Chapter review 6

1 $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} t & 1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} t & 3 \\ -2 & -1 \end{vmatrix}$$

$$= 1(3+1) - 0 + 2(-t+6)$$

$$= 4 - 2t + 12$$

$$= 16 - 2t$$

Since **A** is singular, det(**A**) = 0, therefore:

$16 - 2t = 0 \Rightarrow t = 8$

2 $\mathbf{M}=\begin{pmatrix}1&0&0\\x&2&0\\3&1&1\end{pmatrix}$

Step 1

$$\det(\mathbf{M})=1\begin{vmatrix}2&0\\1&1\end{vmatrix}-0\begin{vmatrix}x&0\\3&1\end{vmatrix}+0\begin{vmatrix}x&2\\3&1\end{vmatrix}$$
$$=1(2-0)$$
$$=2$$

Step 2

$$\mathbf{N}=\begin{pmatrix}\begin{vmatrix}2&0\\1&1\end{vmatrix}&\begin{vmatrix}x&0\\3&1\end{vmatrix}&\begin{vmatrix}x&2\\3&1\end{vmatrix}\\\begin{vmatrix}0&0\\1&1\end{vmatrix}&\begin{vmatrix}1&0\\3&1\end{vmatrix}&\begin{vmatrix}1&0\\3&1\end{vmatrix}\\\begin{vmatrix}0&0\\2&0\end{vmatrix}&\begin{vmatrix}1&0\\x&0\end{vmatrix}&\begin{vmatrix}1&0\\x&2\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}2-0&x-0&x-6\\0-0&1-0&1-0\\0-0&0-0&2-0\end{pmatrix}$$
$$=\begin{pmatrix}2&x&x-6\\0&1&1\\0&0&2\end{pmatrix}$$

Step 3

$$\mathbf{C}=\begin{pmatrix}2&-x&x-6\\0&1&-1\\0&0&2\end{pmatrix}$$

Step 4

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}2&0&0\\-x&1&0\\x-6&-1&2\end{pmatrix}$$

Step 5

$$\mathbf{M}^{-1}=\frac{1}{\det(\mathbf{M})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{2}\begin{pmatrix}2&0&0\\-x&1&0\\x-6&-1&2\end{pmatrix}$$

3 a $\mathbf{M} = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}$, $\lambda_1 = 5$ and $\lambda_2 = -15$

To find an eigenvector corresponding to $\lambda_1 = 5$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements gives:

$x + 8y = 5x \Rightarrow x = 2y$

Setting $y = 1$ gives $x = 2$

Hence, an eigenvector corresponding to $\lambda_1 = 5$ is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

To find an eigenvector corresponding to $\lambda_2 = -15$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -15\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} -15x \\ -15y \end{pmatrix}$$

Equating the upper elements gives:

$x + 8y = -15x \Rightarrow 2x = -y$

Setting $x = 1$ gives $y = -2$

Hence, an eigenvector corresponding to $\lambda_2 = -15$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

b $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has magnitude $\sqrt{2^2 + 1^2} = \sqrt{5}$

Hence, a normalised eigenvector corresponding to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ has magnitude $\sqrt{1^2 + (-2)^2} = \sqrt{5}$

Hence, a normalised eigenvector corresponding to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$$

4 a $\mathbf{A}=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & -1\\ -4 & 2\end{pmatrix}$

$$\mathbf{AB}=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -4 & 2\end{pmatrix}$$
$$=\begin{pmatrix}10-8 & -5+4\\ 4-4 & -2+2\end{pmatrix}$$
$$=\begin{pmatrix}2 & -1\\ 0 & 0\end{pmatrix}$$

b $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$ and $\mathbf{B}^{\mathrm{T}}=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}$

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$$
$$=\begin{pmatrix}10-8 & 4-4\\ -5+4 & -2+2\end{pmatrix}$$
$$=\begin{pmatrix}2 & 0\\ -1 & 0\end{pmatrix}$$
$$\mathbf{AB}=\begin{pmatrix}2 & -1\\ 0 & 0\end{pmatrix}$$

