2y$

An equation satisfied by all the invariant points is $x = 2y$.

The transformation is 3-dimensional and $x = 2y$ represents a plane of points.

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Review Exercise 2
Exercise A, Question 28

Question:

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{M}$.

The vector $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is transformed by T to $\begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$, the vector $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ is transformed to $\begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$

and the vector $\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$ is transformed to $\begin{pmatrix} -\alpha+1 \\ 5 \\ 2\alpha+2 \end{pmatrix}$, where $\alpha(\alpha \neq -1)$ is a constant.

a Find $\mathbf{M}$.

The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, where λ and μ are parameters,

and T transforms Π_1 to the plane Π_2.

b Find a Cartesian equation of Π_2. **[E]**

Solution:

a Let $\mathbf{M}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}=\begin{pmatrix} 2a-b \\ 2d-e \\ 2g-h \end{pmatrix}=\begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$$

This equation expresses the information that $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is transformed by T, the transformation represented by $\mathbf{M}$, to $\begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$. Equating the 3 elements gives 3 equations. The other two vector transformations, similarly, give 6 more equations and all 9 equations are needed to find the 9 elements of $\mathbf{M}$.

Equating the top elements

$2a-b=-5$ ①

Equating the middle elements

$2d-e=-1$ ②

Equating the lowest elements

$2g-h=0$ ③

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}=\begin{pmatrix} -b+2c \\ -e+2f \\ -h+2i \end{pmatrix}=\begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$$

Equating the top elements

$-b+2c=-1$ ④

Equating the middle elements

$-e+2f=9$ ⑤

Equating the lowest elements

$-h+2i=0$ ⑥

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} a\alpha+c \\ d\alpha+f \\ g\alpha+i \end{pmatrix}=\begin{pmatrix} -\alpha+1 \\ 5 \\ 2\alpha+2 \end{pmatrix}$$

Equating the top elements

$a\alpha+c=-\alpha+1$ ⑦

Equating the middle elements

$d\alpha+f=5$ ⑧

Equating the lowest elements

$g\alpha+i=2\alpha+2$ ⑨

Taking equations ①, ④ and ⑦

$2a-b=-5$ ①

$-b+2c=-1$ ④

$a\alpha+c=-\alpha+1$ ⑦

These 3 equations are 3 simultaneous equations in a, b and c. You solve them by eliminating a and b from the equations and finding c.

① − ④

$2a-2c=-4 \Rightarrow a=c-2$

Substitute $a = c - 2$ into ⑦

$$(c-2)\alpha + c = -\alpha + 1$$
$$c\alpha + c - \alpha - 1 = 0$$
$$c(\alpha+1) - 1(\alpha+1) = 0$$
$$(c-1)(\alpha+1) = 0$$

The condition $\alpha \neq -1$ is important in this question. If $\alpha = -1$, the equations could not be solved. You will notice the importance of this condition again later in the question. As frequently happens in mathematics, this special case is of considerable interest and is worth further investigation but this goes beyond the specification for this module.

As $\alpha \neq -1$

$$c = 1$$

$a = c - 2 = 1 - 2 = -1$

From ①

$b = 2a + 5 = -2 + 5 = 3$

Taking equations ②, ⑤ and ⑧

$2d - e = -1$ ②

$-e + 2f = 9$ ⑤

$d\alpha + f = 5$ ⑧

② − ⑤

$2d - 2f = -10 \Rightarrow f = d + 5$

Substitute

$$f = d + 5$$

$d\alpha + d + 5 = 5$

$d(\alpha+1) = 0$

As $\alpha \neq -1$

$d = 0$

$f = d + 5 = 0 + 5 = 5$

From ②

$e = 2d + 1 = 0 + 1 = 1$

Taking equations ③, ⑥ and ⑨

$2g - h = 0$ ③

$-h + 2i = 0$ ⑥

$g\alpha + i = 2\alpha + 2$ ⑨

③ − ⑥

$2g - 2i = 0 \Rightarrow g = i$

Substituting $g = i$ into ⑨

$$i\alpha + i = 2\alpha + 2$$

$i(\alpha+1) - 2(\alpha+1) = 0$

$(i-2)(\alpha+1) = 0$

As $\alpha \neq -1$

$i = 2$

$g = i = 2$

From ③

$h = 2g = 4$

Hence

$$\mathbf{M}=\begin{pmatrix}-1 & 3 & 1\\ 0 & 1 & 5\\ 2 & 4 & 2\end{pmatrix}$$

b Let $\mathbf{r}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, then

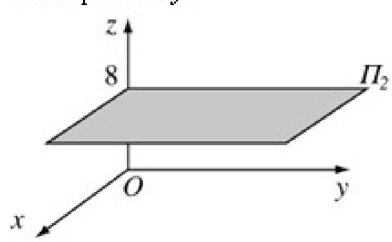
$$\mathbf{r}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}3\\ 0\\ 1\end{pmatrix}+\lambda\begin{pmatrix}2\\ -1\\ 0\end{pmatrix}+\mu\begin{pmatrix}0\\ -1\\ 2\end{pmatrix}=\begin{pmatrix}3+2\lambda\\ -\lambda-\mu\\ 1+2\mu\end{pmatrix}$$

This general point is transformed by T to

$$\begin{pmatrix}-1 & 3 & 1\\ 0 & 1 & 5\\ 2 & 4 & 2\end{pmatrix}\begin{pmatrix}3+2\lambda\\ -\lambda-\mu\\ 1+2\mu\end{pmatrix}=\begin{pmatrix}-1(3+2\lambda)+3(-\lambda-\mu)+1(1+2\mu)\\ 1(-\lambda-\mu)+5(1+2\mu)\\ 2(3+2\lambda)+4(-\lambda-\mu)+2(1+2\mu)\end{pmatrix}$$

$$=\begin{pmatrix}-2-5\lambda-\mu\\ 5-\lambda+9\mu\\ 8\end{pmatrix}$$

As λ and μ are parameters, the x- and y- coordinates of Π_2 can take all real values; there are no restrictions on these coordinates. However, the z-coordinate is 8, so the equation of Π_2 is $z=8$. This is a plane parallel to the plan Oxy.



