Exercise 6E

1 a $\mathbf{T}^{-1}=\begin{pmatrix}2&3&3\\-1&4&5\\2&1&1\end{pmatrix}$

$$\mathbf{T}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}-12\\-7\\8\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix}=\mathbf{T}^{-1}\begin{pmatrix}-12\\-7\\8\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}2&3&3\\-1&4&5\\2&1&1\end{pmatrix}\begin{pmatrix}-12\\-7\\8\end{pmatrix}$$

$$=\begin{pmatrix}-24-21+24\\12-28+40\\-24-7+8\end{pmatrix}$$

$$=\begin{pmatrix}-21\\24\\-23\end{pmatrix}$$

b $l_2:\ \mathbf{r}=t\begin{pmatrix}2\\-2\\1\end{pmatrix}$

$$\mathbf{T}^{-1}\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}2&3&3\\-1&4&5\\2&1&1\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$=\begin{pmatrix}4-6+3\\-2-8+5\\4-2+1\end{pmatrix}$$

$$=\begin{pmatrix}1\\-5\\3\end{pmatrix}$$

Hence:

$$l_1:\ \mathbf{r}=t\begin{pmatrix}1\\-5\\3\end{pmatrix}$$

2 a $\mathbf{T}=\begin{pmatrix}2&0&-3\\0&1&2\\-3&2&8\end{pmatrix}$

Step 1:

$$\det(\mathbf{T})=2\begin{vmatrix}1&2\\2&8\end{vmatrix}-0\begin{vmatrix}0&2\\-3&8\end{vmatrix}-3\begin{vmatrix}0&1\\-3&2\end{vmatrix}$$
$$=2(8-4)-0(0+6)-3(0+3)$$
$$=8-0-9$$
$$=-1$$

Step 2:

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}1&2\\2&8\end{vmatrix}&\begin{vmatrix}0&2\\-3&8\end{vmatrix}&\begin{vmatrix}0&1\\-3&2\end{vmatrix}\\\begin{vmatrix}0&-3\\2&8\end{vmatrix}&\begin{vmatrix}2&-3\\-3&8\end{vmatrix}&\begin{vmatrix}2&0\\-3&2\end{vmatrix}\\\begin{vmatrix}0&-3\\1&2\end{vmatrix}&\begin{vmatrix}2&-3\\0&2\end{vmatrix}&\begin{vmatrix}2&0\\0&1\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}8-4&0+6&0+3\\0+6&16-9&4-0\\0+3&4-0&2-0\end{pmatrix}$$
$$=\begin{pmatrix}4&6&3\\6&7&4\\3&4&2\end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix}4&-6&3\\-6&7&-4\\3&-4&2\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}4&-6&3\\-6&7&-4\\3&-4&2\end{pmatrix}$$

Step 5:

$$\mathbf{T}^{-1}=\frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{-1}\begin{pmatrix}4&-6&3\\-6&7&-4\\3&-4&2\end{pmatrix}$$
$$=\begin{pmatrix}-4&6&-3\\6&-7&4\\-3&4&-2\end{pmatrix}$$

2 b $\mathbf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 6 & -3 \\ 6 & -7 & 4 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} -5 \\ 5 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} 20+30-48 \\ -30-35+64 \\ 15+20-32 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Therefore:

$a = 2$, $b = -1$, $c = 3$

3 a $\mathbf{T}=\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 2\\ -3 & 0 & -4\end{pmatrix}$

Step 1:

$$\det(\mathbf{T})=1\begin{vmatrix}2 & 2\\ 0 & -4\end{vmatrix}-1\begin{vmatrix}0 & 2\\ -3 & -4\end{vmatrix}+2\begin{vmatrix}0 & 2\\ -3 & 0\end{vmatrix}$$
$$=1(-8-0)-1(0+6)+2(0+6)$$
$$=-8-6+12$$
$$=-2$$

Step 2:

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}2 & 2\\ 0 & -4\end{vmatrix} & \begin{vmatrix}0 & 2\\ -3 & -4\end{vmatrix} & \begin{vmatrix}0 & 2\\ -3 & 0\end{vmatrix}\\ \begin{vmatrix}1 & 2\\ 0 & -4\end{vmatrix} & \begin{vmatrix}1 & 2\\ -3 & -4\end{vmatrix} & \begin{vmatrix}1 & 1\\ -3 & 0\end{vmatrix}\\ \begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix} & \begin{vmatrix}1 & 2\\ 0 & 2\end{vmatrix} & \begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}-8-0 & 0+6 & 0+6\\ -4-0 & -4+6 & 0+3\\ 2-4 & 2-0 & 2-0\end{pmatrix}$$
$$=\begin{pmatrix}-8 & 6 & 6\\ -4 & 2 & 3\\ -2 & 2 & 2\end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix}-8 & -6 & 6\\ 4 & 2 & -3\\ -2 & -2 & 2\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-8 & 4 & -2\\ -6 & 2 & -2\\ 6 & -3 & 2\end{pmatrix}$$

Step 5:

$$\mathbf{T}^{-1}=\frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}}$$

$$=\frac{1}{-2}\begin{pmatrix}-8 & 4 & -2\\ -6 & 2 & -2\\ 6 & -3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}4 & -2 & 1\\ 3 & -1 & 1\\ -3 & \frac{3}{2} & -1\end{pmatrix}$$

3 b $l_2: \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

The general point on l_2 is $\begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix}$

Let the general point on l_1 be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\mathbf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 1 \\ 3 & -1 & 1 \\ -3 & \frac{3}{2} & -1 \end{pmatrix}\begin{pmatrix} 2-t \\ 4 \\ 1+t \end{pmatrix}$$

$$= \begin{pmatrix} 4(2-t)-8+1(1+t) \\ 3(2-t)-4+1(1+t) \\ -3(2-t)+6+1(1+t) \end{pmatrix}$$

$$= \begin{pmatrix} 8-4t-8+1+t \\ 6-3t-4+1+t \\ -6+3t+6-1-t \end{pmatrix}$$

$$= \begin{pmatrix} 1-3t \\ 3-2t \\ -1+2t \end{pmatrix}$$

Therefore:

$$l_1: \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$$

4 a $\mathbf{T}=\begin{pmatrix} a & 1 & 2 \\ 4 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Step 1:

$$\det(\mathbf{T}) = a\begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} -4 & 0 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix}$$
$$= a(0-0)-1(-4-0)+2(0-0)$$
$$= 0+4+0$$
$$= 4$$

Step 2:

$$\mathbf{M}=\begin{pmatrix} \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -4 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} a & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} a & 1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} a & 2 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} a & 1 \\ 4 & 0 \end{vmatrix} \end{pmatrix}$$
$$=\begin{pmatrix} 0-0 & -4-0 & 0-0 \\ -1-0 & -a-0 & 0-0 \\ 0-0 & 0-8 & 0-4 \end{pmatrix}$$
$$=\begin{pmatrix} 0 & -4 & 0 \\ -1 & -a & 0 \\ 0 & -8 & -4 \end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix} 0 & 4 & 0 \\ 1 & -a & 0 \\ 0 & 8 & -4 \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix} 0 & 1 & 0 \\ 4 & -a & 8 \\ 0 & 0 & -4 \end{pmatrix}$$

Step 5:

$$\mathbf{T}^{-1} = \frac{1}{\det(\mathbf{T})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 4 & -a & 8 \\ 0 & 0 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -\frac{a}{4} & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

4 b $\mathbf{T}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -\frac{a}{4} & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + \frac{3}{4} + 0 \\ 2 - \frac{3}{4}a - 2 \\ 0 + 0 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4}a \\ 1 \end{pmatrix}$$

Therefore:

$p = \frac{3}{4}$, $q = -\frac{3}{4}a$, $r = 1$

5 a $\mathbf{S}=\begin{pmatrix}1 & -\sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}$

