Exercise 6G

1 a $\mathbf{A}=\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1-\lambda & 3\\ 3 & 1-\lambda\end{pmatrix}$$

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I})&=(1-\lambda)(1-\lambda)-9\\&=1-2\lambda+\lambda^2-9\\&=\lambda^2-2\lambda-8\\&=(\lambda-4)(\lambda+2)\end{aligned}$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$(\lambda-4)(\lambda+2)=0$

$\lambda=-2$ or $\lambda=4$

So the eigenvalues of **A** are –2 and 4

To find an eigenvector of **A** corresponding to the eigenvalue –2:

$$\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-2\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x+3y\\ 3x+y\end{pmatrix}=\begin{pmatrix}-2x\\ -2y\end{pmatrix}$$

Equating the upper elements gives:

$x+3y=-2x\Rightarrow x=-y$

Let $x = 1$, then $y = -1$

Therefore, an eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix}1\\ -1\end{pmatrix}$

The magnitude of $\begin{pmatrix}1\\ -1\end{pmatrix}$ is $\sqrt{1^2+(-1)^2}=\sqrt{2}$

Hence a normalised eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 4:

$$\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=4\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x+3y\\ 3x+y\end{pmatrix}=\begin{pmatrix}4x\\ 4y\end{pmatrix}$$

Equating the upper elements gives:

$x+3y=4x\Rightarrow x=y$

Let $x = 1$, then $y = 1$

Therefore, an eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}1\\ 1\end{pmatrix}$

The magnitude of $\begin{pmatrix}1\\1\end{pmatrix}$ is $\sqrt{1^2+1^2}=\sqrt{2}$

Hence a normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$

Forming the orthogonal matrix **P** from the normalised eigenvectors gives:

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{AP}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 3\\3 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}}+\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}}\\\frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{2}{2}-\frac{2}{2} & -\frac{2}{2}+\frac{2}{2}\\-\frac{4}{2}+\frac{4}{2} & \frac{4}{2}+\frac{4}{2}\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 0\\0 & 4\end{pmatrix}$$

1 b $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(4-\lambda) - 4$$
$$= 4 - 5\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 5\lambda$$
$$= \lambda(\lambda - 5)$$

The eigenvalues are the solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Hence:

$\lambda(\lambda - 5) = 0$

$\lambda = 0$ or $\lambda = 5$

So the eigenvalues of **A** are 0 and 5.

To find an eigenvector of **A** corresponding to the eigenvalue 0:

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x-2y \\ -2x+4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Equating the upper elements gives:

$x - 2y = 0 \Rightarrow x = 2y$

Let $y = 1$, then $x = 2$

Therefore, an eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\sqrt{2^2 + 1^2} = \sqrt{5}$

Hence a normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 5:

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x-2y \\ -2x+4y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements gives:

$x - 2y = 5x \Rightarrow 2x = -y$

Let $y = 2$, then $x = -1$

Therefore, an eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

The magnitude of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is $\sqrt{(-1)^2 + 2^2} = \sqrt{5}$

Hence a normalised eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

Forming the orthogonal matrix **P** from the normalised eigenvectors gives:

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{AP}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{5}} & -\frac{4}{\sqrt{5}} + \frac{4}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -\frac{5}{\sqrt{5}} & \frac{10}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} 0+0 & 0+0 \\ -\frac{10}{5} + \frac{10}{5} & -\frac{5}{5} - \frac{20}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}$$

2 a $\mathbf{A} = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3-\lambda & \sqrt{2} \\ \sqrt{2} & 4-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(4-\lambda) - 2$$
$$= 12 - 7\lambda + \lambda^2 - 2$$
$$= \lambda^2 - 7\lambda + 10$$
$$= (\lambda-2)(\lambda-5)$$

The eigenvalues are the solutions of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Hence:

$(\lambda-2)(\lambda-5) = 0$

$\lambda = 2$ or $\lambda = 5$

So the eigenvalues of **A** are 2 and 5

2 **b** To find an eigenvector of **A** corresponding to the eigenvalue 2:

$$\begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+\sqrt{2}y \\ \sqrt{2}x+4y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Equating the upper elements gives:

$3x+\sqrt{2}y = 2x \Rightarrow x = -\sqrt{2}y$

Let $y = 1$, then $x = -\sqrt{2}$

Therefore, an eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ is $\sqrt{\left(-\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$

Hence a normalised eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} \dfrac{\sqrt{2}}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{3}} \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 5:

$$\begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+\sqrt{2}y \\ \sqrt{2}x+4y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements gives:

$3x+\sqrt{2}y = 5x \Rightarrow x = \dfrac{\sqrt{2}}{2}y$

Let $y = 1$, then $x = \dfrac{\sqrt{2}}{2}$

Therefore, an eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} \dfrac{\sqrt{2}}{2} \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} \dfrac{\sqrt{2}}{2} \\ 1 \end{pmatrix}$ is $\sqrt{\left(\dfrac{\sqrt{2}}{2}\right)^2 + 1^2} = \dfrac{\sqrt{6}}{2}$