An equation of Π_2 is $z=8$.

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Review Exercise 2
Exercise A, Question 29

Question:

The transformation $S:\mathbb{R}^3 \to \mathbb{R}^3$ maps the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ onto the point $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where

$a = x + y - z$

$b = \quad y + z$

$c = \quad\quad z,$

The matrix of this transformation is $\mathbf{A}$.

a By solving the given equations for x, y and z in terms of a, b and c, or otherwise, write down the matrix $\mathbf{A}^{-1}$.

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ has matrix $\mathbf{B} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$

b Given that $\mathbf{BB}^{\mathrm{T}} = k\mathbf{I}$, find the value of k.

U is the composite transformation consisting of T followed by S.

c Find the point whose image under U is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. **[E]**

Solution:

a $a = x + y - z$ ①
$b = y + z$ ②
$c = z$ ③
③ can be written as $z = c$

Substituting $z = c$ into ②
$b = y + c \Rightarrow y = b - c$

If $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then this set of 3 equations can be written as $\mathbf{a} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Substituting $z = c$ and $y = b - c$ into ①
$a = x + b - c - c \Rightarrow x = a - b + 2c$
Hence the three equations can be written as
$x = a - b + 2c$
$y = b - c$
$z = c$
or in vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

As $\mathbf{a} = \mathbf{A}\mathbf{x} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{a}$, then if you can find a matrix, $\mathbf{C}$ say, such that $\mathbf{x} = \mathbf{C}\mathbf{a}$, then $\mathbf{C} = \mathbf{A}^{-1}$.

Hence

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

b $$\mathbf{B}\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 9\mathbf{I}$$

$\mathbf{B}\mathbf{B}^{\mathrm{T}} = 9\mathbf{I}$ can be rewritten as $\mathbf{B}\left(\frac{1}{9}\mathbf{B}^{\mathrm{T}}\right) = \mathbf{I}$. As, by definition, $\mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$, in this case $\mathbf{B}^{-1} = \frac{1}{9}\mathbf{B}^{\mathrm{T}}$.

Hence
$k = 9$

c From part b $\mathbf{B}^{-1} = \frac{1}{9}\mathbf{B}^{\mathrm{T}} = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

The matrix representing U is $\mathbf{AB}$

The order of multiplying matrices is important. The matrix applied first, $\mathbf{B}$ representing the transformation T, is on the right. The matrix applied second, $\mathbf{A}$ representing the transformation S is on the left. You learnt a similar rule, when applying functions, in module C3: fg means 'do g first, then f'.

Let the point whose image under U is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ have vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\mathbf{AB}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

This equation expresses the information that the combined transformation $U = ST$ transforms $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. You use inverse matrices to solve this equation for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Multiplying both sides on the left by $(\mathbf{AB})^{-1}$

$$(\mathbf{AB})^{-1}.\mathbf{AB}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\mathbf{AB})^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Using $(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{B}^{-1}\mathbf{A}^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 3-2+2 \\ -6+1+2 \\ 6+2+1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 \\ -3 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

The point whose image under U is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ has vector $\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$.

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Review Exercise 2
Exercise A, Question 30

Question:

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$$

a Find the eigenvalues of $\mathbf{M}$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There is a line through the origin for which every point on the line is mapped onto itself under T.

b Find the Cartesian equation of this line. **[E]**

Solution:

a

$$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{pmatrix}$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-9-\lambda) - (-5)\times 6$$

$$= -36 - 4\lambda + 9\lambda + \lambda^2 + 30$$

$$= \lambda^2 + 5\lambda - 6 = 0$$

The eigenvalues of a square matrix are found by solving the polynomial $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. You find the determinant of a 2×2 matrix using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

$$(\lambda+6)(\lambda-1) = 0$$

$$\lambda = -6, 1$$

The eigenvalues of $\mathbf{M}$ are -6 and 1.

b Let the line through the origin have equation $y = mx$. If t is a real parameter, the point (t, mt) lies on the line. Under T, the point (t, mt) is mapped onto itself.

$$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} t \\ mt \end{pmatrix}$$

$$\begin{pmatrix} 4t - 5mt \\ 6t - 9mt \end{pmatrix} = \begin{pmatrix} t \\ mt \end{pmatrix}$$

An alternative method is to use the fact that a line of invariant points is determined by the eigenvector corresponding to $\lambda = 1$. The general eigenvector is $t\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $(5t, 3t)$ always lies on $y = \frac{3}{5}x$.

Equating the upper elements

$$4t - 5mt = t$$

$$5mt = 3t \Rightarrow m = \frac{3}{5}$$

$t = 0$ is an answer but that gives you no additional information as the point $(0, 0)$ lies on $y = \frac{3}{5}x$.

An equation of the line of invariant points is $y = \frac{3}{5}x$.

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Review Exercise 2
Exercise A, Question 31

Question:

A transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A} = \begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix}$, where k is a constant.

For the case $k = -4$,

a find the image under T of the line with equation $y = 2x + 1$.