$\mathbf{S}^{\mathrm{T}}=\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}$

$$\mathbf{SS}^{\mathrm{T}}=\begin{pmatrix}1 & -\sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+2+1 & \sqrt{2}+0-\sqrt{2} & 1-2+1\\ \sqrt{2}-0-\sqrt{2} & 2+0+2 & \sqrt{2}+0-\sqrt{2}\\ 1-2+1 & \sqrt{2}+0-\sqrt{2} & 1+2+1\end{pmatrix}$$

$$=\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}$$

$$=4\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=4\mathbf{I}$$

Hence, $k=4$

5 b $\mathbf{S}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

Since:

$\mathbf{S}\mathbf{S}^{\mathrm{T}} = 4\mathbf{I}$

$\mathbf{S}^{\mathrm{T}} = 4\mathbf{S}^{-1}$

$\mathbf{S}^{-1} = \frac{1}{4}\mathbf{S}^{\mathrm{T}}$

Therefore:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{4}\mathbf{S}^{\mathrm{T}}\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2\sqrt{2} + 2 - 2\sqrt{2} \\ -4 + 0 - 4 \\ 2\sqrt{2} - 2 - 2\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 \\ -8 \\ -2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} \frac{1}{2} \\ -2 \\ -\frac{1}{2} \end{pmatrix}$$

Hence:

$a = \frac{1}{2},\ b = -2,\ c = -\frac{1}{2}$

6 a $\mathbf{A} = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & a & -3 \\ b & -1 & -2 \\ -18 & 11 & c \end{pmatrix}$

$\mathbf{AB} = \mathbf{I}$

$$\begin{pmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & -3 \\ b & -1 & -2 \\ -18 & 11 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9+5b-18 & 3a-5+11 & -9-10+c \\ -6+3b+0 & -2a-3+0 & 6-6+0 \\ 12+3b-18 & 4a-3+11 & -12-6+c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: it's not necessary to calculate all nine elements of the matrix multiplication – as long as any three equations involving a, b and c are obtained.

$$\begin{pmatrix} 5b-9 & 3a+6 & c-19 \\ 3b-6 & -2a-3 & 0 \\ 3b-6 & 4a+8 & c-18 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$5b-9=1 \Rightarrow b=2$

$-2a-3=1 \Rightarrow a=-2$

$c-18=1 \Rightarrow c=19$

6 b $\Pi_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Let the plane $\Pi_1 : \mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + s\begin{pmatrix} d \\ e \\ f \end{pmatrix} + t\begin{pmatrix} g \\ h \\ i \end{pmatrix}$

Since $\mathbf{AB} = \mathbf{I}$
$\mathbf{B} = \mathbf{A}^{-1}$

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \mathbf{B}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2+0 \\ 2-1+0 \\ -18+11+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \mathbf{B}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3+0-6 \\ -2+0-4 \\ 18+0+38 \end{pmatrix}$$

$$= \begin{pmatrix} -9 \\ -6 \\ 56 \end{pmatrix}$$

$$\begin{pmatrix} g \\ h \\ i \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \mathbf{B}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -3 \\ 2 & -1 & -2 \\ -18 & 11 & 19 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0-2-3 \\ 0-1-2 \\ 0+11+19 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ -3 \\ 30 \end{pmatrix}$$

Therefore:

$$\Pi_1:\ \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} + s\begin{pmatrix} -9 \\ -6 \\ 56 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 30 \end{pmatrix}$$

7 $\mathbf{T}=\begin{pmatrix}-1 & 3 & 6\\ 1 & 4 & 2\\ 2 & -5 & 1\end{pmatrix}$

$$\mathbf{T}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}-8\\ 0\\ 3\end{pmatrix}$$

$$\begin{pmatrix}-1 & 3 & 6\\ 1 & 4 & 2\\ 2 & -5 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}-8\\ 0\\ 3\end{pmatrix}$$

$$\begin{pmatrix}-a+3b+6c\\ a+4b+2c\\ 2a-5b+c\end{pmatrix}=\begin{pmatrix}-8\\ 0\\ 3\end{pmatrix}$$

