Chapter review 4

1 $y = \tanh x$

$$V = \pi\int_0^1 y^2 \mathrm{d}x$$

$$= \pi\int_0^1 \tanh^2 x \,\mathrm{d}x$$

$$= \pi\int_0^1 \left(1 - \operatorname{sech}^2 x\right) \mathrm{d}x$$

$$= \pi\left[x - \tanh^2 x\right]_0^1$$

$$= \pi\left[\left(1 - \frac{e^2 - 1}{e^2 + 1}\right) - (0 - 0)\right]$$

$$= \pi\left(\frac{e^2 + 1 - e^2 + 1}{e^2 + 1}\right)$$

$$= \frac{2\pi}{e^2 + 1} \text{ as required}$$

2 **a** $4x^2 + 4x + 17 = (2x+1)^2 - 1 + 17$

$$= (2x+1)^2 + 16$$

Therefore:
$a = 2$
$b = 1$
$c = 16$

b $\displaystyle\int_{-0.5}^{1.5} \frac{1}{4x^2 + 4x + 17}\mathrm{d}x = \int_{-0.5}^{1.5} \frac{1}{(2x+1)^2 + 16}\mathrm{d}x$

Let $u = 2x + 1 \Rightarrow \mathrm{d}u = \frac{1}{2}\mathrm{d}x$

When $x = -0.5$, $u = 0$
When $x = 1.5$, $u = 4$

$$\int_{-0.5}^{1.5} \frac{1}{4x^2 + 4x + 17}\mathrm{d}x = \frac{1}{2}\int_0^4 \frac{1}{u^2 + 4^2}\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{1}{4}\arctan\left(\frac{u}{4}\right)\right]_0^4$$

$$= \frac{1}{8}\left[\arctan(1) - \arctan(0)\right]$$

$$= \frac{1}{8}\arctan(1)$$

$$= \frac{\pi}{32}$$

3 a $\sinh 4x \cosh 6x = \frac{1}{2}\left(e^{4x} - e^{-4x}\right) \times \frac{1}{2}\left(e^{6x} + e^{-6x}\right)$

$$= \frac{1}{4}\left(e^{4x} - e^{-4x}\right)\left(e^{6x} + e^{-6x}\right)$$

$$= \frac{1}{4}\left(e^{10x} + e^{-2x} - e^{2x} - e^{-10x}\right)$$

$$= \frac{1}{2}(\sinh 10x - \sinh 2x)$$

$$\int \sinh 4x \cosh 6x \, dx = \frac{1}{2}\int (\sinh 10x - \sinh 2x) dx$$

$$= \frac{1}{20}\cosh 10x - \frac{1}{4}\cosh 2x + c$$

b $\frac{d}{dx}\left(1 + 2\operatorname{sech} x\right) = 2(-1)(\cosh x)^{-2}(\sinh x)$

$$= -\frac{2\sinh x}{\cosh^2 x}$$

$$= -2\operatorname{sech} x \tanh x$$

$$\int \frac{\operatorname{sech} x \tanh x}{1 + 2\operatorname{sech} x} dx = -\frac{1}{2}\int \frac{-2\operatorname{sech} x \tanh x}{1 + 2\operatorname{sech} x} dx$$

$$= -\frac{1}{2}\ln\left|1 + 2\operatorname{sech} x\right| + c$$

c $e^x \sinh x = e^x\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)$

$$= \frac{1}{2}\left(e^{2x} - 1\right)$$

$$\int e^x \sinh x \, dx = \frac{1}{2}\int \left(e^{2x} - 1\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2}e^{2x} - x\right) + c$$

$$= \frac{1}{4}e^{2x} - \frac{1}{2}x + c$$

4 $y = \dfrac{10}{\sqrt{4x^2+9}}$

$= \dfrac{10}{\sqrt{4\left(x^2+\frac{9}{4}\right)}}$

$= \dfrac{10}{2\sqrt{x^2+\frac{9}{4}}}$

$= \dfrac{5}{\sqrt{x^2+\frac{9}{4}}}$

R' is the area under the curve.

$$R' = 5\int_0^5 \frac{1}{\sqrt{x^2+\left(\frac{3}{2}\right)^2}}\,dx$$

$$= 5\left[\operatorname{arsinh}\left(\frac{2}{3}x\right)\right]_0^5$$

$$= 5\left[\ln\left(\frac{2}{3}x+\sqrt{\left(\frac{2}{3}x\right)^2+1}\right)\right]_0^5$$

$$= 5\left[\ln\left(\frac{10}{3}+\sqrt{\left(\frac{10}{3}\right)^2+1}\right)-\ln\left(0+\sqrt{0^2+1}\right)\right]$$

$$= 5\left[\ln\left(\frac{10}{3}+\sqrt{\frac{100}{9}+1}\right)-\ln(1)\right]$$

$$= 5\ln\left(\frac{10+\sqrt{109}}{3}\right)$$

$= 9.59$ (3 s.f.)

Each unit is $10 \times 10\ \text{m}^2$, therefore:

$R = 100R'$

$= 959$ (3 s.f.)

$= 960$ (2 s.f.)