For the case $k = 2$, find

b the two eigenvalues of A,

c a Cartesian equation of the two lines passing through the origin which are invariant under T. **[E]**

Solution:

a If t is a real parameter, the point $(t, 2t+1)$ lies on the line with equation $y = 2x+1$, for all t.
With $k = -4$,

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} t \\ 2t+1 \end{pmatrix} = \begin{pmatrix} -4t+4t+2 \\ 2t-2t-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

The image under T of the line with equation $y = 2x+1$ is the point with coordinates $(2, -1)$.

> The whole of the line is mapped onto a single point. Usually a line is mapped onto a line but it is not always the case. Here the determinant of the matrix is 0 and the matrix is singular. With a singular matrix, a line may map onto a single point.

b With $k = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) - 4 = -2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda+2)(\lambda-3) = 0$$

$$\lambda = -2, 3$$

The eigenvalues of $\mathbf{A}$ are -2 and 3.

> $\lambda^2 - \lambda - 6$ is the **characteristic polynomial** of the matrix $\mathbf{A}$.

c Using the eigenvalues from part **b**
With $\lambda = -2$,

$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating the upper elements
$2x + 2y = -2x \Rightarrow y = -2x$
With $\lambda = 3$,

$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating the upper elements

$$2x + 2y = 3x \Rightarrow y = \frac{1}{2}x$$

The Cartesian equations of the lines are

$$y = \frac{1}{2}x \text{ and } y = -2x.$$

> Vectors directed along the invariant lines are eigenvectors.
> A Cartesian equation of the invariant line corresponding to an eigenvalue λ can be found by writing the equation for an eigenvector,
> $$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix},$$
> and equating either the upper or the lower elements. Either of the elements will give you the same equation.

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Review Exercise 2
Exercise A, Question 32

Question:

The eigenvalues of the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, are λ_1 and λ_2, where $\lambda_1 < \lambda_2$.

a Find the value of λ_1 and the value of λ_2.

b Find $\mathbf{M}^{-1}$.

c Verify that the eigenvalues of $\mathbf{M}^{-1}$ are λ_1^{-1} and λ_2^{-1}.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There are two lines, passing through the origin, each of which is mapped onto itself under the transformation T.

d Find Cartesian equations for each of these lines. **[E]**

Solution:

a $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{vmatrix}=0$$

$$(4-\lambda)(1-\lambda)+2=4-4\lambda-\lambda+\lambda^2+2=0$$

$$\lambda^2-5\lambda+6=0$$

$$(\lambda-2)(\lambda-3)=0$$

$$\lambda=2,3$$

As $\lambda_1<\lambda_2$

$\lambda_1=2, \lambda_2=3$

b $\mathbf{M}^{-1}=\frac{1}{4+2}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}=\frac{1}{6}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}$

> You use the formula for the inverse of a 2×2 matrix, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$. The formulae for the determinant and the inverse of a 2×2 matrix are parts of the FP1 specification and these formulae may be tested on an FP3 paper.

c $\mathbf{M}^{-1}-\lambda\mathbf{I}=\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{6}-\lambda & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\lambda\end{pmatrix}$

$$\begin{vmatrix}\frac{1}{6}-\lambda & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\lambda\end{vmatrix}=\left(\frac{1}{6}-\lambda\right)\left(\frac{2}{3}-\lambda\right)+\frac{1}{18}=0$$

$$\frac{1}{9}-\frac{1}{6}\lambda-\frac{2}{3}\lambda+\lambda^2+\frac{1}{18}=0$$

$$2-3\lambda-12\lambda+18\lambda^2+1=0$$

$$18\lambda^2-15\lambda+3=0$$

$$6\lambda^2-5\lambda+1=(2\lambda-1)(3\lambda-1)=0$$

$\lambda=\frac{1}{2},\frac{1}{3}=\lambda_1^{-1},\lambda_2^{-1}$, as required

> It would also be acceptable to substitute $\lambda=\frac{1}{2}$ and $\lambda=\frac{1}{3}$ into this determinant, evaluate the determinant and show that both substitutions gave 0. For example
>
> $$\begin{vmatrix}\frac{1}{6}-\frac{1}{2} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\frac{1}{2}\end{vmatrix}=\begin{vmatrix}-\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{6} & \frac{1}{6}\end{vmatrix}$$
>
> $$=-\frac{1}{18}+\frac{1}{18}=0.$$

d Using the eigenvalues from part **a**

With $\lambda = 2$, $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements

$4x - 2y = 2x \Rightarrow y = x$

With $\lambda = 3$, $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements

$4x - 2y = 3x \Rightarrow y = \frac{1}{2}x$

The Cartesian equations of the lines are

$y = \frac{1}{2}x$ and $y = -2x$.

If you equated the lower elements, you would obtain $x + y = 2y \Rightarrow y = x$. Equating the upper and the lower elements gives the same answer.

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Review Exercise 2
Exercise A, Question 33

Question:

Find the eigenvalues and corresponding eigenvectors for the matrix $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

[E]

Solution:

Let $\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$, then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{pmatrix}$$

As the eigenvalues are the solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, the first step is to find $(\mathbf{A} - \lambda\mathbf{I})$. With practice, this step can just be written down and this working will not be shown in the working to later questions.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ -5 & -4-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & 1-\lambda \\ -5 & 2 \end{vmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)(-4-\lambda)-6\right] + 3(-12-3\lambda+15) + (6+5-5\lambda)$$

$$= (2-\lambda)(\lambda^2+3\lambda-10) - 9\lambda + 9 + 11 - 5\lambda$$

$$= 2\lambda^2 + 6\lambda - 20 - \lambda^3 - 3\lambda^2 + 10\lambda + 20 - 14\lambda$$

$$= -\lambda^3 - \lambda^2 + 2\lambda$$

This cubic is the **characteristic polynomial** of the matrix $\mathbf{A}$.