Therefore:

$$(\mathbf{AB})^{\mathrm{T}}=\begin{pmatrix}2 & 0\\ -1 & 0\end{pmatrix}$$

Hence $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ as required

5 a $\mathbf{A}=\begin{pmatrix}-5 & 8\\ 3 & -7\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}-5-\lambda & 8\\ 3 & -7-\lambda\end{pmatrix}$$
$$\det(\mathbf{A}-\lambda\mathbf{I})=(-5-\lambda)(-7-\lambda)-24$$
$$=(5+\lambda)(7+\lambda)-24$$
$$=35+12\lambda+\lambda^2-24$$
$$=\lambda^2+12\lambda+11$$
$$=(\lambda+1)(\lambda+11)$$
$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$
$$(\lambda+1)(\lambda+11)=0$$
$$\lambda=-1 \text{ or } \lambda=-11$$

5 b When $\lambda = -1$

$$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Equating the upper elements gives:

$$-5x+8y = -x \Rightarrow y = \frac{1}{2}x$$

When $\lambda = -11$

$$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -11\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -11x \\ -11y \end{pmatrix}$$

Equating the upper elements gives:

$$-5x+8y = -11x \Rightarrow y = -\frac{3}{4}x$$

6 a $\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and 1 is an eigenvalue of $\mathbf{A}$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x+y \\ 2x+4y \\ x+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the lower elements gives:

$x+z = z \Rightarrow x = 0$

Equating the middle elements and setting $x = 0$ gives:

$0+4y = y \Rightarrow y = 0$

Equating the lower elements again and setting $x = 0$ gives:

$z = z$

Hence, $z = 1$

6 b $\mathbf{A}=\begin{pmatrix}3&1&0\\2&4&0\\1&0&1\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda&1&0\\2&4-\lambda&0\\1&0&1-\lambda\end{pmatrix}$$

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I})&=(3-\lambda)\left[(4-\lambda)(1-\lambda)-0\right]-1\left[2(1-\lambda)-0\right]+0\left[0-(4-\lambda)\right]\\&=(1-\lambda)(3-\lambda)(4-\lambda)-2(1-\lambda)\\&=(1-\lambda)\left[(3-\lambda)(4-\lambda)-2\right]\\&=(1-\lambda)\left(12-7\lambda+\lambda^2-2\right)\\&=(1-\lambda)\left(\lambda^2-7\lambda+10\right)\\&=(1-\lambda)(\lambda-2)(\lambda-5)\end{aligned}$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0$

$(1-\lambda)(\lambda-2)(\lambda-5)=0$

$\lambda=1$ or $\lambda=2$ or $\lambda=5$

Hence, the remaining eigenvalues are 2 and 5

7 $\mathbf{T}=\begin{pmatrix}4 & 3 & 0\\0 & -2 & 1\\3 & 1 & -2\end{pmatrix}$

$l_1:\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+t\begin{pmatrix}2\\-3\\0\end{pmatrix}$

$$\begin{pmatrix}4 & 3 & 0\\0 & -2 & 1\\3 & 1 & -2\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}4+0+0\\0+0+2\\3+0-4\end{pmatrix}$$

$$=\begin{pmatrix}4\\2\\-1\end{pmatrix}$$

$$\begin{pmatrix}4 & 3 & 0\\0 & -2 & 1\\3 & 1 & -2\end{pmatrix}\begin{pmatrix}2\\-3\\0\end{pmatrix}=\begin{pmatrix}8-9+0\\0+6+0\\6-3+0\end{pmatrix}$$

$$=\begin{pmatrix}-1\\6\\3\end{pmatrix}$$

$l_2:\mathbf{r}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+t\begin{pmatrix}-1\\6\\3\end{pmatrix}$