5 a $\int \frac{1+2x}{1+4x^2}\,dx = \int \frac{1}{1+4x^2}\,dx + \int \frac{2x}{1+4x^2}\,dx$

$= \int \frac{1}{4\left(\frac{1}{4}+x^2\right)}\,dx + \frac{1}{4}\int \frac{8x}{1+4x^2}\,dx$

$= \frac{1}{4}\int \frac{1}{\frac{1}{4}+x^2}\,dx + \frac{1}{4}\int \frac{8x}{1+4x^2}\,dx$

$= \frac{2}{4}\arctan(2x) + \frac{1}{4}\ln\left(1+4x^2\right) + c$

$= \frac{1}{2}\arctan(2x) + \frac{1}{4}\ln\left(1+4x^2\right) + c$

b $\int_0^{0.5} \frac{1+2x}{1+4x^2}\,dx = \left[\frac{1}{2}\arctan(2x) + \frac{1}{4}\ln\left(1+4x^2\right)\right]_0^{0.5}$

$= \left[\left(\frac{1}{2}\arctan(1) + \frac{1}{4}\ln\left(1+4(0.5)^2\right) - \frac{1}{2}\arctan(0) + \frac{1}{4}\ln\left(1+4(0)^2\right)\right)\right]$

$= \left[\left(\frac{1}{2}\times\frac{\pi}{4} + \frac{1}{4}\ln(2) + \frac{1}{4}\ln(1)\right)\right]$

$= \frac{\pi}{8} + \frac{1}{4}\ln(2)$

$= \frac{1}{8}\left(\pi + 2\ln(2)\right)$

6 $y = 4\cosh\left(\frac{x}{4}\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{x}{4}\right) \Rightarrow \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \sinh^2\left(\frac{x}{4}\right)$

Using $s = \int_{x_A}^{x_B} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ gives:

$$L = \int_{-20}^{20} \sqrt{1+\sinh^2\left(\frac{x}{4}\right)}\,\mathrm{d}x$$

$$= \int_{-20}^{20} \sqrt{\cosh^2\left(\frac{x}{4}\right)}\,\mathrm{d}x$$

$$= \int_{-20}^{20} \cosh\left(\frac{x}{4}\right)\mathrm{d}x$$

$$= 4\left[\sinh\left(\frac{x}{4}\right)\right]_{-20}^{20}$$

$$= 4\left[\frac{\mathrm{e}^{\frac{x}{4}} - \mathrm{e}^{-\frac{x}{4}}}{2}\right]_{-20}^{20}$$

$$= 2\left[\mathrm{e}^{\frac{x}{4}} - \mathrm{e}^{-\frac{x}{4}}\right]_{-20}^{20}$$

$$= 2\left[\left(\mathrm{e}^5 - \mathrm{e}^{-5}\right) - \left(\mathrm{e}^{-5} - \mathrm{e}^5\right)\right]$$

$$= 2\left(2\mathrm{e}^5 - 2\mathrm{e}^{-5}\right)$$

$$= 4\left(\mathrm{e}^5 - \mathrm{e}^{-5}\right)$$

$$= 4\left(\mathrm{e}^5 - \frac{1}{\mathrm{e}^5}\right)$$

$$= 4\left(\frac{\mathrm{e}^{10} - 1}{\mathrm{e}^5}\right)$$

$$= 594 \text{ (3 s.f.)}$$

7 $\int_0^{\frac{1}{2}} \operatorname{artanh} x \, dx$

Let $u = \operatorname{artanh}(x) \Rightarrow \frac{du}{dx} = \frac{1}{1-x^2}$

Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\int_0^{\frac{1}{2}} \operatorname{artanh} x \, dx = [x \operatorname{artanh} x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1-x^2} dx$$

$$= [x \operatorname{artanh} x]_0^{\frac{1}{2}} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x}{1-x^2} dx$$

$$= [x \operatorname{artanh} x]_0^{\frac{1}{2}} + \frac{1}{2}\left[\ln\left(1-x^2\right)\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\operatorname{artanh}\left(\frac{1}{2}\right) + \frac{1}{2}\left[\ln\left(1-\left(\frac{1}{2}\right)^2\right) - \ln\left(1-0^2\right)\right]$$

$$= \frac{1}{2}\operatorname{artanh}\left(\frac{1}{2}\right) + \frac{1}{2}\left[\ln\left(\frac{3}{4}\right) - \ln(1)\right]$$

$$= \frac{1}{2}\operatorname{artanh}\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

Using $\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$$\int_0^{\frac{1}{2}} \operatorname{artanh} x \, dx = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\ln(3) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\ln(3) + \frac{2}{4}\ln\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\ln(3) + \frac{1}{4}\ln\left(\frac{3}{4}\right)^2$$

$$= \frac{1}{4}\ln(3) + \frac{1}{4}\ln\left(\frac{9}{16}\right)$$

$$= \frac{1}{4}\ln\left(\frac{27}{16}\right)$$

8 a i $I_0 = \int_0^{\frac{\pi}{2}} x^0 \cos x \, dx$

$= \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= [\sin x]_0^{\frac{\pi}{2}}$

$= \sin\left(\frac{\pi}{2}\right)$

$= 1$

ii $I_1 = \int_0^{\frac{\pi}{2}} x^1 \cos x \, dx$

$= \int_0^{\frac{\pi}{2}} x \cos x \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$\int_0^{\frac{\pi}{2}} x \cos x \, dx = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$

