Exercise 6D

1 a $\mathbf{T}:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x-y\\y+z\\2x-3z\end{pmatrix}$

$\mathbf{T}:\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}1-0\\0+0\\2-0\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$

$\mathbf{T}:\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0-1\\1+0\\0-0\end{pmatrix}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\mathbf{T}:\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}0-0\\0+1\\0-3\end{pmatrix}=\begin{pmatrix}0\\1\\-3\end{pmatrix}$

Therefore, the matrix representing **T** is $\begin{pmatrix}1&-1&0\\0&1&1\\2&0&-3\end{pmatrix}$

b $\mathbf{U}:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}2x-3y-z\\2y+3z\\5z\end{pmatrix}$

$\mathbf{U}:\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}2-0-0\\0+0\\0\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$

$\mathbf{U}:\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0-3-0\\2+0\\0\end{pmatrix}=\begin{pmatrix}-3\\2\\0\end{pmatrix}$

$\mathbf{U}:\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}0-0-1\\0+3\\5\end{pmatrix}=\begin{pmatrix}-1\\3\\5\end{pmatrix}$

Therefore, the matrix representing **U** is $\begin{pmatrix}2&-3&-1\\0&2&3\\0&0&5\end{pmatrix}$

1 c $\mathbf{TU} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & -3 \end{pmatrix}\begin{pmatrix} 2 & -3 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2+0+0 & -3-2+0 & -1-3+0 \\ 0+0+0 & 0+2+0 & 0+3+5 \\ 4+0+0 & -6+0+0 & -2+0-15 \end{pmatrix}$

$= \begin{pmatrix} 2 & -5 & -4 \\ 0 & 2 & 8 \\ 4 & -6 & -17 \end{pmatrix}$

2 $\begin{pmatrix} 4 & -1 & 0 \\ -2 & 2 & 3 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix} = \begin{pmatrix} 4-3+0 \\ -2+6+3a \\ 5-6+a \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 3a+4 \\ a-1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 3a+4 \\ a-1 \end{pmatrix} = \begin{pmatrix} b \\ -5 \\ c \end{pmatrix}$

Hence:
$a = -3$, $b = 1$, $c = -4$

3 Let $\mathbf{T} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ 2d \\ 2g \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$

So $a = 3$, $d = 1$, $g = 2$

$\begin{pmatrix} 3 & b & c \\ 1 & e & f \\ 2 & h & i \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 9+0-c \\ 3+0-f \\ 6+0-i \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

So $c = 11$, $f = 0$, $i = 1$

$\begin{pmatrix} 3 & b & 11 \\ 1 & e & 0 \\ 2 & h & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0+b-11 \\ 0+e+0 \\ 0+h-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

So $b = 13$, $e = -1$, $h = -1$

Hence:

$\mathbf{T} = \begin{pmatrix} 3 & 13 & 11 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$

4 $\mathbf{T} = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 5 & -4 \\ 3 & 2 & 1 \end{pmatrix}$

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2-t \\ 4-2t \\ 1+3t \end{pmatrix}$$

The image of l_1 is given by:

$$\mathbf{r} = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 5 & -4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2-t \\ 4-2t \\ 1+3t \end{pmatrix}$$

$$= \begin{pmatrix} -1(4-2t)+2(1+3t) \\ 2(2-t)+5(4-2t)-4(1+3t) \\ 3(2-t)+2(4-2t)+1(1+3t) \end{pmatrix}$$

$$= \begin{pmatrix} -2+8t \\ 20-24t \\ 15-4t \end{pmatrix}$$

Hence, a vector equation of l_2 is:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 20 \\ 15 \end{pmatrix} + t \begin{pmatrix} 8 \\ -24 \\ -4 \end{pmatrix}$$

5 a $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$

$$\mathbf{T} = \begin{pmatrix} 1 & -3 & 4 \\ 2 & 3 & -2 \\ 0 & 2 & 5 \end{pmatrix}$$

$$\mathbf{a}' = \mathbf{Ta}$$

$$= \begin{pmatrix} 1 & -3 & 4 \\ 2 & 3 & -2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2-3+0 \\ 4+3+0 \\ 0+2+0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

$$\mathbf{b}' = \mathbf{Tb}$$

$$= \begin{pmatrix} 1 & -3 & 4 \\ 2 & 3 & -2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2-9+16 \\ -4+9-8 \\ 0+6+20 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -3 \\ 26 \end{pmatrix}$$

b $l = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} + t\begin{pmatrix} 5-(-1) \\ -3-7 \\ 26-2 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} + t\begin{pmatrix} 6 \\ -10 \\ 24 \end{pmatrix}$$

6 $\Pi_1 : x - 2y + z = 0$

Fix $x = 0$ and $y = 0$, then:

$0 - 0 + z = 0 \Rightarrow z = 0$

Hence (0, 0, 0) lies on the plane.