In Cartesian form this is the equation:

$$\frac{x-4}{-1}=\frac{y-2}{6}=\frac{z+1}{3}$$

8 a $\mathbf{A}=\begin{pmatrix}3 & 4 & -4\\ 4 & 5 & 0\\ -4 & 0 & 1\end{pmatrix}$

$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 4 & -4\\ 4 & 5-\lambda & 0\\ -4 & 0 & 1-\lambda\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= (3-\lambda)\left[(5-\lambda)(1-\lambda)-0\right]-4\left[4(1-\lambda)-0\right]-4\left[0+4(5-\lambda)\right]\\ &= (1-\lambda)(3-\lambda)(5-\lambda)-16(1-\lambda)-16(5-\lambda)\\ &= (1-\lambda)(3-\lambda)(5-\lambda)-16+16\lambda-80+16\lambda\\ &= (1-\lambda)(3-\lambda)(5-\lambda)-96+32\lambda\\ &= (1-\lambda)(3-\lambda)(5-\lambda)-32(3-\lambda)\\ &= (3-\lambda)\left[(1-\lambda)(5-\lambda)-32\right]\\ &= (3-\lambda)(5-6\lambda+\lambda^2-32)\\ &= (3-\lambda)(\lambda^2-6\lambda-27)\\ &= (3-\lambda)(\lambda-9)(\lambda+3)\end{aligned}$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0$

$(3-\lambda)(\lambda-9)(\lambda+3)=0$

$\lambda=-3$ or $\lambda=3$ or $\lambda=9$

Hence, 3 is an eigenvalue (as required) and the remaining eigenvalues are −3 and 9

b To find an eigenvector corresponding to the eigenvalue 3:

$$\begin{pmatrix}3 & 4 & -4\\ 4 & 5 & 0\\ -4 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}3x+4y-4z\\ 4x+5y\\ -4x+z\end{pmatrix}=\begin{pmatrix}3x\\ 3y\\ 3z\end{pmatrix}$$

Equating the elements of the middle row gives:

$4x+5y=3y \Rightarrow 2x=-y$

Setting $x=1$ gives $y=-2$

Equating the elements of the bottom row gives:

$-4x+z=3z \Rightarrow 2x=-z$

Hence $z=-2$

Therefore, an eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\ -2\\ -2\end{pmatrix}$

8 c The remaining two eigenvectors are $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\2\end{pmatrix}$

$\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ has magnitude $\sqrt{1^2+(-2)^2+(-2)^2}=3$

Hence a normalised eigenvector corresponding to $\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{pmatrix}$

$\begin{pmatrix}2\\2\\-1\end{pmatrix}$ has magnitude $\sqrt{2^2+2^2+(-1)^2}=3$

Hence a normalised eigenvector corresponding to $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ is $\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}$

$\begin{pmatrix}2\\-1\\2\end{pmatrix}$ has magnitude $\sqrt{2^2+(-1)^2+2^2}=3$

Hence a normalised eigenvector corresponding to $\begin{pmatrix}2\\-1\\2\end{pmatrix}$ is $\begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$

Therefore:

$$\mathbf{P}=\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix}$$

Note that **P** could also be formed with any reordering of the columns, and with any column(s) multiplied by −1

9 a $\mathbf{A} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = k\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 4-6+0 \\ -4+3-2 \\ 0+6-5 \end{pmatrix} = k\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = k\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

Hence $k = -1$ and therefore -1 is an eigenvalue of **A**, corresponding to $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = k\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4+2+0 \\ -4-1+2 \\ 0-2+5 \end{pmatrix} = k\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = k\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Hence $k = 3$ and therefore 3 is an eigenvalue of **A**, corresponding to $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

9 b $\mathbf{A} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$ and 6 is the third eigenvalue.

To find an eigenvector of **A** corresponding to eigenvalue 6:

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-2y \\ -2x+y+2z \\ 2y+5z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the upper elements gives:

$2x-2y=6x \Rightarrow 2x=-y$

Setting $x=1$ gives $y=-2$

Equating the lower elements and substituting $y=-2$ gives:
gives:

$-4+5z=6z \Rightarrow z=-4$

Hence an eigenvector of **A** corresponding to eigenvalue 6 is $\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$

9 c The eigenvectors of **A** are $\begin{pmatrix}2\\3\\-1\end{pmatrix}$, $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-4\end{pmatrix}$

$\begin{pmatrix}2\\3\\-1\end{pmatrix}$ has magnitude $\sqrt{2^2+3^2+(-1)^2}=\sqrt{14}$