Using $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$-\lambda^3 - \lambda^2 + 2\lambda = 0$$

$$-\lambda(\lambda^2 + \lambda - 2) = -\lambda(\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 0, 1, -2$$

The eigenvalues of the matrix are $-2, 0$ and 1.

To find an eigenvector corresponding to -2.

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

An eigenvector corresponding to an eigenvalue λ is a solution of $\mathbf{A}x = \lambda\mathbf{x}$. There are always an infinite number of such solutions but they are all parallel.

$$\begin{pmatrix} 2x - 3y + z \\ 3x + y + 3z \\ -5x + 2y - 4z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

Equating the top elements

$2x-3y+z=-2x \Rightarrow 4x-3y+z=0$ ①

Equating the middle elements

Equating the three elements would give three equations. However two of the equations will usually be sufficient to find an eigenvector.

$3x+y+3z=-2y \Rightarrow 3x+3y+3z=0$ ②

① + ②

$7x+4z=0$

Let $x=4$, then $z=-7$

From ②

$y=-x-z=-4+7=3$

At this stage there is a free choice of one variable and the other variables can then be evaluated. Here x has been chosen as 4 as this avoids fractions but any value, other than 0, could be chosen.

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix}$.

Any non-zero multiple of this vector is also a correct answer.

To find an eigenvector corresponding to 0

You now repeat the procedure you used to find an eigenvector corresponding to -2 to find eigenvectors corresponding to 0 and 1.

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the top elements

$2x-3y+z=0$ ①

Equating the middle elements

$3x+y+3z=0$ ②

① $+3\times$ ②

$11x+10z=0$

Let $x=10$, then $z=-11$

From ②

$y=-3x-3z=-30+33=3$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 10 \\ 3 \\ -11 \end{pmatrix}$.

Again any non-zero multiple of the vector is correct. Zero is impossible as, by definition, eigenvectors are non-zero but note that, as here, an eigenvalue may be zero.

To find an eigenvector corresponding to 1

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements

$2x-3y+z = x \Rightarrow x-3y+z=0$ ①

Equating the middle elements

$3x+y+3z = y \Rightarrow 3x+3z=0 \Rightarrow x+z=0$ ②

In this case there is no y in equation 2 so it is not necessary to eliminate a variable between the equations and you can make a choice of x (or z) immediately.

Let $x=1$, then $z=-1$

From ①

$1-3y-1=0 \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

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Review Exercise 2
Exercise A, Question 34

Question:

Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix}$,

a find the eigenvalue of $\mathbf{A}$ corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

b find the value of p and the value of q.

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by $\mathbf{A}$ is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

c Using the values of p and q from part **b**, find the values of the constants l, m and n.

[E]

Solution:

a $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

If the column vector $\mathbf{x}$ is an eigenvector of the matrix $\mathbf{A}$ then for some number $\lambda, \mathbf{Ax} = \lambda\mathbf{x}$.

$\begin{pmatrix} 4-p \\ q+4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ -\lambda \end{pmatrix}$

Equating the three elements of these column vectors gives you three equations from which you can find the values of λ, p and q.

Equating the lowest elements
$-2 = -\lambda \Rightarrow \lambda = 2$

b Equating the top elements
$4 - p = 0 \Rightarrow p = 4$
Equating the middle elements
$q + 4 = \lambda = 2 \Rightarrow q = -2$

c Using the results of part **b**

$\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 3l+4m+4n \\ -l-2m-4n \\ l+m+3n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$

As $\mathbf{A}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$, then an alternative method is to find $\mathbf{A}^{-1}$ and use $\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$. However an inverse matrix is quite complicated to work out and, in this question, you have not been asked to find it. If the question does not require you to find the inverse, the method illustrated here, using simultaneous equations and the ordinary processes of algebra, is often carried out more quickly. If you use the inverse matrix then

$$\mathbf{A}^{-1} = -\frac{1}{6}\begin{pmatrix} -2 & -8 & -8 \\ -1 & 5 & 8 \\ 1 & 1 & -2 \end{pmatrix}.$$

Equating the elements of the vectors

$3l + 4m + 4n = 10$ ①

$-l - 2m - 4n = -4$ ②

$l + m + 3n = 3$ ③

① + 3 × ②

$-2m - 8n = -2$ ④

② + ③

$-m - n = -1$ ⑤

2 × ⑤ − ④

$6n = 0 \Rightarrow n = 0$

Substitute $n = 0$ into ⑤

$-m = -1 \Rightarrow m = 1$

Substitute $n = 0$ and $m = 1$ into ③

$l + 1 + 0 = 3 \Rightarrow l = 2$

$l = 2, m = 1, n = 0$

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Review Exercise 2
Exercise A, Question 35

Question:

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & -2 \\ -1 & 6 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

a Show that 3 is an eigenvalue of **A**.

b Find the other two eigenvalues of **A**.

c Find also a normalised eigenvector corresponding to the eigenvalue 3. **[E]**

Solution:

a Substitute $\lambda = 3$ into $\begin{vmatrix} 5-\lambda & 1 & -2 \\ -1 & 6-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$

If 3 is an eigenvalue of **A**, then $\lambda = 3$ must satisfy the equation $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$. So to solve part **a**, it is sufficient to substitute $\lambda = 3$ into this determinant and show that its value is 0.