$= [x \sin x]_0^{\frac{\pi}{2}} + [\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - \cos(0)$

$= \frac{\pi}{2} - 1$

8 b $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$

Let $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

Let $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$$\int_0^{\frac{\pi}{2}} x^n \cos x \, dx = \left[x^n \sin x \right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

Let $u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)nx^{n-2}$

Let $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$$\int_0^{\frac{\pi}{2}} x^n \cos x \, dx = \left(\frac{\pi}{2}\right)^n - n\left(\left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx \right)$$

$$= \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

c $I_3 = \left(\frac{\pi}{2}\right)^3 - 3(3-1)I_1$

$$= \left(\frac{\pi}{2}\right)^3 - 6I_1$$

$$= \left(\frac{\pi}{2}\right)^3 - 6\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi^3}{8} - 3\pi + 6$$

$$= \frac{1}{8}\left(\pi^3 - 24\pi + 48\right) \text{ as required}$$

8 d $I_4 = \left(\frac{\pi}{2}\right)^4 - 4(4-1)I_2$

$= \left(\frac{\pi}{2}\right)^4 - 12I_2$

$= \left(\frac{\pi}{2}\right)^4 - 12\left(\left(\frac{\pi}{2}\right)^2 - 2(2-1)I_0\right)$

$= \left(\frac{\pi}{2}\right)^4 - 12\left(\left(\frac{\pi}{2}\right)^2 - 2I_0\right)$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$

9 a $x^2 - 2x + 10 = (x-1)^2 + 9$

Let $u = x - 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$

$\int \frac{1}{\sqrt{x^2 - 2x + 10}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x-1)^2 + 9}}\,\mathrm{d}x$

$= \int \frac{1}{\sqrt{u^2 + 3^2}}\,\mathrm{d}x$

$= \operatorname{arsinh}\left(\frac{u}{3}\right) + c$

$= \operatorname{arsinh}\left(\frac{x-1}{3}\right) + c$

b $x^2 - 2x + 10 = (x-1)^2 + 9$

Let $u = x - 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$

$\int \frac{1}{x^2 - 2x + 10}\,\mathrm{d}x = \int \frac{1}{(x-1)^2 + 9}\,\mathrm{d}x$

$= \int \frac{1}{u^2 + 3^2}\,\mathrm{d}x$

$= \frac{1}{3}\arctan\left(\frac{u}{3}\right) + c$

$= \frac{1}{3}\arctan\left(\frac{x-1}{3}\right) + c$

9 c $\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

When $x = 0$, $\theta = 0$

When $x = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$$\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^4\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sin^4\theta}{\sqrt{\cos^2\theta}} \times \cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^4\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \left(\sin^2\theta\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2}(1-\cos 2\theta)\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \left(1 - 2\cos 2\theta + \frac{1}{2}(\cos 4\theta + 1)\right) d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \left(1 - 2\cos 2\theta + \frac{1}{2}\cos 4\theta + \frac{1}{2}\right) d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left[\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[\left(\frac{3}{2}\times\frac{\pi}{6} - \sin\left(2\times\frac{\pi}{6}\right) + \frac{1}{8}\sin\left(4\times\frac{\pi}{6}\right)\right) - \left(\frac{3}{2}\times 0 - \sin(2\times 0) + \frac{1}{8}\sin(4\times 0)\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \sin\left(\frac{\pi}{3}\right) + \frac{1}{8}\sin\left(\frac{2\pi}{3}\right)\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{16}\right)$$

$$= \frac{\pi}{16} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{64}$$

$$= \frac{4\pi}{64} - \frac{8\sqrt{3}}{64} + \frac{\sqrt{3}}{64}$$

$$= \frac{4\pi - 7\sqrt{3}}{64} \text{ as required}$$

10 a $I_n = \int_0^1 x^n (1-x)^{\frac{1}{3}}\,dx$

Let $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

Let $\frac{dv}{dx} = (1-x)^{\frac{1}{3}} \Rightarrow v = -\frac{3}{4}(1-x)^{\frac{4}{3}}$

$$\int_0^1 x^n (1-x)^{\frac{1}{3}}\,dx = \left[-\frac{3}{4}x^n(1-x)^{\frac{4}{3}}\right]_0^1 + \frac{3n}{4}\int_0^1 x^{n-1}(1-x)^{\frac{4}{3}}\,dx$$

$$= \frac{3n}{4}\int_0^1 x^{n-1}(1-x)^{\frac{4}{3}}\,dx$$

$$= \frac{3n}{4}\int_0^1 x^{n-1}(1-x)(1-x)^{\frac{1}{3}}\,dx$$

$$= \frac{3n}{4}\int_0^1 x^{n-1}(1-x)^{\frac{1}{3}}\,dx - \frac{3n}{4}\int_0^1 x^n(1-x)^{\frac{1}{3}}\,dx$$

Therefore:

$$I_n = \frac{3n}{4}I_{n-1} - \frac{3n}{4}I_n$$

$$I_n + \frac{3n}{4}I_n = \frac{3n}{4}I_{n-1}$$

$$I_n\left(1 + \frac{3n}{4}\right) = \frac{3n}{4}I_{n-1}$$

$$I_n\left(\frac{4+3n}{4}\right) = \frac{3n}{4}I_{n-1}$$

$$I_n = \frac{3n}{4+3n}I_{n-1} \text{ as required.}$$

10 b $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,dx = \int_0^1 (1+x)(1-x)(1-x)^{\frac{1}{3}}\,dx$