Therefore, a normalised eigenvector corresponding to $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ is $\begin{pmatrix}\frac{2}{\sqrt{14}}\\\frac{3}{\sqrt{14}}\\-\frac{1}{\sqrt{14}}\end{pmatrix}$

$\begin{pmatrix}2\\-1\\1\end{pmatrix}$ has magnitude $\sqrt{2^2+(-1)^2+1^2}=\sqrt{6}$

Therefore, a normalised eigenvector corresponding to $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ is $\begin{pmatrix}\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$

$\begin{pmatrix}1\\-2\\-4\end{pmatrix}$ has magnitude $\sqrt{1^2+(-2)^2+(-4)^2}=\sqrt{21}$

Therefore, a normalised eigenvector corresponding to $\begin{pmatrix}1\\-2\\-4\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{21}}\\-\frac{2}{\sqrt{21}}\\-\frac{4}{\sqrt{21}}\end{pmatrix}$

$$\mathbf{P}=\begin{pmatrix}\frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}}\\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}}\\ -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{21}}\end{pmatrix}$$

Note that **P** could also be formed with any reordering of the columns, and with any column(s) multiplied by −1

10 a $\mathbf{A}=\begin{pmatrix}1 & x & -1\\ 3 & 0 & 2\\ 1 & 1 & 0\end{pmatrix}$

Step 1

$$\det(\mathbf{A})=1\begin{vmatrix}0 & 2\\ 1 & 0\end{vmatrix}-x\begin{vmatrix}3 & 2\\ 1 & 0\end{vmatrix}-1\begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}$$
$$=1(0-2)-x(0-2)-1(3-0)$$
$$=-2+2x-3$$
$$=2x-5$$

Step 2

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}0 & 2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}3 & 2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}\\ \begin{vmatrix}x & -1\\ 1 & 0\end{vmatrix} & \begin{vmatrix}1 & -1\\ 1 & 0\end{vmatrix} & \begin{vmatrix}1 & x\\ 1 & 1\end{vmatrix}\\ \begin{vmatrix}x & -1\\ 0 & 2\end{vmatrix} & \begin{vmatrix}1 & -1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}1 & x\\ 3 & 0\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}0-2 & 0-2 & 3-0\\ 0+1 & 0+1 & 1-x\\ 2x-0 & 2+3 & 0-3x\end{pmatrix}$$
$$=\begin{pmatrix}-2 & -2 & 3\\ 1 & 1 & 1-x\\ 2x & 5 & -3x\end{pmatrix}$$

Step 3

$$\mathbf{C}=\begin{pmatrix}-2 & 2 & 3\\ -1 & 1 & x-1\\ 2x & -5 & -3x\end{pmatrix}$$

Step 4

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-2 & -1 & 2x\\ 2 & 1 & -5\\ 3 & x-1 & -3x\end{pmatrix}$$

Step 5

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{2x-5}\begin{pmatrix}-2 & -1 & 2x\\ 2 & 1 & -5\\ 3 & x-1 & -3x\end{pmatrix}$$

10 b $\mathbf{A} = \begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{A}^{-1} = \frac{1}{2x-5}\begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$

Therefore:

$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{A}^{-1} = \begin{pmatrix} -2 & -1 & 6 \\ 2 & 1 & -5 \\ 3 & 2 & -9 \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 & 6 \\ 2 & 1 & -5 \\ 3 & 2 & -9 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -8-3+30 \\ 8+3-25 \\ 12+6-45 \end{pmatrix} \\ &= \begin{pmatrix} 19 \\ -14 \\ -27 \end{pmatrix} \end{aligned}$$

So

$a = 19,\ b = -14,\ c = -27$

11 a $\mathbf{A} = \begin{pmatrix} \alpha & 0 & 2 \\ 4 & 3 & 0 \\ -2 & -1 & 1 \end{pmatrix}$