$$\begin{vmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 2\times(-1) - 1\times 0 + (-2)\times(-1) = -2 + 2 = 0$$

Hence 3 is an eigenvalue of **A**.

b $\begin{vmatrix} 5-\lambda & 1 & -2 \\ -1 & 6-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$

$$= (5-\lambda)\begin{vmatrix} 6-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 0 & 3-\lambda \end{vmatrix} + (-2)\begin{vmatrix} -1 & 6-\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (5-\lambda)\left[(6-\lambda)(3-\lambda)-1\right] + (3-\lambda) + 2$$

$$= (5-\lambda)\left[18 - 9\lambda + \lambda^2 - 1\right] + 5 - \lambda$$

$$= (5-\lambda)\left[\lambda^2 - 9\lambda + 17\right] + 5 - \lambda$$

$$= 5\lambda^2 - 45\lambda + 85 - \lambda^3 + 9\lambda^2 - 17\lambda + 5 - \lambda$$

$$= 90 - 63\lambda + 14\lambda^2 - \lambda^3$$

The eigenvalues of **A** are the solutions of

$\lambda^3 - 14\lambda^2 + 63\lambda - 90 = 0$

Let

$\lambda^3 - 14\lambda^2 + 63\lambda - 90 = (\lambda-3)(\lambda^2 + a\lambda + 30)$

As you know $\lambda = 3$ is a solution of this equation, you can factorise this cubic either by long division or, as is shown here, by equating coefficients.

Equating the coefficients of λ^2

$-14 = -3 + a \Rightarrow a = -11$

Hence

$(\lambda-3)(\lambda^2 - 11\lambda + 30) = (\lambda-3)(\lambda-5)(\lambda-6) = 0$

$$\lambda = 3, 5, 6$$

The other two eigenvalues of **A** are 5 and 6.

c To find an eigenvector corresponding to 3.

$$\begin{pmatrix} 5 & 1 & -2 \\ -1 & 6 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x+y-2z \\ -x+6y+z \\ y+3z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the lowest elements

$y+3z=3z \Rightarrow y=0$ ①

Equating any 2 of the 3 elements will give you sufficient information to solve the question. Here the lowest elements give a particularly simple equation, so these have been used first.

Equating the top elements and substituting $y=0$

$5x-2z=3x \Rightarrow 2x=2z \Rightarrow x=z$ ②

Let $z=1$, then $x=1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The length of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $\sqrt{(1^2+0^2+1^2)}=\sqrt{2}$

A normalised eigenvector is an eigenvector of length 1. To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{(x^2+y^2+z^2)}$.

A normalised eigenvector corresponding to the eigenvalue 3 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Either of these forms is acceptable as an answer.

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Review Exercise 2
Exercise A, Question 36

Question:

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}$$

a Show that $\det A = 20 - 4k$.

b Find A^{-1}.

Given that $k = 3$ and that $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of A,

c find the corresponding eigenvalue.

Given that the only other distinct eigenvalue of A is 8,

d find a corresponding eigenvector. **[E]**

Solution:

a $\det \mathbf{A} = 3\begin{vmatrix} 0 & 2 \\ 2 & k \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & k \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix}$

$= 3(0-4) - 2(2k-8) + 4(4-0)$

$= -12 - 4k + 16 + 16 = 20 - 4k$, as required.

b The matrix of the minors, $\mathbf{M}$ say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 2 & k \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 4 & k \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} \\ \cdots & \begin{vmatrix} 3 & 4 \\ 4 & k \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} \\ \cdots & \cdots & \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2k-8 & 4 \\ 2k-8 & 3k-16 & -2 \\ 4 & -2 & -4 \end{pmatrix}$$

As the matrix $\mathbf{A}$ is symmetric, the matrix of the minors is symmetric and you do not have to work out separately the three minors marked … in this matrix.

The matrix of the cofactors, $\mathbf{C}$ say, is given by

$$\mathbf{C} = \begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

The matrix of the cofactors is obtained from the matrix of the minors by changing the signs of the elements marked with a negative sign in this pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

As $\mathbf{C}$ is symmetric $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}} = \frac{1}{\det \mathbf{A}}\mathbf{C}$$

$$= \frac{1}{20-4k}\begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

c If $k = 3, \mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

The eigenvalue corresponding to $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is given by

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\lambda \\ -\lambda \end{pmatrix}$$

> If the column vector $\mathbf{x}$ is an eigenvector of the matrix $\mathbf{A}$ then the corresponding eigenvalue λ is given by $\mathbf{Ax} = \lambda\mathbf{x}$.

Equating the middle elements
$-2 = 2\lambda \Rightarrow \lambda = -1$
The corresponding eigenvalue is -1.

d To find an eigenvector corresponding to 8

$$\begin{pmatrix} 3x+2y+4z \\ 2x+2z \\ 4x+2y+3z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

> Equating any two of the three elements will enable you to find an eigenvector.

Equating the top elements
$3x+2y+4z = 8x \Rightarrow -5x+2y+4z = 0$ ①

Equating the lowest elements
$4x+2y+3z = 8z \Rightarrow 4x+2y-5z = 0$ ②
② − ①
$9x - 9z = 0 \Rightarrow x = z$
Let $z = 2$, then $x = 2$
Substitute $x = 2$ and $z = 2$ into ①
$-10+2y+8 = 0 \Rightarrow 2y = 2 \Rightarrow y = 1$

> Here you have a free choice of either x or z. This choice has been made to avoid fractions but any value of z could be chosen.