$= \int_0^1 (1-x^2)(1-x)^{\frac{1}{3}}\,dx$

$= \int_0^1 (1-x)^{\frac{1}{3}}\,dx - \int_0^1 x^2(1-x)^{\frac{1}{3}}\,dx$

$= \int_0^1 (1-x)^{\frac{1}{3}}\,dx - I_2$

$= \int_0^1 (1-x)^{\frac{1}{3}}\,dx - \frac{3\times 2}{4+3\times 2}I_1$

$= \int_0^1 (1-x)^{\frac{1}{3}}\,dx - \frac{3}{5}I_1$

$= \int_0^1 (1-x)^{\frac{1}{3}}\,dx - \frac{3}{5}\left(\frac{3\times 1}{4+3\times 1}I_0\right)$

$= \int_0^1 (1-x)^{\frac{1}{3}}\,dx - \frac{9}{35}I_0$

Since $I_0 = \int_0^1 x^0(1-x)^{\frac{1}{3}}\,dx$

$\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,dx = \int_0^1 (1-x)^{\frac{1}{3}}\,dx - \frac{9}{35}\int_0^1 (1-x)^{\frac{1}{3}}\,dx$

$= \frac{26}{35}\int_0^1 (1-x)^{\frac{1}{3}}\,dx$

$= \frac{26}{35}\left[-\frac{3}{4}(1-x)^{\frac{4}{3}}\right]_0^1$

$= -\frac{39}{70}\left[(1-1)^{\frac{4}{3}} - (1-0)^{\frac{4}{3}}\right]$

$= \frac{39}{70}$

11 a $x = t - \ln t \Rightarrow \frac{dx}{dt} = 1 - \frac{1}{t} \Rightarrow \left(\frac{dx}{dt}\right)^2 = 1 - \frac{2}{t} + \frac{1}{t^2}$

$y = 4t^{\frac{1}{2}} \Rightarrow \frac{dy}{dt} = 2t^{-\frac{1}{2}} \Rightarrow \left(\frac{dy}{dt}\right)^2 = \frac{4}{t}$

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t}} \\ &= \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} \\ &= \sqrt{\left(1 + \frac{1}{t}\right)^2} \\ &= 1 + \frac{1}{t}\end{aligned}$$

Using $s = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ gives:

$$\begin{aligned}s &= \int_1^4 \left(1 + \frac{1}{t}\right) dt \\ &= [t + \ln t]_1^4 \\ &= (4 + \ln 4) - (1 + \ln 1) \\ &= 3 + \ln 4 \text{ as required}\end{aligned}$$

b Using $S = 2\pi \int_{t_A}^{t_B} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ gives:

$$\begin{aligned}S &= 2\pi \int_1^4 4t^{\frac{1}{2}} \left(1 + \frac{1}{t}\right) dt \\ &= 8\pi \int_1^4 \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) dt \\ &= 8\pi \left[\frac{2}{3}t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right]_1^4 \\ &= 8\pi \left[\left(\frac{2}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}} + 2(1)^{\frac{1}{2}}\right)\right] \\ &= 8\pi \left[\left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 2\right)\right] \\ &= \frac{160\pi}{3}\end{aligned}$$

12 $\int_0^3 x^2 \text{arsinh}\, x \, dx$

Let $u = \text{arsinh}(x) \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1+x^2}}$

Let $\frac{dv}{dx} = x^2 \Rightarrow v = \frac{1}{3}x^3$

$$\int_0^3 x^2 \text{arsinh}\, x \, dx = \left[\frac{1}{3}x^3 \text{arsinh}\, x\right]_0^3 - \frac{1}{3}\int_0^3 \frac{x^3}{\sqrt{1+x^2}} dx$$

$$= \frac{1}{3}\left[\left(3^3 \text{arsinh}\, 3\right) - \left(0^3 \text{arsinh}\, 0\right)\right] - \frac{1}{3}\int_0^3 \frac{x^3}{\sqrt{1+x^2}} dx$$

$$= 9 \text{arsinh}\, 3 - \frac{1}{3}\int_0^3 \frac{x^3}{\sqrt{1+x^2}} dx$$

$$= 9\ln\left(3+\sqrt{3^2+1}\right) - \frac{1}{3}\int_0^3 \frac{x^3}{\sqrt{1+x^2}} dx$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\int_0^3 \frac{x^3}{\sqrt{1+x^2}} dx$$

Let $x = \sinh\theta \Rightarrow dx = \cosh\theta\, d\theta$

When $x = 0$, $\theta = \text{arsinh}(0) = 0$

When $x = 3$, $\theta = \text{arsinh}(3)$

$$\int_0^3 x^2 \text{arsinh}\, x \, dx = 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\int_0^{\text{arsinh}(3)} \frac{\sinh^3\theta}{\sqrt{1+\sinh^2\theta}} \times \cosh\theta\, d\theta$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\int_0^{\text{arsinh}(3)} \frac{\sinh^3\theta}{\sqrt{\cosh^2\theta}} \times \cosh\theta\, d\theta$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\int_0^{\text{arsinh}(3)} \sinh^3\theta\, d\theta$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\int_0^{\text{arsinh}(3)} \sinh\theta \sinh^2\theta\, d\theta$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\int_0^{\text{arsinh}(3)} \sinh\theta\left(\cosh^2\theta - 1\right) d\theta$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\left(\int_0^{\text{arsinh}(3)} \cosh^2\theta \sinh\theta\, d\theta - \int_0^{\text{arsinh}(3)} \sinh\theta\, d\theta\right)$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\left[\frac{1}{3}\cosh^3\theta - \cosh\theta\right]_0^{\text{arsinh}(3)}$$