If $\det(\mathbf{A}-\mathbf{I}) = 0$ then 1 must be an eigenvalue of $\mathbf{A}$. So calculate $\det(\mathbf{A}-\mathbf{I})$:

$$\mathbf{A}-\mathbf{I} = \begin{pmatrix} \alpha-1 & 0 & 2 \\ 4 & 2 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A}-\mathbf{I}) &= (\alpha-1)(0-0) - 0(0-0) + 2(-4+4) \\ &= 0+0+0 \\ &= 0 \end{aligned}$$

Therefore, for all values of α an eigenvalue is 1

11 b $\begin{pmatrix} \alpha & 0 & 2 \\ 4 & 3 & 0 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \beta\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2\alpha+0+2 \\ 8-6+0 \\ -4+2+1 \end{pmatrix} = \beta\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha+2 \\ 2 \\ -1 \end{pmatrix} = \beta\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Equating the lower elements gives:
$\beta = -1$
Equating the upper elements gives:
$2\alpha + 2 = -2 \Rightarrow \alpha = -2$

c When $\alpha = -2$

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 2 \\ 4 & 3 & 0 \\ -2 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (-2-\lambda)(3-\lambda)(1-\lambda) + 0 + 2(-4 + 2(3-\lambda)) \\ &= -(2+\lambda)(3-\lambda)(1-\lambda) - 8 + 12 + 4\lambda \\ &= -(2+\lambda)(3-\lambda)(1-\lambda) + 4(1-\lambda) \\ &= (1-\lambda)[4 - (2+\lambda)(3-\lambda)] \\ &= (1-\lambda)[\lambda^2 - \lambda - 2] \\ &= (1-\lambda)(\lambda-2)(\lambda+1) \end{aligned}$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$(1-\lambda)(\lambda-2)(\lambda+1) = 0$

$\lambda = -1$ or $\lambda = 1$ or $\lambda = 2$

Hence when $\alpha = -2$, the third eigenvalue of **A** is 2

12 a $\mathbf{A}=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & u\\ 0 & 1 & 1\end{pmatrix}$

Step 1

$$\det(\mathbf{A})=1\begin{vmatrix}1 & u\\ 1 & 1\end{vmatrix}+1\begin{vmatrix}2 & u\\ 0 & 1\end{vmatrix}+3\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}$$
$$=1(1-u)+1(2-0)+3(2-0)$$
$$=1-u+2+6$$
$$=9-u$$

Step 2

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}1 & u\\ 1 & 1\end{vmatrix} & \begin{vmatrix}2 & u\\ 0 & 1\end{vmatrix} & \begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}-1 & 3\\ 1 & 1\end{vmatrix} & \begin{vmatrix}1 & 3\\ 0 & 1\end{vmatrix} & \begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}-1 & 3\\ 1 & u\end{vmatrix} & \begin{vmatrix}1 & 3\\ 2 & u\end{vmatrix} & \begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}1-u & 2-0 & 2-0\\ -1-3 & 1-0 & 1-0\\ -u-3 & u-6 & 1+2\end{pmatrix}$$
$$=\begin{pmatrix}1-u & 2 & 2\\ -4 & 1 & 1\\ -u-3 & u-6 & 3\end{pmatrix}$$

Step 3

$$\mathbf{C}=\begin{pmatrix}1-u & -2 & 2\\ 4 & 1 & -1\\ -u-3 & 6-u & 3\end{pmatrix}$$

Step 4

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}1-u & 4 & -u-3\\ -2 & 1 & 6-u\\ 2 & -1 & 3\end{pmatrix}$$

Step 5

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{9-u}\begin{pmatrix}1-u & 4 & -u-3\\ -2 & 1 & 6-u\\ 2 & -1 & 3\end{pmatrix}\quad u\neq 9$$

12 b $\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix}$

When $u = 4$,

$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} -0.6 & 0.8 & -1.4 \\ -0.4 & 0.2 & 0.4 \\ 0.4 & -0.2 & 0.6 \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 & -1.4 \\ -0.4 & 0.2 & 0.4 \\ 0.4 & -0.2 & 0.6 \end{pmatrix}\begin{pmatrix} -2.8 \\ 5.3 \\ 2.3 \end{pmatrix}$