Hence a normalised eigenvector corresponding to the eigenvalue 5 is:

$$\frac{1}{\frac{\sqrt{6}}{2}}\begin{pmatrix} \dfrac{\sqrt{2}}{2} \\ 1 \end{pmatrix} = \frac{\sqrt{6}}{3}\begin{pmatrix} \dfrac{\sqrt{2}}{2} \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

2 c Forming the orthogonal matrix **P** from the normalised eigenvectors gives:

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

$$\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$$

3 a $$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\mathbf{P}\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{2}{6}-\frac{1}{3}+0 \\ \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{2}{6}-\frac{1}{3}+0 \\ \frac{2}{6}-\frac{1}{3}+0 & \frac{2}{6}-\frac{1}{3}+0 & \frac{4}{6}+\frac{1}{3}+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence **P** is an orthogonal matrix.

3 b

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & \frac{3}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2\sqrt{6}} - \frac{3}{2\sqrt{6}} + \frac{2}{\sqrt{6}} & -\frac{3}{2\sqrt{6}} + \frac{3}{2\sqrt{6}} + \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} \\ -\frac{3}{2\sqrt{3}} + \frac{3}{2\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{3}{2\sqrt{3}} - \frac{3}{2\sqrt{3}} + \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ \frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} + 0 & -\frac{3}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} + 0 & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{6} + \frac{2}{6} + \frac{8}{6} & -\frac{2}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{4}{\sqrt{18}} & \frac{2}{\sqrt{12}} - \frac{2}{\sqrt{12}} + 0 \\ \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} & -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} & \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} + 0 \\ \frac{3}{\sqrt{12}} + \frac{3}{\sqrt{12}} - \frac{6}{\sqrt{12}} & -\frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} + 0 & \frac{3}{2} + \frac{3}{2} + 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix.

4 $\mathbf{A}=\begin{pmatrix}2&0&2\\0&2&0\\2&0&2\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&0&2\\0&2&0\\2&0&2\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}2-\lambda&0&2\\0&2-\lambda&0\\2&0&2-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)\begin{vmatrix}2-\lambda&0\\0&2-\lambda\end{vmatrix}-0\begin{vmatrix}2-\lambda&2\\2&2-\lambda\end{vmatrix}+2\begin{vmatrix}0&2-\lambda\\2&0\end{vmatrix}$$

$$=(2-\lambda)\left[(2-\lambda)(2-\lambda)-0\right]+2\left[0-2(2-\lambda)\right]$$

$$=(2-\lambda)(2-\lambda)(2-\lambda)-4(2-\lambda)$$

$$=(2-\lambda)\left[(2-\lambda)(2-\lambda)-4\right]$$

$$=(2-\lambda)\left(4-4\lambda+\lambda^2-4\right)$$

$$=(2-\lambda)\left(\lambda^2-4\lambda\right)$$

$$=\lambda(2-\lambda)(\lambda-4)$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence:

$\lambda(2-\lambda)(\lambda-4)=0$

Therefore $\lambda=0, \lambda=2, \lambda=4$

To find an eigenvector of **A** corresponding to the eigenvalue 0:

$$\begin{pmatrix}2&0&2\\0&2&0\\2&0&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}2x+2z\\2y\\2x+2z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Equating the top elements gives:

$2x+2z=0\Rightarrow x=-z$

Setting $x=1$ gives $z=-1$

Equating the middle elements gives:

$2y=0\Rightarrow y=0$

Therefore, an eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

The magnitude of $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ is $\sqrt{1^2+0^2+(-1)^2}=\sqrt{2}$

Hence a normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 2:

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2z \\ 2y \\ 2x+2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements gives:

$2x+2z=2x \Rightarrow z=0$

Equating the bottom elements gives:

$2x+2z=2z \Rightarrow z=0$

Equating the middle elements gives:

$2y=2y$

So set $y = 1$

Therefore, an eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is $\sqrt{0^2+1^2+0^2}=1$

Hence a normalised eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 4:

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2z \\ 2y \\ 2x+2z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements gives:

$2x+2z=4x \Rightarrow x=z$

Setting $x = 1$ gives $z = 1$

Equating the middle elements gives:

$2y=4y \Rightarrow y=0$

Therefore, an eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is $\sqrt{1^2+0^2+1^2}=\sqrt{2}$

Hence a normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$

Forming the orthogonal matrix **P** from the normalised eigenvectors gives:

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\0 & 1 & 0\\-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix} \text{ and } \mathbf{P}^{\mathrm{T}}=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\0 & 1 & 0\\\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

Hence:

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP}=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\0 & 1 & 0\\\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2 & 0 & 2\\0 & 2 & 0\\2 & 0 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\0 & 1 & 0\\-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{\sqrt{2}}+0-\frac{2}{\sqrt{2}} & 0+0+0 & \frac{2}{\sqrt{2}}-\frac{2}{\sqrt{2}}\\0+0+0 & 0+2+0 & 0+0+0\\\frac{2}{\sqrt{2}}+0+\frac{2}{\sqrt{2}} & 0+0+0 & \frac{2}{\sqrt{2}}+0+\frac{2}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\0 & 1 & 0\\-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\0 & 2 & 0\\\frac{4}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\0 & 1 & 0\\-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}0+0+0 & 0+0+0 & 0+0+0\\0+0+0 & 0+2+0 & 0+0+0\\\frac{4}{2}+0-\frac{4}{2} & 0+0+0 & \frac{4}{2}+0+\frac{4}{2}\end{pmatrix}$$

$$\mathbf{D}=\begin{pmatrix}0 & 0 & 0\\0 & 2 & 0\\0 & 0 & 4\end{pmatrix}$$

5 a $\mathbf{A}=\begin{pmatrix}5&3&3\\3&1&1\\3&1&1\end{pmatrix}$ has eigenvalues 0, –1 and 8

To find an eigenvector of **A** corresponding to the eigenvalue 0:

$$\begin{pmatrix}5&3&3\\3&1&1\\3&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}5x+3y+3z\\3x+y+z\\3x+y+z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Equating the top elements gives:

$5x+3y+3z=0\Rightarrow 3z=-5x-3y$ **(1)**

Equating the middle elements gives:

$3x+y+z=0\Rightarrow z=-3x-y$ **(2)**

Substituting **(2)** into **(1)** gives:

$3(-3x-y)=-5x-3y$

$-9x-3y=-5x-3y$

$4x=0\Rightarrow x=0$

Substituting $x=0$ into **(1)** gives:

$3z=0-3y\Rightarrow z=-y$

Setting $y=1$ gives $z=-1$

Therefore, an eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}0\\1\\-1\end{pmatrix}$

The magnitude of $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ is $\sqrt{0^2+1^2+(-1)^2}=\sqrt{2}$

Hence a normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$

5 b An eigenvector of **A** corresponding to the eigenvalue -1 is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is $\sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$

Hence a normalised eigenvector of **A** corresponding to the eigenvalue -1 is $\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

An eigenvector of **A** corresponding to the eigenvalue 8 is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

Hence a normalised eigenvector of **A** corresponding to the eigenvalue -1 is $\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$

Forming the orthogonal matrix **P** from the normalised eigenvectors gives:

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Step 1:

$$\det(\mathbf{P}) = \begin{vmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{vmatrix}$$

$$= -\frac{1}{\sqrt{3}}\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{vmatrix} - 0\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{vmatrix} + \frac{2}{\sqrt{6}}\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}}\right) - 0 + \frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\right)$$

$$= -\frac{2}{\sqrt{36}} - \frac{4}{\sqrt{36}}$$

$$= -1$$

Step 2:

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{vmatrix} & \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{vmatrix} & \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{vmatrix} & \begin{vmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{vmatrix} & \begin{vmatrix} -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{vmatrix} & \begin{vmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{vmatrix} & \begin{vmatrix} -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} \\ 0 + \frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} & \frac{1}{\sqrt{6}} - 0 \\ 0 - \frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} & -\frac{1}{\sqrt{6}} - 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{12}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{12}} & -\frac{3}{\sqrt{18}} & \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{12}} & -\frac{3}{\sqrt{18}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix} \frac{2}{\sqrt{12}} & 0 & -\frac{2}{\sqrt{6}} \\ -\frac{2}{\sqrt{12}} & -\frac{3}{\sqrt{18}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{12}} & \frac{3}{\sqrt{18}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix} \frac{2}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{2}{\sqrt{12}} \\ 0 & -\frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

Step 5:

$$\mathbf{P}^{-1}=\frac{1}{\det(\mathbf{P})}\mathbf{C}^{\mathrm{T}}$$

$$=\frac{1}{-1}\begin{pmatrix} \frac{2}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{2}{\sqrt{12}} \\ 0 & -\frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ 0 & \frac{3}{\sqrt{18}} & -\frac{3}{\sqrt{18}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\mathbf{P}^{-1}\mathbf{AP}=\begin{pmatrix}-\frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}}\\ 0 & \frac{3}{\sqrt{18}} & -\frac{3}{\sqrt{18}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}5 & 3 & 3\\ 3 & 1 & 1\\ 3 & 1 & 1\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{10}{\sqrt{12}}+\frac{6}{\sqrt{12}}+\frac{6}{\sqrt{12}} & -\frac{6}{\sqrt{12}}+\frac{2}{\sqrt{12}}+\frac{2}{\sqrt{12}} & -\frac{6}{\sqrt{12}}+\frac{2}{\sqrt{12}}+\frac{2}{\sqrt{12}}\\ 0+\frac{9}{\sqrt{18}}-\frac{9}{\sqrt{18}} & 0+\frac{3}{\sqrt{18}}-\frac{3}{\sqrt{18}} & 0+\frac{3}{\sqrt{18}}-\frac{3}{\sqrt{18}}\\ \frac{10}{\sqrt{6}}+\frac{3}{\sqrt{6}}+\frac{3}{\sqrt{6}} & \frac{6}{\sqrt{6}}+\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}} & \frac{6}{\sqrt{6}}+\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{2}{\sqrt{12}}\\ 0 & 0 & 0\\ \frac{16}{\sqrt{6}} & \frac{8}{\sqrt{6}} & \frac{8}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{2}{\sqrt{36}}-\frac{2}{\sqrt{36}}-\frac{2}{\sqrt{36}} & 0-\frac{2}{\sqrt{24}}+\frac{2}{\sqrt{24}} & \frac{4}{\sqrt{72}}-\frac{2}{\sqrt{72}}-\frac{2}{\sqrt{72}}\\ 0+0+0 & 0+0+0 & 0+0+0\\ -\frac{16}{\sqrt{18}}+\frac{8}{\sqrt{18}}+\frac{8}{\sqrt{18}} & 0+\frac{8}{\sqrt{12}}-\frac{8}{\sqrt{12}} & \frac{32}{6}+\frac{8}{6}+\frac{8}{6}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{6}{\sqrt{36}} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{48}{6}\end{pmatrix}$$

$$\mathbf{D}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 8\end{pmatrix}$$

6 a $\mathbf{A}=\begin{pmatrix}7 & 0 & -2\\ 0 & 5 & -2\\ -2 & -2 & 6\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}7 & 0 & -2\\ 0 & 5 & -2\\ -2 & -2 & 6\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(7-\lambda)\begin{vmatrix}5-\lambda & 0\\ -2 & 6-\lambda\end{vmatrix}-0\begin{vmatrix}0 & -2\\ -2 & 6-\lambda\end{vmatrix}-2\begin{vmatrix}0 & 5-\lambda\\ -2 & -2\end{vmatrix}$$

$$=(7-\lambda)\left[(5-\lambda)(6-\lambda)-4\right]-0[0-4]-2\left[0+2(5-\lambda)\right]$$

$$=(7-\lambda)\left(\lambda^2-11\lambda+30-4\right)-4(5-\lambda)$$

$$=-\left[(\lambda-7)\left(\lambda^2-11\lambda+26\right)-4\lambda+20\right]$$

$$=-\left[\lambda^3-11\lambda^2+26\lambda-7\lambda^2+77\lambda-182-4\lambda+20\right]$$

$$=-\left[\lambda^3-18\lambda^2+99\lambda-162\right]$$

As $\lambda=9$ is an eigenvalue, factorize by inspection with $(\lambda-9)$ in one bracket:

$$=-\left[(\lambda-9)\left(\lambda^2-9\lambda+18\right)\right]$$

$$=-\left[(\lambda-9)(\lambda-6)(\lambda-3)\right]$$

The eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$

Hence $(\lambda-9)(\lambda-6)(\lambda-3)=0$

Therefore $\lambda=3$, $\lambda=6$ and $\lambda=9$

6 b To find an eigenvector of **A** corresponding to the eigenvalue 3:

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2z \\ 2y-2z \\ -2x-2y+3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the upper elements:

$4x-2z=0 \Rightarrow 2x=z$

Setting $x=1$ gives $z=2$

Equating the middle elements and substituting $z=2$:

$2y-2\times 2=0 \Rightarrow y=2$

Therefore, an eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 6:

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-2z \\ -y-2z \\ 2x-2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the upper elements:

$x-2z=0 \Rightarrow x=2z$

Setting $z=1$ gives $x=2$

Equating the middle elements and substituting $z=1$:

$-y-2\times 2=0 \Rightarrow y=-2$

Therefore, an eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

To find an eigenvector of **A** corresponding to the eigenvalue 9:

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} -2x-2z \\ -4y-2z \\ -2x-2y-3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the middle elements:

$-4y-2z=0 \Rightarrow z=-2y$

Setting $y=1$ gives $z=-2$

Equating the upper elements and substituting $z = -2$:
$-2x - 2 \times (-2) = 0 \Rightarrow x = 2$

Therefore, an eigenvector corresponding to the eigenvalue 9 is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

6 c Construct **P** from the normalized eigenvectors (in any order).
Construct **D** from the eigenvalues (in the corresponding order).
Then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$
But as **A** is a symmetric matrix, **P** will be an orthogonal matrix, so $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$ as required.