An eigenvector corresponding to the eigenvalue 8 is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

> Any non-zero multiple of this vector is also a correct answer.

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Review Exercise 2
Exercise A, Question 37

Question:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}$$

a Verify that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

b Show that 9 is another eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

c Given that the third eigenvector of $\mathbf{A}$ is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[E]**

Solution:

a $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+4 \\ -10+4 \\ 8-8+3 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix}$

$= 3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

An eigenvector is a vector whose direction is not changed by the transformation. So to verify that a column vector **x** is an eigenvector of **A**, you have to show that for some constant $\lambda, \mathbf{Ax} = \lambda\mathbf{x}$. λ, the magnification factor of the vector under the transformation, is the eigenvalue corresponding to **x**.

Hence $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and the corresponding eigenvalue is 3.

b Substitute $\lambda = 9$ into

$\begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix}$

An eigenvalue is a solution of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. To verify that 9 is an eigenvalue it is sufficient to substitute $\lambda = 9$ into this determinant and show that its value is 0.

$\begin{vmatrix} 1-9 & 0 & 4 \\ 0 & 5-9 & 4 \\ 4 & 4 & 3-9 \end{vmatrix} = \begin{vmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{vmatrix}$

$= (-8)\begin{vmatrix} -4 & 4 \\ 4 & -6 \end{vmatrix} - 0\begin{vmatrix} 0 & 4 \\ 4 & -6 \end{vmatrix} + 4\begin{vmatrix} 0 & -4 \\ 4 & 4 \end{vmatrix}$

$= (-8)(24-16) - 0 + 4(0+16)$

$= -8\times 8 + 4\times 16 = -64 + 64 = 0$

Hence 9 is an eigenvalue of **A**.

To find an eigenvector corresponding to 9.

$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} x+4z \\ 5y+4z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$

Equating the top elements

$x + 4z = 9x \Rightarrow -8x + 4z = 0 \Rightarrow z = 2x$

Let $x = 1$, then $z = 2$

Equating the middle elements

$5y + 4z = 9y \Rightarrow 4z = 4y \Rightarrow y = z$

As $z = 2, y = 2$

At this stage there is a free choice of one variable and the other variables can then be evaluated. Here x has been chosen as 1, as this avoids fractions, but any value, other than 0, could have been chosen,

An eigenvector corresponding to the eigenvalue 9 is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Any non-zero multiple of this vector is also a correct answer.

c $\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}2-8\\5-8\\8+4-6\end{pmatrix}=\begin{pmatrix}-6\\-3\\6\end{pmatrix}$

$=-3\begin{pmatrix}2\\1\\-2\end{pmatrix}$

The diagonal elements of the matrix **D** are the eigenvalues and you will need the eigenvalue corresponding to $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ to write down **D**.

The eigenvalue corresponding to $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ is -3.

The lengths of the vector $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is

$\sqrt{\left(2^2+(-2)^2+1^2\right)}=\sqrt{9}=3$

Similarly the lengths of $\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ are 3.

A normalised eigenvector is an eigenvector of length 1.
To normalise an eigenvector $\begin{pmatrix}x\\y\\z\end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{\left(x^2+y^2+z^2\right)}$. In this case the length of all three vectors is 3.

Normalised eigenvectors are $\begin{pmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix},\begin{pmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$ and $\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$.

$\mathbf{P}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}3&0&0\\0&9&0\\0&0&-3\end{pmatrix}$

If normalised eigenvectors are used to form **P** then the diagonal elements of **D** are the corresponding eigenvalues and this is the safest way to complete the question. However, any three distinct eigenvectors of the same magnitude can be used and the diagonal elements will be multiples of the eigenvalues. There are infinitely many possible answers. One other possible answer is

$\mathbf{P}=\begin{pmatrix}2&1&2\\-2&2&1\\1&2&-2\end{pmatrix},\mathbf{D}=\begin{pmatrix}27&0&0\\0&81&0\\0&0&-27\end{pmatrix}$.

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Review Exercise 2
Exercise A, Question 38

Question:

$$A = \begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

Given that $\lambda = -1$ and $\lambda = 8$ are two eigenvalues of $\mathbf{A}$,

a find the third eigenvalue of $\mathbf{A}$.

b Find the normalised eigenvector corresponding to the eigenvalue $\lambda = 8$.

Given that $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ are normalised eigenvectors corresponding to the other two eigenvalues,

c find a matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix.

d Find $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$. **[E]**

Solution:

a $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\begin{vmatrix} 6-\lambda & 2 & -3 \\ 2 & -\lambda & 0 \\ -3 & 0 & 2-\lambda \end{vmatrix}$$

$$=(6-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ -3 & 2-\lambda \end{vmatrix} + (-3)\begin{vmatrix} 2 & -\lambda \\ -3 & 0 \end{vmatrix}$$

$$=(6-\lambda)(-2\lambda+\lambda^2)-2(4-2\lambda)+(-3)(-3\lambda)$$

$$=-12\lambda+6\lambda^2+2\lambda^2-\lambda^3-8+4\lambda+9\lambda$$

$$=-\lambda^3+8\lambda^2+\lambda-8=0$$

$$\lambda^3-8\lambda^2-\lambda+8=0$$

$$\lambda^2(\lambda-8)-1(\lambda-8)=0$$

$$(\lambda^2-1)(\lambda-8)=0$$

$$(\lambda-1)(\lambda+1)(\lambda-8)=0$$

$$\lambda=1,-1,8$$

The third eigenvalue is 1.