$= \begin{pmatrix} 1.68 + 4.24 - 3.22 \\ 1.12 + 1.06 + 0.92 \\ -1.12 - 1.06 + 1.38 \end{pmatrix}$

$= \begin{pmatrix} 2.7 \\ 3.1 \\ -0.8 \end{pmatrix}$

$a = 2.7$, $b = 3.1$ and $c = -0.8$

13 a $\mathbf{M} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & -1 & 3 \end{pmatrix}$

$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 4 & -1 & 3-\lambda \end{pmatrix}$

$\det(\mathbf{M} - \lambda\mathbf{I}) = (3-\lambda)[(1-\lambda)(3-\lambda)+1] + 0[1(3-\lambda)-4] + 0[-1-4(1-\lambda)]$

$= (3-\lambda)[(1-\lambda)(3-\lambda)+1]$

$= (3-\lambda)(3-4\lambda+\lambda^2+1)$

$= (3-\lambda)(\lambda^2-4\lambda+4)$

$= (3-\lambda)(\lambda-2)(\lambda-2)$

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

$(3-\lambda)(\lambda-2)(\lambda-2) = 0$

$\lambda = 3$ or $\lambda = 2$ repeated

Hence **M** has only two distinct eigenvalues.

13 b To find an eigenvector corresponding to eigenvalue 3:

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ x+y+z \\ 4-y+3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the lower elements gives:

$4x - y + 3z = 3z \Rightarrow y = 4x$

Setting $x = 1$ gives $y = 4$

Equating the middle elements and substituting $x = 1$ and $y = 4$ gives:

$1+4+z = 12 \Rightarrow z = 7$

Hence, an eigenvector corresponding to eigenvalue 3 is $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

To find an eigenvector corresponding to eigenvalue 2:

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ x+y+z \\ 4-y+3z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements gives:

$3x = 2x \Rightarrow x = 0$

Equating the middle elements and substituting $x = 0$ gives:

$0 + y + z = 2y \Rightarrow y = z$

Setting $y = 1$ gives $z = 1$

Hence, an eigenvector corresponding to eigenvalue 2 is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

14 a $\mathbf{P}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}}\\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}$

$\mathbf{P}^{\mathrm{T}}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}$

$$\mathbf{P}\mathbf{P}^{\mathrm{T}}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}}\\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{4}+\frac{1}{4}+\frac{1}{2} & \frac{1}{4}+\frac{1}{4}-\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0\\ \frac{1}{4}+\frac{1}{4}-\frac{1}{2} & \frac{1}{4}+\frac{1}{4}+\frac{1}{2} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0\\ \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}+0 & \frac{1}{2}+\frac{1}{2}+0\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Therefore, **P** is orthogonal.

14 b $\Pi_2: x + y - \sqrt{2}z = 0$

Find two non-parallel position vectors **v** and **w** inside Π_2:

Set $x = 0$ and $z = 1$: then $y = \sqrt{2}$

So $\mathbf{v} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}$ are position vectors inside Π_2

To find Π_1, determine **a** and **b** such that $\mathbf{Pa} = \mathbf{v}$ and $\mathbf{Pb} = \mathbf{w}$ as these will be non-parallel position vectors inside Π_1

So $\mathbf{a} = \mathbf{P}^{-1}\mathbf{v}$ and $\mathbf{b} = \mathbf{P}^{-1}\mathbf{w}$

As **P** is orthogonal, $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$ so $\mathbf{a} = \mathbf{P}^{\mathrm{T}}\mathbf{v}$ and $\mathbf{b} = \mathbf{P}^{\mathrm{T}}\mathbf{w}$

$$\mathbf{a} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}$$

$\mathbf{a} \times \mathbf{b}$ is in the normal direction of the plane Π_1

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 + 0 \\ \sqrt{2} + \sqrt{2} \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

Therefore Π_1 is $2\sqrt{2}y = 0$

Or simply, $y = 0$

15 a $\mathbf{A}=\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}$

$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=(3-\lambda)\left[(2-\lambda)(-2-\lambda)-0\right]+3\left[0(-2-\lambda)-0\right]+6(0-0)$