An eigenvector of **A** corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is $\sqrt{1^2 + 2^2 + 2^2} = 3$

Hence a normalised eigenvector of **A** corresponding to the eigenvalue 3 is $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

An eigenvector of **A** corresponding to the eigenvalue 6 is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is $\sqrt{2^2 + (-2)^2 + 1^2} = 3$

Hence a normalised eigenvector of **A** corresponding to the eigenvalue 6 is $\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

An eigenvector of **A** corresponding to the eigenvalue 9 is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is $\sqrt{2^2 + 2^2 + (-2)^2} = 3$

Hence a normalised eigenvector of **A** corresponding to the eigenvalue 9 is $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$

Then $\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

7 a $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & \sqrt{5} \\ 0 & \sqrt{5} & 1-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)\begin{vmatrix} 1-\lambda & \sqrt{5} \\ \sqrt{5} & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & \sqrt{5} \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 1-\lambda \\ 0 & \sqrt{5} \end{vmatrix}$$
$$= (1-\lambda)\left[(1-\lambda)(1-\lambda)-5\right] - 2\left[2(1-\lambda)-0\right] + 0\left(2\sqrt{5}+0\right)$$

If 4 is an eigenvalue of **A** then:

$(1-4)\left[(1-4)(1-4)-5\right] - 2\left[2(1-4)\right] = 0$

$-12+12=0$

Hence 4 is an eigenvalue of **A**.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)\left[(1-\lambda)(1-\lambda)-5\right] - 2\left[2(1-\lambda)-0\right]$$
$$= (1-\lambda)\left[(1-\lambda)(1-\lambda)-5\right] - 4(1-\lambda)$$
$$= \left[(1-\lambda)\left[(1-\lambda)(1-\lambda)-5-4\right]\right]$$
$$= (1-\lambda)\left(\lambda^2-2\lambda-8\right)$$
$$= (1-\lambda)(\lambda-4)(\lambda+2)$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$(1-\lambda)(\lambda-4)(\lambda+2) = 0$

$\lambda = -2$ or $\lambda = 1$ or $\lambda = 4$

Hence the eigenvalues of **A** are −2, 1 and 4

7 b To find an eigenvector corresponding to eigenvalue 4:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+2y \\ 2x+y+\sqrt{5}z \\ \sqrt{5}y+z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the upper elements gives:

$x+2y=4x \Rightarrow 2y=3x$

Setting $x=2$ gives $y=3$

Equating the lower elements and substituting $y=3$ gives:

$3\sqrt{5}+z=4z \Rightarrow z=\sqrt{5}$

Hence an eigenvector corresponding to eigenvalue 4 is $\begin{pmatrix} 2 \\ 3 \\ \sqrt{5} \end{pmatrix}$

$\begin{pmatrix} 2 \\ 3 \\ \sqrt{5} \end{pmatrix}$ has magnitude $\sqrt{2^2+3^2+\left(\sqrt{5}\right)^2}=3\sqrt{2}$

Hence a normalised eigenvector corresponding to eigenvalue 4 is $\begin{pmatrix} \frac{2}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{\sqrt{5}}{3\sqrt{2}} \end{pmatrix}$

7 c $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ has magnitude $\sqrt{(-2)^2 + 3^2 + \left(-\sqrt{5}\right)^2} = 3\sqrt{2}$

Hence a normalised eigenvector corresponding to $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ is $\begin{pmatrix} -\frac{2}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{5}}{3\sqrt{2}} \end{pmatrix}$

$\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ has magnitude $\sqrt{\left(\sqrt{5}\right)^2 + (-2)^2} = 3$

Hence a normalised eigenvector corresponding to $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ is $\begin{pmatrix} \frac{\sqrt{5}}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}$

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3\sqrt{2}} & -\frac{2}{3\sqrt{2}} & \frac{\sqrt{5}}{3} \\ \frac{1}{\sqrt{2}} & \frac{3}{3\sqrt{2}} & 0 \\ \frac{\sqrt{5}}{3\sqrt{2}} & -\frac{\sqrt{5}}{3\sqrt{2}} & -\frac{2}{3} \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{2}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{5}}{3\sqrt{2}} \\ -\frac{2}{3\sqrt{2}} & \frac{3}{3\sqrt{2}} & -\frac{\sqrt{5}}{3\sqrt{2}} \\ \frac{\sqrt{5}}{3} & 0 & -\frac{2}{3} \end{pmatrix}$$