$-a+3b+6c=-8$ **(1)**
$a+4b+2c=0$ **(2)**
$2a-5b+c=3$ **(3)**
Adding **(1)** and **(2)** gives:
$7b+8c=-8$ **(4)**
Adding 2 × **(1)** and **(3)** gives:
$b+13c=-13$ **(5)**
Subtracting 7 × **(5)** from **(4)** gives:
$-83c=83 \Rightarrow c=-1$
Substituting $c=-1$ into **(4)** gives:
$7b+8(-1)=-8 \Rightarrow b=0$
Substituting $b=0$ and $c=-1$ into **(1)** gives:
$-a+3(0)+6(-1)=-8$
$-a-6=-8 \Rightarrow a=2$
Hence:
$a=2, b=0, c=-1$
Note that this could also be solved by finding $\mathbf{T}^{-1}$ then calculating:

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\mathbf{T}^{-1}\begin{pmatrix}-8\\ 0\\ 3\end{pmatrix}$$

8 a $\mathbf{S}=\begin{pmatrix}2&-1&2\\0&2&1\\1&0&1\end{pmatrix}$

Step 1:

$$\det(\mathbf{S})=2\begin{vmatrix}2&1\\0&1\end{vmatrix}+1\begin{vmatrix}0&1\\1&1\end{vmatrix}+2\begin{vmatrix}0&2\\1&0\end{vmatrix}$$
$$=2(2-0)+1(0-1)+2(0-2)$$
$$=4-1-4$$
$$=-1$$

Step 2:

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}2&1\\0&1\end{vmatrix}&\begin{vmatrix}0&1\\1&1\end{vmatrix}&\begin{vmatrix}0&2\\1&0\end{vmatrix}\\\begin{vmatrix}-1&2\\0&1\end{vmatrix}&\begin{vmatrix}2&2\\1&1\end{vmatrix}&\begin{vmatrix}2&-1\\1&0\end{vmatrix}\\\begin{vmatrix}-1&2\\2&1\end{vmatrix}&\begin{vmatrix}2&2\\0&1\end{vmatrix}&\begin{vmatrix}2&-1\\0&2\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}2-0&0-1&0-2\\-1-0&2-2&0+1\\-1-4&2-0&4-0\end{pmatrix}$$
$$=\begin{pmatrix}2&-1&-2\\-1&0&1\\-5&2&4\end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix}2&1&-2\\1&0&-1\\-5&-2&4\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}2&1&-5\\1&0&-2\\-2&-1&4\end{pmatrix}$$

Step 5:

$$\mathbf{S}^{-1}=\frac{1}{\det(\mathbf{S})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{-1}\begin{pmatrix}2&1&-5\\1&0&-2\\-2&-1&4\end{pmatrix}$$
$$=\begin{pmatrix}-2&-1&5\\-1&0&2\\2&1&-4\end{pmatrix}$$

8 b $\mathbf{T} = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}$

$$\mathbf{T}^2 = \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9-24+16 & 12-28+16 & 12-24+12 \\ -18+42-24 & -24+49-24 & -24+42-18 \\ 12-24+12 & 16-28+12 & 16-24+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= \mathbf{I}$ as required

8 c $\mathbf{ST}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$

$$\mathbf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}^{-1}\mathbf{S}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

But since $\mathbf{T}^2 = \mathbf{I}$, $\mathbf{T}^{-1} = \mathbf{T}$ so:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{TS}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$\mathbf{S}^{-1}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 5 \\ -1 & 0 & 2 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -12+3+10 \\ -6+0+4 \\ 12-3-8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{T}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 4 \\ -6 & -7 & -6 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-8+4 \\ -6+14-6 \\ 4-8+3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Therefore:

$a = -1$, $b = 2$, $c = -1$