As you know that -1 and 8 are roots of this equation you could just write down that the factors of the cubic are $(\lambda+1)(\lambda-8)(\lambda-1)$. However it is a good idea to factorise fully the expression, as shown here, to check that you have not made an error.

b

$$\begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x+2y-3z \\ 2x \\ -3x+2z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

Equating the middle elements

$2x=8y \Rightarrow x=4y$

Let $y=1$, then $x=4$

Equating the lowest elements

$-3x+2z=8z \Rightarrow 3x=-6z \Rightarrow x=-2z$

As $x=4$

$4=-2z \Rightarrow z=-2$

In general, the simplest equations are those with the fewest variables in them, so it is sensible to equate the middle and lowest terms.

An eigenvector corresponding to $\lambda = 8$ is $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

The length of $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(4^2 + 1^2 + (-2)^2\right)} = \sqrt{21}$.

A normalised vector corresponding to $\lambda = 8$ is

$$\begin{pmatrix} \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ \frac{-2}{\sqrt{21}} \end{pmatrix}$$

A normalised eigenvector is an eigenvector of length 1.

To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{\left(x^2 + y^2 + z^2\right)}$. The vector $\begin{pmatrix} \frac{-4}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \end{pmatrix}$ is also correct.

c $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{21}} \end{pmatrix}$

d To find the eigenvalue corresponding to $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$

$\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix with the eigenvalues on the diagonals in the order corresponding to the order of the normalised eigenvectors used to form $\mathbf{P}$. At this stage, you do not know if $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$ corresponds to 1 or -1, and you must establish which before proceeding.

$$\begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{6+4-9}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3+6}{\sqrt{14}} \end{pmatrix} = 1\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$$

1 is the eigenvalue corresponding to $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$

Hence $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

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Review Exercise 2
Exercise A, Question 39

Question:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}$$

a Find the eigenvalues and corresponding eigenvectors of $\mathbf{M}$.

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{M}$.

b Find Cartesian equations of the image of the line $\frac{x}{2} = y = \frac{z}{-1}$ under this transformation. **[E]**

Solution:

a $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 4 & 3 & 1-\lambda\end{pmatrix}$

The eigenvalues are the solutions of the equation $\det(\mathbf{M}-\lambda\mathbf{I})=0$.

$$\det(\mathbf{M}-\lambda\mathbf{I})=(1-\lambda)\begin{vmatrix}2-\lambda & 0\\ 3 & 1-\lambda\end{vmatrix}-0\begin{vmatrix}0 & 0\\ 4 & 1-\lambda\end{vmatrix}+1\begin{vmatrix}0 & 2-\lambda\\ 4 & 3\end{vmatrix}$$
$$=(1-\lambda)(2-\lambda)(1-\lambda)-0-4(2-\lambda)$$
$$=(2-\lambda)(1-\lambda)^2-4(2-\lambda)$$
$$=(2-\lambda)\left((1-\lambda)^2-4\right)=(2-\lambda)(\lambda^2-2\lambda-3)$$
$$=(2-\lambda)(\lambda-3)(\lambda+1)=0$$
$$\lambda=-1,2,3$$

$(2-\lambda)$ is a common factor of this expression. Taking this factor outside the expression at this stage avoids having to factorise a cubic later.

The eigenvalues of $\mathbf{M}$ are $-1, 2$ and 3.

To find an eigenvector corresponding to -1

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 4 & 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-1\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$
$$\begin{pmatrix}x+z\\ 2y\\ 4x+3y+z\end{pmatrix}=\begin{pmatrix}-x\\ -y\\ -z\end{pmatrix}$$

Equating the middle elements
$2y=-y \Rightarrow 3y=0 \Rightarrow y=0$
Equating the top elements
$x+z=-x \Rightarrow z=-2x$
Let $x=1$, then $z=-2$

In general, start by equating the elements with the fewest variables. Here the middle elements contain only y.

An eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}$.

To find an eigenvector corresponding to 2

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 4 & 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$
$$\begin{pmatrix}x+z\\ 2y\\ 4x+3y+z\end{pmatrix}=\begin{pmatrix}2x\\ 2y\\ 2z\end{pmatrix}$$

Equating the middle elements here gives $2y=2y$. This is a simple identity and gives you no information so you must use the other elements.

Equating the top elements
$x+z=2x \Rightarrow x=z$
Let $z=1$, then $x=1$
Equating the lowest elements
$4x+3y+z=2z \Rightarrow 3y=z-4x$
As $x=1$ and $z=1$
$3y=1-4 \Rightarrow y=-1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

To find an eigenvector corresponding to 3

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+z \\ 2y \\ 4x+3y+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the middle elements

$2y = 3y \Rightarrow y = 0$

Equating the top elements

$x + z = 3x \Rightarrow z = 2x$

Let $x = 1$, then $z = 2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

b Let $\frac{x}{2} = y = \frac{z}{-1} = t$

The $x = 2t, y = t, z = -t$

$(2t, t, -t)$ is the parametric form of the general point on the line

Although there are other methods, when finding the images of lines under three-dimensional linear transformations, it is usually sensible to find the parametric form of a general point on the line and to obtain the image of the general point by matrix multiplication.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2t \\ t \\ -t \end{pmatrix} = \begin{pmatrix} 2t-t \\ 2t \\ 8t+3t-t \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 10t \end{pmatrix}$$

The image of the line under this transformation is $x = t, y = 2t, z = 10t$

Hence

$$x = \frac{y}{2} = \frac{z}{10} = t$$

Eliminating t gives Cartesian equations of the line.