When $\theta = 0$, $\cosh\theta = 1$
When $\theta = \text{arsinh}(3)$,

$$\sinh\theta = 3 \text{ and } \cosh\theta = \sqrt{1+\sinh^2\theta} = \sqrt{1+3^2} = \sqrt{10}$$

So:

$$\int_0^3 x^2 \text{arsinh}\, x \,dx = 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\left[\frac{1}{3}\cosh^3\theta - \cosh\theta\right]_0^{\text{arsinh}(3)}$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\left[\left(\frac{1}{3}\left(\sqrt{10}\right)^3 - \sqrt{10}\right) - \left(\frac{1}{3}\times 1^3 - 1\right)\right]$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\left[\left(\frac{10}{3}\sqrt{10} - \sqrt{10}\right) + \frac{2}{3}\right]$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{3}\left[\frac{7}{3}\sqrt{10} + \frac{2}{3}\right]$$

$$= 9\ln\left(3+\sqrt{10}\right) - \frac{1}{9}\left(2+7\sqrt{10}\right)$$

13 a $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$

Let $u = x^2 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow x dx = \dfrac{1}{2}du$

When $x = 0$, $u = 0$
When $x = 1$, $u = 1$

$$\int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \frac{1}{1+u^2}\times\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\left[\arctan(u)\right]_0^1$$

$$= \frac{1}{2}\times\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

b i $\displaystyle\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\,dx$

Let $u = x - 2 \Rightarrow du = dx$

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-u^2}}\,du$$

$$= \arcsin\left(\frac{u}{2}\right) + c$$

$$= \arcsin\left(\frac{x-2}{2}\right) + c$$

13 b ii $\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx$

$$\frac{d}{dx}\left(\left(4x-x^2\right)^{\frac{1}{2}}\right)=\frac{1}{2}(4-2x)\left(4x-x^2\right)^{-\frac{1}{2}}$$

$$=\frac{\frac{1}{2}(4-2x)}{\sqrt{4x-x^2}}$$

Therefore:

$$\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx = 2\sqrt{4x-x^2}+c$$

iii $\int_3^4 \frac{5-2x}{\sqrt{4x-x^2}}\,dx = \int_3^4 \frac{1+(4-2x)}{\sqrt{4x-x^2}}\,dx$

$$=\int_3^4 \frac{1}{\sqrt{4x-x^2}}\,dx+\int_3^4 \frac{4-2x}{\sqrt{4x-x^2}}\,dx$$

$$=\left[\arcsin\left(\frac{x-2}{2}\right)\right]_3^4+\left[2\sqrt{4x-x^2}\right]_3^4$$

$$=\left[\arcsin(1)-\arcsin\left(\frac{1}{2}\right)\right]+\left[2\sqrt{16-16}-2\sqrt{12-9}\right]$$

$$=\frac{\pi}{2}-\frac{\pi}{6}-2\sqrt{3}$$

$$=\frac{\pi}{3}-2\sqrt{3}$$

$$=\frac{\pi-6\sqrt{3}}{3}$$

14 a $y^2=4x$

$$2y\frac{dy}{dx}=4$$

$$\frac{dy}{dx}=\frac{4}{2y}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{16}{4y^2}$$

$$=\frac{1}{x}$$

Using $S=2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ gives:

$$S=2\pi\int_0^1 2\sqrt{x}\sqrt{1+\frac{1}{x}}\,dx$$

$$=4\pi\int_0^1 \sqrt{x}\sqrt{\frac{x+1}{x}}\,dx=4\pi\int_0^1 \sqrt{x+1}\,dx$$

14 b $S = 4\pi\int_0^1 \sqrt{x+1}\,\mathrm{d}x$

$= 4\pi\left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_0^1$

$= \frac{8\pi}{3}\left[(1+1)^{\frac{3}{2}} - (0+1)^{\frac{3}{2}}\right]$

$= \frac{8\pi}{3}\left(2^{\frac{3}{2}} - 1\right)$

$= \frac{8\pi}{3}\left(2\sqrt{2} - 1\right)$

c The length, L, from (0, 0) to (1, 2)

$L = \int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$

By symmetry, the total length, L_{tot} is $2\int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$:

From 14 part a:

$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \frac{1}{x}}$

Therefore:

$L_{\text{tot}} = 2\int_0^1 \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x$

$= 2\int_0^1 \sqrt{\frac{x+1}{x}}\,\mathrm{d}x$

14 d Let $x=\sinh^2\theta \Rightarrow \frac{dx}{d\theta}=2\sinh\theta\cosh\theta$