Fix $x = 0$ and $y = 1$, then:

$0 - 2 + z = 0 \Rightarrow z = 2$

Hence (0, 1, 2) lies on the plane.

Fix $x = 1$ and $y = 1$, then:

$1 - 2 + z = 0 \Rightarrow z = 1$

Hence (1, 1, 1) lies on the plane.

So the equation of Π_1 can be written as:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{T}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{T}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 - 2 - 4 \\ 0 - 8 + 8 \\ 0 + 4 + 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 4 \end{pmatrix}$$

$$\mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - 2 - 2 \\ -2 - 8 + 4 \\ -2 + 4 + 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 2 \end{pmatrix}$$

So the equation of Π_2 can be written as:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -6 \\ 2 \end{pmatrix}$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 0 & 4 \\ -1 & -6 & 2 \end{vmatrix}$$

$$= \mathbf{i}(0+24) - \mathbf{j}(-12+4) + \mathbf{k}(36+0)$$

$$= 24\mathbf{i} + 8\mathbf{j} + 36\mathbf{k}$$

$$\mathbf{r} \cdot \mathbf{n} = 0$$

Hence:

$$24x + 8y + 36z = 0$$

$$6x + 2y + 9z = 0$$

7 $$\mathbf{T} = \begin{pmatrix} 4 & 5 & -3 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} s+3t \\ 1-s \\ 1+2s+4t \end{pmatrix}$$

$$\mathbf{Tx} = \begin{pmatrix} 4 & 5 & -3 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} s+3t \\ 1-s \\ 1+2s+4t \end{pmatrix}$$

$$= \begin{pmatrix} 4(s+3t)+5(1-s)-3(1+2s+4t) \\ -1(s+3t)+2(1-s)+1(1+2s+4t) \\ 1(s+3t)+0(1-s)+1(1+2s+4t) \end{pmatrix}$$

$$= \begin{pmatrix} 4s+12t+5-5s-3-6s-12t \\ -s-3t+2-2s+1+2s+4t \\ s+3t+1+2s+4t \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-7s \\ 3-s+t \\ 1+3s+7t \end{pmatrix}$$

Hence a vector equation of Π_2 is:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & -1 & 3 \\ 0 & 1 & 7 \end{vmatrix}$$

$$= \mathbf{i}(-7-3) - \mathbf{j}(-49-0) + \mathbf{k}(-7+0)$$

$$= -10\mathbf{i} + 49\mathbf{j} - 7\mathbf{k}$$

Therefore for some value of p:

$$\mathbf{r} \cdot \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix} = p$$

Substitute in a known point (2, 3, 1) on the plane to find p:

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix} = -20 + 147 - 7 = 120$$

Hence the equation of Π_2 can be written:

$$\mathbf{r} \cdot \begin{pmatrix} -10 \\ 49 \\ -7 \end{pmatrix} = 120$$

8 $\mathbf{T}=\begin{pmatrix}4 & 1 & -2\\ -2 & 3 & 4\\ -1 & 0 & 2\end{pmatrix}$ and $\mathbf{T}:\begin{pmatrix}a\\ b\\ c\end{pmatrix}\mapsto\begin{pmatrix}a\\ b\\ c\end{pmatrix}$

Let a general point on l be $\begin{pmatrix}a\\ b\\ c\end{pmatrix}$

$$\begin{pmatrix}4 & 1 & -2\\ -2 & 3 & 4\\ -1 & 0 & 2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}4a+b-2c\\ -2a+3b+4c\\ -a+2c\end{pmatrix}$$

Since each point on l is mapped to itself:

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}4a+b-2c\\ -2a+3b+4c\\ -a+2c\end{pmatrix}$$

Hence:

$a=4a+b-2c\Rightarrow 3a=-b+2c$ **(1)**

$b=-2a+3b+4c\Rightarrow 2a=2b+4c$ **(2)**

$c=-a+2c\Rightarrow a=c$ **(3)**

Substituting $a=c$ into **(1)** gives:

$3a=-b+2a\Rightarrow a=-b$

Therefore, the general point can be written as:

$\begin{pmatrix}a\\ -a\\ a\end{pmatrix}$ or $\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$

And the equation of the line can be written as:

$\mathbf{r}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}+\lambda\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$ or $\mathbf{r}=\lambda\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$