$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$

$$\mathbf{D}=\begin{pmatrix}\frac{2}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{5}}{3\sqrt{2}}\\ -\frac{2}{3\sqrt{2}} & \frac{3}{3\sqrt{2}} & -\frac{\sqrt{5}}{3\sqrt{2}}\\ \frac{\sqrt{5}}{3} & 0 & -\frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 2 & 1 & \sqrt{5}\\ 0 & \sqrt{5} & 1\end{pmatrix}\begin{pmatrix}\frac{2}{3\sqrt{2}} & -\frac{2}{3\sqrt{2}} & \frac{\sqrt{5}}{3}\\ \frac{1}{\sqrt{2}} & \frac{3}{3\sqrt{2}} & 0\\ \frac{\sqrt{5}}{3\sqrt{2}} & -\frac{\sqrt{5}}{3\sqrt{2}} & -\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{3\sqrt{2}}+\frac{6}{3\sqrt{2}}+0 & \frac{4}{3\sqrt{2}}+\frac{3}{3\sqrt{2}}+\frac{5}{3\sqrt{2}} & 0+\frac{3\sqrt{5}}{3\sqrt{2}}+\frac{\sqrt{5}}{3\sqrt{2}}\\ -\frac{2}{3\sqrt{2}}+\frac{6}{3\sqrt{2}}+0 & -\frac{4}{3\sqrt{2}}+\frac{3}{3\sqrt{2}}-\frac{5}{3\sqrt{2}} & 0+\frac{3\sqrt{5}}{3\sqrt{2}}-\frac{\sqrt{5}}{3\sqrt{2}}\\ \frac{\sqrt{5}}{3}+0+0 & \frac{2\sqrt{5}}{3}+0-\frac{2\sqrt{5}}{3} & 0+0-\frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{3\sqrt{2}} & -\frac{2}{3\sqrt{2}} & \frac{\sqrt{5}}{3}\\ \frac{1}{\sqrt{2}} & \frac{3}{3\sqrt{2}} & 0\\ \frac{\sqrt{5}}{3\sqrt{2}} & -\frac{\sqrt{5}}{3\sqrt{2}} & -\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{8}{3\sqrt{2}} & \frac{12}{3\sqrt{2}} & \frac{4\sqrt{5}}{3\sqrt{2}}\\ \frac{4}{3\sqrt{2}} & -\frac{6}{3\sqrt{2}} & \frac{2\sqrt{5}}{3\sqrt{2}}\\ \frac{\sqrt{5}}{3} & 0 & -\frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{3\sqrt{2}} & -\frac{2}{3\sqrt{2}} & \frac{\sqrt{5}}{3}\\ \frac{1}{\sqrt{2}} & \frac{3}{3\sqrt{2}} & 0\\ \frac{\sqrt{5}}{3\sqrt{2}} & -\frac{\sqrt{5}}{3\sqrt{2}} & -\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{16}{18}+\frac{36}{18}+\frac{20}{18} & -\frac{16}{18}+\frac{36}{18}-\frac{20}{18} & \frac{8\sqrt{5}}{9\sqrt{2}}+0-\frac{8\sqrt{5}}{9\sqrt{2}}\\ \frac{8}{18}-\frac{18}{18}+\frac{10}{18} & -\frac{8}{18}-\frac{18}{18}-\frac{10}{18} & \frac{4\sqrt{5}}{9\sqrt{2}}+0-\frac{4\sqrt{5}}{9\sqrt{2}}\\ \frac{2\sqrt{5}}{9\sqrt{2}}+0-\frac{2\sqrt{5}}{9\sqrt{2}} & -\frac{2\sqrt{5}}{9\sqrt{2}}+0+\frac{2\sqrt{5}}{9\sqrt{2}} & \frac{5}{9}+0+\frac{4}{9}\end{pmatrix}$$

$$=\begin{pmatrix}4 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

8 a $\mathbf{A}=\begin{pmatrix}2&2&-3\\2&2&3\\-3&3&3\end{pmatrix}$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&2&-3\\2&2&3\\-3&3&3\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}2-\lambda&2&-3\\2&2-\lambda&3\\-3&3&3-\lambda\end{pmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)\begin{vmatrix}2-\lambda&3\\3&3-\lambda\end{vmatrix}-2\begin{vmatrix}2&3\\-3&3-\lambda\end{vmatrix}-3\begin{vmatrix}2&2-\lambda\\-3&3\end{vmatrix}$$

$$=(2-\lambda)\left[(2-\lambda)(3-\lambda)-9\right]-2\left[2(3-\lambda)+9\right]-3\left[6+3(2-\lambda)\right]$$

If 6 is an eigenvalue of **A** then:

$$(2-6)\left[(2-6)(3-6)-9\right]-2\left[2(3-6)+9\right]-3\left[6+3(2-6)\right]=0$$

$$-12-6+18=0$$

Hence 6 is an eigenvalue of **A**.