When $x = 0$, $\theta=0$

When $x = 1$, $\theta=\text{arsinh}(1)$

$$\int_0^1\sqrt{\frac{x+1}{x}}dx=\int_0^{\text{arsinh}(1)}\sqrt{\frac{\sinh^2\theta+1}{\sinh^2\theta}}\times 2\sinh\theta\cosh\theta\, d\theta$$

$$=2\int_0^{\text{arsinh}(1)}\sqrt{\frac{\cosh^2\theta}{\sinh^2\theta}}\times\sinh\theta\cosh\theta\, d\theta$$

$$=2\int_0^{\text{arsinh}(1)}\frac{\cosh\theta}{\sinh\theta}\times\sinh\theta\cosh\theta\, d\theta$$

$$=2\int_0^{\text{arsinh}(1)}\cosh^2\theta\, d\theta$$

$$=2\int_0^{\text{arsinh}(1)}\frac{\cosh 2\theta+1}{2}d\theta$$

$$=\int_0^{\text{arsinh}(1)}(\cosh 2\theta+1)\, d\theta$$

$$=\left[\frac{1}{2}\sinh 2\theta+\theta\right]_0^{\text{arsinh}(1)}$$

$$=\left[\frac{1}{2}\times 2\sinh\theta\cosh\theta+\theta\right]_0^{\text{arsinh}(1)}$$

$$=\left[\sinh\theta\sqrt{1+\sinh^2\theta}+\theta\right]_0^{\text{arsinh}(1)}$$

When $\theta=0$, $\sinh\theta=0$

When $\theta=\text{arsinh}(1)$, $\sinh\theta=1$

Also:

$$\text{arsinh}(\theta)=\ln\left(\theta+\sqrt{\theta^2+1}\right)$$

So:

$$\int_0^1\sqrt{\frac{x+1}{x}}dx=\left(1\sqrt{1+1^2}+\text{arsinh}(1)\right)-(0+0)$$

$$=\sqrt{2}+\ln\left(1+\sqrt{1^2+1}\right)$$

$$=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

Therefore:

$$L_{tot}=2\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

15 a $\int x \operatorname{arcosh} x \, \mathrm{d}x$

Let $u = \operatorname{arcosh} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}$

Let $\frac{\mathrm{d}v}{\mathrm{d}x} = x \Rightarrow v = \frac{1}{2}x^2$

$$\int x \operatorname{arcosh} x \, \mathrm{d}x = \frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2 - 1}} \mathrm{d}x$$

Consider:

$$\int \frac{x^2}{\sqrt{x^2 - 1}} \mathrm{d}x$$

Let $x = \cosh\theta \Rightarrow \mathrm{d}x = \sinh\theta \, \mathrm{d}\theta$

$$\int \frac{x^2}{\sqrt{x^2 - 1}} \mathrm{d}x = \int \frac{\cosh^2\theta}{\sqrt{\cosh^2\theta - 1}} \times \sinh\theta \, \mathrm{d}\theta$$

$$= \int \frac{\cosh^2\theta}{\sqrt{\sinh^2\theta}} \times \sinh\theta \, \mathrm{d}\theta$$

$$= \int \cosh^2\theta \, \mathrm{d}\theta$$

$$= \int \frac{\cosh 2\theta + 1}{2} \mathrm{d}\theta$$

$$= \frac{1}{2}\int (\cosh 2\theta + 1) \, \mathrm{d}\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2\theta + \theta\right) + c$$

$$= \frac{1}{4}\sinh 2\theta + \frac{1}{2}\theta + c$$

$$= \frac{1}{4} \times 2\cosh\theta \sinh\theta + \frac{1}{2}\theta + c$$

$$\frac{1}{2}\cosh\theta\sqrt{\cosh^2\theta - 1} + \frac{1}{2}\theta + c$$

As $x = \cosh\theta$,

$$\int \frac{x^2}{\sqrt{x^2 - 1}} \mathrm{d}x = \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\operatorname{arcosh} x + c$$

And:

$$\int x \operatorname{arcosh} x \, \mathrm{d}x = \frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{2}\left(\frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\operatorname{arcosh} x + c\right)$$

$$= \frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{4}x\sqrt{x^2 - 1} - \frac{1}{4}\operatorname{arcosh} x + c$$

$$= \frac{1}{4}(2x^2 - 1)\operatorname{arcosh} x - \frac{1}{4}x\sqrt{x^2 - 1} + c \text{ as required}$$

15 b Let $u^2 = x \Rightarrow \mathrm{d}x = 2u\mathrm{d}u$

$$\int \operatorname{arcosh}\left(\sqrt{x}\right)\mathrm{d}x = \int \operatorname{arcosh}(u)\times 2u\,\mathrm{d}u$$
$$= 2\int u\operatorname{arcosh}(u)\,\mathrm{d}u$$
$$= 2\left[\frac{1}{4}\left(2u^2-1\right)\operatorname{arcosh}u - \frac{1}{4}u\sqrt{u^2-1}\right]+c$$
$$= \frac{1}{2}\left(2u^2-1\right)\operatorname{arcosh}u - \frac{1}{2}u\sqrt{u^2-1}+c$$
$$= \frac{1}{2}(2x-1)\operatorname{arcosh}\left(\sqrt{x}\right) - \frac{1}{2}\sqrt{x}\sqrt{x-1}+c$$