Cartesian equations of the image of the line are

$$x = \frac{y}{2} = \frac{z}{10}.$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 40

Question:

a Show that 9 is an eigenvalue of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}$.

b Find the other two eigenvalues of the matrix.

c Find also normalised eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ corresponding to each of these eigenvalues.

d Verify that the matrix **P** with columns $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ is an orthogonal matrix. **[E]**

Solution:

a $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{vmatrix}$

$= (6-\lambda)\begin{vmatrix} 5-\lambda & 0 \\ 0 & 7-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & 0 \\ 2 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 5-\lambda \\ 2 & 0 \end{vmatrix}$

$= (6-\lambda)(5-\lambda)(7-\lambda) + 2(-14+2\lambda) + 2(-10+2\lambda)$

$= (6-\lambda)(5-\lambda)(7-\lambda) - 28 + 4\lambda - 20 + 4\lambda$

$= (6-\lambda)(5-\lambda)(7-\lambda) + 8\lambda - 48$

$= (6-\lambda)(5-\lambda)(7-\lambda) + 8(\lambda - 6)$

$= (6-\lambda)(35 - 12\lambda + \lambda^2 - 8)$

$= (6-\lambda)(27 - 12\lambda + \lambda^2)$

$= (6-\lambda)(3-\lambda)(9-\lambda)$

$(6-\lambda)(3-\lambda)(9-\lambda) = 0$

$\lambda = 3, 6, 9$

9 is an eigenvalue of the matrix.

b The other eigenvalues are 3 and 6

c **To find an eigenvector corresponding to 3**

$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 6x - 2y + 2z \\ -2x + 5y \\ 2x + 7z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$

Equating the middle terms

$-2x + 5y = 3y \Rightarrow 2y = 2x \Rightarrow y = x$

Let $x = 2$, then $y = 2$

Equating the lowest terms

$2x + 7z = 3z \Rightarrow 2x = -4z \Rightarrow x = -2z$

As $x = 2$, $z = -1$

You can find an eigenvector by equating any two of the elements. You can then choose a non-zero value for any one of the variables and use it to calculate the values of the other variables. Here x has been chosen to be 2 in order to avoid fractions.

An eigenvector corresponding to 3 is $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

The length of $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is $\sqrt{\left(2^2 + 2^2 + (-1)^2\right)} = \sqrt{9} = 3$

A normalised eigenvector is an eigenvector of length 1. To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{\left(x^2 + y^2 + z^2\right)}$.

A normalised eigenvector, $\mathbf{x}_1$ say, corresponding to 3 is $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$.

$\begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ is also correct.

To find an eigenvector corresponding to 6

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the middle terms

$-2x+5y=6y \Rightarrow y=-2x$

Let $x=-1$, then $y=2$

Equating the lowest terms

$2x+7z=6z \Rightarrow z=-2x$

As $x=-1$, $z=2$

An eigenvector corresponding to 6 is $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

The length of $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ is $\sqrt{\left((-1)^2+2^2+2^2\right)}=\sqrt{9}=3$

A normalised eigenvector, $\mathbf{x}_2$ say, corresponding to 6 is

$$\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Again the negative of this vector is correct. There are always two normalised vectors corresponding to an eigenvalue. These two are essentially the same vector in opposite directions.

To find an eigenvector corresponding to 9.

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$$

Equating the middle terms

$-2x+5y=9y \Rightarrow -2x=4y \Rightarrow x=-2y$

Let $y=-1$, then $x=2$

Equating the lowest terms

$2x+7z=9z \Rightarrow x=z$

As $x=2$, $z=2$

An eigenvector corresponding to 9 is $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

The length of $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is $\sqrt{\left(2^2+(-1)^2+2^2\right)} = \sqrt{9} = 3$

A normalised eigenvector, $\mathbf{x}_3$ say, corresponding to 9 is $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

d $\mathbf{P} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

> There are two ways of testing that a matrix is orthogonal. One is to show that $\mathbf{PP}^{\mathrm{T}} = \mathbf{I}$ and the other is to show that the 3 normalised column vectors are orthogonal to each other. The second method has been used here. The scalar product of each of the three pairs of vectors is shown to be zero and, as all of the vectors are non-zero, this shows the vectors are mutually orthogonal.

$$\mathbf{x}_1.\mathbf{x}_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} . \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3}\times\left(-\frac{1}{3}\right)+\frac{2}{3}\times\frac{2}{3}+\left(-\frac{1}{3}\right)\times\frac{2}{3}$$

$$= -\frac{2}{9}+\frac{4}{9}-\frac{2}{9}=0$$

Hence $\mathbf{x}_1$ is orthogonal (perpendicular) to $\mathbf{x}_2$.

$$\mathbf{x}_1 \cdot \mathbf{x}_3 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3}\times\frac{2}{3}+\frac{2}{3}\times\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\times\frac{2}{3}$$

$$= \frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0$$

Hence $\mathbf{x}_1$ is orthogonal (perpendicular) to $\mathbf{x}_3$.

Orthogonal and perpendicular have the same meaning.

$$\mathbf{x}_2 \cdot \mathbf{x}_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \left(-\frac{1}{3}\right)\times\frac{2}{3}+\frac{2}{3}\times\left(-\frac{1}{3}\right)+\frac{2}{3}\times\frac{2}{3}$$

$$= -\frac{2}{9}-\frac{2}{9}+\frac{4}{9}=0$$

Hence $\mathbf{x}_2$ is orthogonal (perpendicular) to $\mathbf{x}_3$.

The matrix $\mathbf{P}$ is an orthogonal matrix.