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)\left[(2-\lambda)(3-\lambda)-9\right]-2\left[2(3-\lambda)+9\right]-3\left[6+3(2-\lambda)\right]$$

$$=(2-\lambda)(\lambda^2-5\lambda-3)-2(15-2\lambda)-3(12-3\lambda)$$

$$=2\lambda^2-10\lambda-6-\lambda^3+5\lambda^2+3\lambda-30+4\lambda-36+9\lambda$$

$$=-\lambda^3+7\lambda^2+6\lambda-72$$

$$=-\lambda^3+6\lambda^2+\lambda^2-6\lambda+12\lambda-72$$

$$=-\lambda^2(\lambda-6)+\lambda(\lambda-6)+12(\lambda-6)$$

$$=(\lambda-6)(-\lambda^2+\lambda+12)$$

$$=(\lambda-6)(-\lambda-3)(\lambda-4)$$

$$=-(\lambda-6)(\lambda+3)(\lambda-4)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$(\lambda-6)(\lambda+3)(\lambda-4)=0$$

Hence the eigenvalues of **A** are $\lambda_1=6,\ \lambda_2=4,\ \lambda_3=-3$

b $\det(\mathbf{A})=2\begin{vmatrix}2&3\\3&3\end{vmatrix}-2\begin{vmatrix}2&3\\-3&3\end{vmatrix}-3\begin{vmatrix}2&2\\-3&3\end{vmatrix}$

$$\det(\mathbf{A})=2(6-9)-2(6+9)-3(6+6)$$

$$=-6-30-36$$

$$=-72$$

But also: $\lambda_1\lambda_2\lambda_3=6\times4\times(-3)=-72$

Therefore $\det(\mathbf{A})=\lambda_1\lambda_2\lambda_3$ as required

8 c To find an eigenvector corresponding to eigenvalue 6 is

$$\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y-3z \\ 2x+2y+3z \\ -3x+3y+3z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the upper elements gives:
$2x+2y-3z=6x \Rightarrow 2y=4x+3z$
Equating the middle elements gives:
$2x+2y+3z=6y \Rightarrow 4y=2x+3z$
Hence by substituting for $2y$ in the second equation:
$2(4x+3z)=2x+3z$
$8x+6z=2x+3z$
$6x=-3z$
Setting $x = 1$ gives $z = -2$
Equating the lower elements and setting $x = 1$ and $z = -2$ gives:
$-3+3y-6=-12 \Rightarrow y=-1$

Hence an eigenvector corresponding to eigenvalue 6 is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

8 d **A** has eigenvectors $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$

$\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ has magnitude $\sqrt{1^2+(-1)^2+(-2)^2}=\sqrt{6}$

Hence a normalised eigenvector corresponding to $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\-\frac{2}{\sqrt{6}}\end{pmatrix}$

$\begin{pmatrix}1\\1\\0\end{pmatrix}$ has magnitude $\sqrt{1^2+1^2}=\sqrt{2}$

Hence a normalised eigenvector corresponding to $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$

$\begin{pmatrix}1\\-1\\1\end{pmatrix}$ has magnitude $\sqrt{1^2+(-1)^2+1^2}=\sqrt{3}$

Hence a normalised eigenvector corresponding to $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}}\\-\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}}\end{pmatrix}$$

$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} + \frac{6}{\sqrt{6}} & \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} - \frac{6}{\sqrt{6}} & -\frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{6}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 & -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} + 0 \\ \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} & -\frac{3}{\sqrt{3}} - \frac{3}{\sqrt{3}} + \frac{3}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{\sqrt{6}} & -\frac{6}{\sqrt{6}} & -\frac{12}{\sqrt{6}} \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} & 0 \\ -\frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & -\frac{3}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{6} + \frac{6}{6} + \frac{24}{6} & \frac{6}{\sqrt{12}} - \frac{6}{\sqrt{12}} + 0 & \frac{6}{\sqrt{18}} + \frac{6}{\sqrt{18}} - \frac{12}{\sqrt{18}} \\ \frac{4}{\sqrt{12}} - \frac{4}{\sqrt{12}} + 0 & \frac{4}{2} + \frac{4}{2} + 0 & \frac{4}{\sqrt{6}} - \frac{4}{\sqrt{6}} + 0 \\ -\frac{3}{\sqrt{18}} - \frac{3}{\sqrt{18}} + \frac{6}{\sqrt{18}} & -\frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} + 0 & -\frac{3}{3} - \frac{3}{3} - \frac{3}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Challenge

$$\mathbf{M}=\begin{pmatrix}\frac{1}{9} & \frac{8}{9} & -\frac{4}{9}\\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9}\\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9}\end{pmatrix}$$

$$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}\frac{1}{9}-\lambda & \frac{8}{9} & -\frac{4}{9}\\ \frac{8}{9} & \frac{1}{9}-\lambda & \frac{4}{9}\\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9}-\lambda\end{pmatrix}$$