16 a $I_n = \int \frac{\sin(2n+1)x}{\sin x}\,\mathrm{d}x$

Therefore:

$$I_{n-1} = \int \frac{\sin\left(2[n-1]+1\right)x}{\sin x}\,\mathrm{d}x$$
$$= \int \frac{\sin(2n-2+1)x}{\sin x}\,\mathrm{d}x$$
$$= \int \frac{\sin(2n-1)x}{\sin x}\,\mathrm{d}x$$

Hence:

$$I_n - I_{n-1} = \int \frac{\sin(2n+1)x}{\sin x}\,\mathrm{d}x - \int \frac{\sin(2n-1)x}{\sin x}\,\mathrm{d}x$$
$$= \int \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,\mathrm{d}x$$
$$= \int \frac{\sin 2nx\cos x + \cos 2x\sin x - (\sin 2nx\cos x - \cos 2nx\sin x)}{\sin x}\,\mathrm{d}x$$
$$= \int \frac{2\cos 2nx\sin x}{\sin x}\,\mathrm{d}x$$
$$= 2\int \cos 2nx\,\mathrm{d}x$$
$$= \frac{2\sin 2nx}{2n}$$
$$= \frac{\sin 2nx}{n} \text{ as required}$$

16 b $I_5 - I_4 = \frac{1}{5}\sin 10x$

$I_4 - I_3 = \frac{1}{4}\sin 8x$

$I_3 - I_2 = \frac{1}{3}\sin 6x$

$I_2 - I_1 = \frac{1}{2}\sin 4x$

$I_1 - I_0 = \sin 2x$

$I_5 = \frac{1}{5}\sin 10x + I_4$

$= \frac{1}{5}\sin 10x + \frac{1}{4}\sin 8x + I_3$

$= \frac{1}{5}\sin 10x + \frac{1}{4}\sin 8x + \frac{1}{3}\sin 6x + I_2$

$= \frac{1}{5}\sin 10x + \frac{1}{4}\sin 8x + \frac{1}{3}\sin 6x + \frac{1}{2}\sin 4x + I_1$

$= \frac{1}{5}\sin 10x + \frac{1}{4}\sin 8x + \frac{1}{3}\sin 6x + \frac{1}{2}\sin 4x + \sin 2x + I_0$

$I_n = \int \frac{\sin(2n+1)x}{\sin x}\,dx$

Therefore:

$I_0 = \int \frac{\sin x}{\sin x}\,dx$

$= \int dx$

$= x + c$

Hence:

$I_5 = \frac{1}{5}\sin 10x + \frac{1}{4}\sin 8x + \frac{1}{3}\sin 6x + \frac{1}{2}\sin 4x + \sin 2x + x + c$

16 c $$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\,dx = \left[\frac{\sin 2nx}{n}\right]_0^{\frac{\pi}{2}}$$
$$= 0$$

Therefore:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\,dx$$

Likewise:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-3)x}{\sin x}\,dx = \left[\frac{\sin(2n-2)x}{n}\right]_0^{\frac{\pi}{2}}$$

$\left[\frac{\sin(2n-2)x}{n}\right]_0^{\frac{\pi}{2}}$ is always an even number multiplied by π (as the result is zero)

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx$$

Write I_n as I_{2p+1}

Then:

$$I_{2p+1} = I_{2p-1} = I_{2p-3} = \ldots = I_0$$

Where:

$$I_0 = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x}\,dx$$
$$= \frac{\pi}{2}$$

Thus:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx = \frac{\pi}{2} \text{ for positive integers}$$

17 a The equation $y^2 = \frac{1}{3}x(x-1)^2$ has two roots, namely $x = 0$ and $x = 1$

Therefore:

$$\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

gives half the length of the loop.

$$y^2 = \frac{1}{3}x(x-1)^2$$

$$3y^2 = x(x^2-2x+1)$$

$$= x^3 - 2x^2 + x$$

$$6y\frac{dy}{dx} = 3x^2 - 4x + 1$$

$$= 3x(x-1) - x + 1$$

$$= 3x(x-1) - (x-1)$$

$$= (3x-1)(x-1)$$

$$\frac{dy}{dx} = \frac{(3x-1)(x-1)}{6y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(3x-1)^2(x-1)^2}{36y^2}$$

$$= \frac{(3x-1)^2(x-1)^2}{36\left(\frac{1}{3}x(x-1)^2\right)}$$

$$= \frac{(3x-1)^2}{12x}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{12x+(3x-1)^2}{12x}$$

$$= \frac{12x+9x^2-6x+1}{12x}$$

$$= \frac{9x^2+6x+1}{12x}$$

$$= \frac{(3x+1)^2}{12x}$$

Let the length of the loop be L, therefore:

$$\frac{1}{2}L = \int_0^1 \sqrt{\frac{(3x+1)^2}{12x}}\,dx$$

$$= \frac{1}{\sqrt{12}}\int_0^1 \frac{3x+1}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{12}}\int_0^1 \frac{3x}{\sqrt{x}}\,dx + \frac{1}{\sqrt{12}}\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \frac{3}{\sqrt{12}}\int_0^1 \sqrt{x}\,dx + \frac{1}{\sqrt{12}}\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \frac{3}{\sqrt{12}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 + \frac{1}{\sqrt{12}}\left[2x^{\frac{1}{2}}\right]_0^1$$

$$= \frac{3}{\sqrt{12}}\left(\frac{2}{3}\right) + \frac{1}{\sqrt{12}}(2)$$

$$= \frac{6}{3\sqrt{12}} + \frac{2}{\sqrt{12}}$$

$$= \frac{6}{6\sqrt{3}} + \frac{2}{2\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

