Exercise 5D

1 a The equation of the line is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0 \Rightarrow \mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$. This gives:

$$\mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=(2\mathbf{i}+\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 2\\ 3 & 1 & -2\end{vmatrix}$$

$$\Rightarrow \mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=-4\mathbf{i}+10\mathbf{j}-\mathbf{k}$$

b The equation of the line is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0 \Rightarrow \mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$. This gives:

$$\mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=(2\mathbf{i}-3\mathbf{k})\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -3\\ 1 & 1 & 5\end{vmatrix}$$

$$\Rightarrow \mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=3\mathbf{i}-13\mathbf{j}+2\mathbf{k}$$

c The equation of the line is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0 \Rightarrow \mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$. This gives:

$$\mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=(4\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -2 & 1\\ -1 & -2 & 3\end{vmatrix}$$

$$\Rightarrow \mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=-4\mathbf{i}-13\mathbf{j}-10\mathbf{k}$$

2 Let $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

Then the Cartesian form of the equation of the line that passes through a point with position vector **a** and is parallel to the vector **b** is $\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=\lambda$ (from Core Pure Book 1, Section 9.1)

a $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{-2}=\lambda$

b $\frac{x-2}{1}=\frac{y}{1}=\frac{z+3}{5}=\lambda$

c $\frac{x-4}{-1}=\frac{y+2}{-2}=\frac{z-1}{3}=\lambda$

3 a The line is in the direction $\begin{pmatrix}6\\4\\2\end{pmatrix}-\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}1\\3\\5\end{pmatrix}\right)\times\begin{pmatrix}5\\1\\-3\end{pmatrix}=0$

3 b The line is in the direction $\begin{pmatrix}4\\3\\5\end{pmatrix}-\begin{pmatrix}3\\4\\12\end{pmatrix}=\begin{pmatrix}1\\-1\\-7\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}3\\4\\12\end{pmatrix}\right)\times\begin{pmatrix}1\\-1\\-7\end{pmatrix}=0$

c The line is in the direction $\begin{pmatrix}3\\7\\11\end{pmatrix}-\begin{pmatrix}-2\\2\\6\end{pmatrix}=\begin{pmatrix}5\\5\\5\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}-2\\2\\6\end{pmatrix}\right)\times\begin{pmatrix}5\\5\\5\end{pmatrix}=0$

d The line is in the direction $\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}4\\2\\-4\end{pmatrix}=\begin{pmatrix}-3\\-1\\5\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\times\begin{pmatrix}-3\\-1\\5\end{pmatrix}=0$

In each part of question **3** one solution is illustrated, but there are alternatives. Either given point may be used for **a** in the equation $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ and any multiple of the direction vector may be used as **b**.

4 The solutions for question **3** are in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, where **a** is the position vector of a point on the line and **b** is vector parallel to the line. Use the standard formula for the Cartesian form of the equation of the line (see solution to question **2**). As with question **3**, there are alternative solutions.

a $\dfrac{x-1}{5}=\dfrac{y-3}{1}=\dfrac{z-5}{-3}=\lambda$

b $\dfrac{x-3}{1}=\dfrac{y-4}{-1}=\dfrac{z-12}{-7}=\lambda$

c $\dfrac{x+2}{5}=\dfrac{y-2}{5}=\dfrac{z-6}{5}=\lambda$

Or $x+2=y-2=z-6=\mu$ as **i** + **j** + **k** is also in the direction of the line.

4 d $\dfrac{x-1}{-3}=\dfrac{y-1}{-1}=\dfrac{z-1}{5}=\lambda$

Or $\dfrac{x-4}{3}=\dfrac{y-2}{1}=\dfrac{z+4}{-5}=\lambda$ as (4, 2, –4) is a point on the line and $3\mathbf{i}+\mathbf{j}-5\mathbf{k}$ is also in the direction of the line.

5 A straight line with the equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ passes through the point with position vector **a** and is parallel to the vector **b**. The equation of the line can be written $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$.

a $(\mathbf{r}-(\mathbf{i}+\mathbf{j}-2\mathbf{k}))\times(2\mathbf{i}-\mathbf{k})=0$

b $(\mathbf{r}-(\mathbf{i}+4\mathbf{j}))\times(3\mathbf{i}+\mathbf{j}-5\mathbf{k})=0$

c $(\mathbf{r}-(3\mathbf{i}+4\mathbf{j}-4\mathbf{k}))\times(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})=0$

6 i $\dfrac{x-3}{2}=\dfrac{y+1}{5}=\dfrac{2z-3}{3}=\lambda$ can be written as $\dfrac{x-3}{2}=\dfrac{y+1}{5}=\dfrac{z-\frac{3}{2}}{\frac{3}{2}}=\lambda$

The direction of the line is parallel $2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}$

A point on the line has position vector $3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}$

Therefore the vector equation of the line can be written as

$$\mathbf{r}\times\left(2\mathbf{i}+5\mathbf{j}+\tfrac{3}{2}\mathbf{k}\right)=\left(3\mathbf{i}-\mathbf{j}+\tfrac{3}{2}\mathbf{k}\right)\times\left(2\mathbf{i}+5\mathbf{j}+\tfrac{3}{2}\mathbf{k}\right)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & \frac{3}{2}\\ 2 & 5 & \frac{3}{2}\end{vmatrix}$$

$$\Rightarrow \mathbf{r}\times\left(2\mathbf{i}+5\mathbf{j}+\tfrac{3}{2}\mathbf{k}\right)=-9\mathbf{i}-\tfrac{3}{2}\mathbf{j}+17\mathbf{k}$$

ii $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}+t\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)$

Or $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}+t(4\mathbf{i}+10\mathbf{j}+3\mathbf{k})$

7 As $(p, q, 1)$ lies on the line with equation $\mathbf{r}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$ then $\begin{pmatrix}p\\q\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$

$$\begin{pmatrix}p\\q\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ p & q & 1\\ 2 & 1 & 3\end{vmatrix}==(3q-1)\mathbf{i}-(3p-2)\mathbf{j}+(p-2q)\mathbf{k}=\begin{pmatrix}3q-1\\2-3p\\p-2q\end{pmatrix}$$

So $3q-1=8\Rightarrow q=3$ and $2-3p=-7\Rightarrow p=3$

Solution: $p = 3$ and $q = 3$

8 The line with equation $\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$ has direction $\mathbf{i}+\mathbf{j}-\mathbf{k}$, i.e. $\begin{pmatrix}1\\1\\-1\end{pmatrix}$

If the line passes through a point (a_1, a_2, a_3) then $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\1&1&-1\end{vmatrix}=(-a_2-a_3)\mathbf{i}+(a_1+a_3)\mathbf{j}+(a_1-a_2)\mathbf{k}=\begin{pmatrix}-a_2-a_3\\a_1+a_3\\a_1-a_2\end{pmatrix}$$

So $\begin{pmatrix}-a_2-a_3\\a_1+a_3\\a_1-a_2\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

These equations have an infinite number of solutions so let $a_1=0$, then as $a_1+a_3=2$ and $a_1-a_2=1$ this gives $a_3=2$ and $a_2=-1$, therefore (0, –1, 2) lies on the line.

So the line equation may be written as $\mathbf{r}=-\mathbf{j}+2\mathbf{k}+t(\mathbf{i}+\mathbf{j}-\mathbf{k})$