$$\begin{aligned}\det(\mathbf{M}-\lambda\mathbf{I})&=\left(\frac{1}{9}-\lambda\right)\left[\left(\frac{1}{9}-\lambda\right)\left(\frac{7}{9}-\lambda\right)-\frac{16}{81}\right]-\frac{8}{9}\left[\frac{8}{9}\left(\frac{7}{9}-\lambda\right)+\frac{16}{81}\right]-\frac{4}{9}\left[\frac{32}{81}+\frac{4}{9}\left(\frac{1}{9}-\lambda\right)\right]\\
&=\left(\frac{1}{9}-\lambda\right)\left(\frac{7}{81}-\frac{8}{9}\lambda+\lambda^2-\frac{16}{81}\right)-\frac{8}{9}\left(\frac{56}{81}-\frac{8}{9}\lambda+\frac{16}{81}\right)-\frac{4}{9}\left(\frac{36}{81}-\frac{4}{9}\lambda\right)\\
&=\left(\frac{1}{9}-\lambda\right)\left(\lambda^2-\frac{8}{9}\lambda-\frac{9}{81}\right)-\frac{8}{9}\left(\frac{72}{81}-\frac{8}{9}\lambda\right)-\frac{4}{9}\left(\frac{36}{81}-\frac{4}{9}\lambda\right)\\
&=\frac{1}{9}\lambda^2-\frac{8}{81}\lambda-\frac{9}{729}-\lambda^3+\frac{8}{9}\lambda^2+\frac{9}{81}\lambda-\frac{576}{729}+\frac{64}{81}\lambda-\frac{144}{729}+\frac{16}{81}\lambda\\
&=-\lambda^3+\lambda^2+\lambda-1\\
&=(\lambda-1)(\lambda-1)(\lambda+1)\end{aligned}$$

$\det(\mathbf{M}-\lambda\mathbf{I})=0$

$(\lambda-1)(\lambda-1)(\lambda+1)=0$

$\lambda=-1$ or $\lambda=1$

Hence the eigenvalues of **A** are −1, and 1 repeated

To find an eigenvector corresponding to eigenvalue −1:

$$\begin{pmatrix}\frac{1}{9} & \frac{8}{9} & -\frac{4}{9}\\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9}\\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{9}x+\frac{8}{9}y-\frac{4}{9}z\\ \frac{8}{9}x+\frac{1}{9}y+\frac{4}{9}z\\ -\frac{4}{9}x+\frac{4}{9}y+\frac{7}{9}z\end{pmatrix}=\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$$

Equating the upper elements gives:

$\frac{1}{9}x+\frac{8}{9}y-\frac{4}{9}z=-x\Rightarrow 4z=10x+8y$

Equating the middle elements gives:

$\frac{8}{9}x+\frac{1}{9}y+\frac{4}{9}z=-y \Rightarrow 4z=-8x-10y$

Hence:

$10x+8y=-8x-10y$

$2x=-2y$

$x=-y$

Setting $x = 1$ gives $y = -1$

Equating the lower elements and setting $x = 1$ and $y = -1$ gives:

$-\frac{4}{9}-\frac{4}{9}+\frac{7}{9}z=-z \Rightarrow \frac{16}{9}z=\frac{8}{9} \Rightarrow z=\frac{1}{2}$

Hence an eigenvector corresponding to eigenvalue -1 is $\begin{pmatrix}2\\-2\\1\end{pmatrix}$

To find an eigenvector corresponding to eigenvalue 1

$$\begin{pmatrix}\frac{1}{9} & \frac{8}{9} & -\frac{4}{9}\\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9}\\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{9}x+\frac{8}{9}y-\frac{4}{9}z\\ \frac{8}{9}x+\frac{1}{9}y+\frac{4}{9}z\\ -\frac{4}{9}x+\frac{4}{9}y+\frac{7}{9}z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating the upper elements gives:

$\frac{1}{9}x+\frac{8}{9}y-\frac{4}{9}z=x \Rightarrow -\frac{8}{9}x+\frac{8}{9}y-\frac{4}{9}z=0$

Equating the middle elements gives:

$\frac{8}{9}x+\frac{1}{9}y+\frac{4}{9}z=y \Rightarrow \frac{8}{9}x-\frac{8}{9}y+\frac{4}{9}z=0 \Rightarrow -\frac{8}{9}x+\frac{8}{9}y-\frac{4}{9}z=0$

Equating the lower elements gives:

$-\frac{4}{9}x+\frac{4}{9}y+\frac{7}{9}z=z \Rightarrow -\frac{4}{9}x+\frac{4}{9}y-\frac{2}{9}z=0 \Rightarrow -\frac{8}{9}x+\frac{8}{9}y-\frac{4}{9}z=0$

Setting $y = 0$ gives $2x = -z$ and setting $x = -1$ gives $z = 2$

Hence an eigenvector corresponding to eigenvalue 1 is $\begin{pmatrix}-1\\0\\2\end{pmatrix}$

Setting $z = 0$ gives $x = y$ and setting $x = 1$ gives $y = 1$

Hence another eigenvector corresponding to eigenvalue 1 is $\begin{pmatrix}1\\1\\0\end{pmatrix}$

Geometrically, as **M** represents a reflection, the eigenvector with eigenvalue –1 must be normal to the plane, and all eigenvectors with eigenvalue 1 must be parallel to the plane.
Note that there are many possible eigenvectors that can be found within the plane.

The equation of the plane is $\mathbf{r}.\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 0$

In Cartesian form, this is:
$2x - 2y + z = 0$