$=-(2-\lambda)(2+\lambda)(3-\lambda)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0$

$-(2-\lambda)(2+\lambda)(3-\lambda)=0$

$\lambda=-2$ or $\lambda=2$ or $\lambda=3$

b $\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=k\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}9-3+0\\0+2+0\\0+0+0\end{pmatrix}=k\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}6\\2\\0\end{pmatrix}=k\begin{pmatrix}3\\1\\0\end{pmatrix}$

Therefore $k=2$

Hence $\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of A

15 c $\mathbf{B}=\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}$

$$\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}=k\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}21-6+0\\ 3+2+0\\ 3-3+0\end{pmatrix}=k\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}15\\ 5\\ 0\end{pmatrix}=k\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}$$

Therefore $k=5$

Hence $\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}$ is an eigenvector of **B**

d So $\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}$ is an eigenvector of **AB** corresponding to the eigenvalue 10

16 a $\mathbf{A}=\begin{pmatrix}1&0&1\\3&1&1\\4&2&7\end{pmatrix}$

Step 1

$$\det(\mathbf{A})=1\begin{vmatrix}1&1\\2&7\end{vmatrix}-0\begin{vmatrix}3&1\\4&7\end{vmatrix}+1\begin{vmatrix}3&1\\4&2\end{vmatrix}$$
$$=1(7-2)-0+1(6-4)$$
$$=5+2$$
$$=7$$

Step 2

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}1&1\\2&7\end{vmatrix}&\begin{vmatrix}3&1\\4&7\end{vmatrix}&\begin{vmatrix}3&1\\4&2\end{vmatrix}\\\begin{vmatrix}0&1\\2&7\end{vmatrix}&\begin{vmatrix}1&1\\4&7\end{vmatrix}&\begin{vmatrix}1&0\\4&2\end{vmatrix}\\\begin{vmatrix}0&1\\1&1\end{vmatrix}&\begin{vmatrix}1&1\\3&1\end{vmatrix}&\begin{vmatrix}1&0\\3&1\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}7-2&21-4&6-4\\0-2&7-4&2-0\\0-1&1-3&1-0\end{pmatrix}$$
$$=\begin{pmatrix}5&17&2\\-2&3&2\\-1&-2&1\end{pmatrix}$$

Step 3

$$\mathbf{C}=\begin{pmatrix}5&-17&2\\2&3&-2\\-1&2&1\end{pmatrix}$$

Step 4

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}5&2&-1\\-17&3&2\\2&-2&1\end{pmatrix}$$

Step 5

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{7}\begin{pmatrix}5&2&-1\\-17&3&2\\2&-2&1\end{pmatrix}$$

16 b $x = \frac{y}{4} = \frac{z}{3}$

written in vector form this is the line:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

Let the line mapped onto $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ be $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 5+8-3 \\ -17+12+6 \\ 2-8+3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 10 \\ 1 \\ -3 \end{pmatrix}$$

So the mapped line is:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{10}{7} \\ \frac{1}{7} \\ -\frac{3}{7} \end{pmatrix}$$

In Cartesian form this is the line:

$$\frac{x}{\frac{10}{7}} = \frac{y}{\frac{1}{7}} = \frac{z}{-\frac{3}{7}}$$

$$\frac{x}{10} = \frac{y}{1} = \frac{z}{-3}$$

Challenge

a $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\text{tr}(\mathbf{AB}) = ae+bg+cf+dh$$

$$\mathbf{BA} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$$\text{tr}(\mathbf{BA}) = ae+bg+cf+dh$$

So tr(**AB**) = tr(**BA**) as required

b $\text{tr}(\mathbf{P}^{-1}\mathbf{MP}) = \text{tr}(\mathbf{P}^{-1}(\mathbf{MP}))$

$= \text{tr}((\mathbf{MP})\mathbf{P}^{-1})$

$= \text{tr}(\mathbf{M})$

Since $\text{tr}(\mathbf{P}^{-1}\mathbf{MP}) = p + q$ then $\text{tr}(\mathbf{M}) = p + q$ as required.