Hence:

$$L = \frac{4\sqrt{3}}{3}$$

17 b $y^2 = \frac{1}{3}x(x-1)^2$

From part **a**

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{(3x+1)^2}{12x}$$

And using:

$$S = 2\pi\int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_0^1 \sqrt{y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$= 2\pi\int_0^1 \sqrt{\frac{\frac{1}{3}x(x-1)^2(3x+1)^2}{12x}}\,dx$$

$$= 2\pi\int_0^1 \sqrt{\frac{(x-1)^2(3x+1)^2}{36}}\,dx$$

$$= \frac{\pi}{3}\int_0^1 (x-1)(3x+1)\,dx \text{ (choose the positive root)}$$

$$= \frac{\pi}{3}\int_0^1 (3x^2-2x-1)\,dx$$

$$= \frac{\pi}{3}\left[x^3-x^2-x\right]_0^1$$

$$= -\frac{\pi}{3}$$

Note that the answer is negative as the positive root was chosen above.
Had the negative root been chosen then the answer obtained would have been positive.

The area of the surface is $\frac{\pi}{3}$

18 a $\frac{1}{\sinh x + 2\cosh x} = \frac{1}{\frac{1}{2}\left(e^x - e^{-x}\right) + 2\times\frac{1}{2}\left(e^x + e^{-x}\right)}$

$$= \frac{1}{\frac{1}{2}e^x - \frac{1}{2}e^{-x} + e^x + e^{-x}}$$

$$= \frac{1}{\frac{3}{2}e^x + \frac{1}{2}e^{-x}}$$

$$= \frac{2}{3e^x + e^{-x}}$$

$$= \frac{2e^x}{3e^{2x} + 1}$$

$$= \frac{2}{3}\left(\frac{e^x}{e^{2x} + \frac{1}{3}}\right)$$

$$\int \frac{1}{\sinh x + 2\cosh x}\,dx = \frac{2}{3}\int \frac{e^x}{e^{2x} + \frac{1}{3}}\,dx$$

Let $u = e^x \Rightarrow du = e^x dx$

$$\int \frac{1}{\sinh x + 2\cosh x}\,dx = \frac{2}{3}\int \frac{1}{u^2 + \left(\frac{1}{\sqrt{3}}\right)^2}\,du$$

$$= \frac{2}{3}\times\frac{1}{\left(\frac{1}{\sqrt{3}}\right)}\arctan\left(\frac{u}{\left(\frac{1}{\sqrt{3}}\right)}\right) + c$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}u\right) + c$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}e^x\right) + c$$

18 b $$\int_1^4 \frac{3x-1}{\sqrt{x^2-2x+10}}\,dx = \int_1^4 \frac{3x-3}{\sqrt{x^2-2x+10}}\,dx + \int_1^4 \frac{2}{\sqrt{x^2-2x+10}}\,dx$$
$$= 3\int_1^4 \frac{x-1}{\sqrt{x^2-2x+10}}\,dx + 2\int_1^4 \frac{1}{\sqrt{(x-1)^2+3^2}}\,dx$$

Let $u = x-1 \Rightarrow du = dx$
When $x = 1$, $u = 0$
When $x = 4$, $u = 3$

$$\int_1^4 \frac{3x-1}{\sqrt{x^2-2x+10}}\,dx = 3\int_1^4 \frac{x-1}{\sqrt{x^2-2x+10}}\,dx + 2\int_0^3 \frac{1}{\sqrt{u^2+3^2}}\,du$$
$$= 3\left[\sqrt{x^2-2x+10}\right]_1^4 + 2\left[\text{arsinh}\left(\frac{u}{3}\right)\right]_0^3$$
$$= 3\left[\sqrt{16-8+10} - \sqrt{1-2+10}\right] + 2\left[\text{arsinh}(1) - \text{arsinh}(0)\right]$$
$$= 3\left[\sqrt{18} - \sqrt{9}\right] + 2\text{arsinh}(1)$$
$$= 3\left[3\sqrt{2} - 3\right] + 2\text{arsinh}(1)$$
$$= 9\left(\sqrt{2}-1\right) + 2\text{arsinh}(1) \text{ as required}$$

19 a $$I_n = \int \sec^n x\,dx$$
$$= \int \sec^{n-2} x \sec^2 x\,dx$$

Let $u = \sec^{n-2} x \Rightarrow \frac{du}{dx} = (n-2)\sec^{n-3} x \sec x \tan x \Rightarrow \frac{du}{dx} = (n-2)\sec^{n-2} x \tan x$

Let $\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$$\int \sec^{n-2} x \sec^2 x\,dx = \sec^{n-2} x \tan x - (n-2)\int \tan x \sec^{n-2} x \tan x\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$
$$I_n + (n-2)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$
$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2} \text{ as required}$$

19 b $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \left(\dfrac{n-2}{n-1}\right) I_{n-2}$

$$I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4} I_3$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left(\frac{\sec x \tan x}{2} + \left(\frac{1}{2}\right) I_1\right)$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8} I_1$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\int \sec x \, \mathrm{d}x$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln\left|\sec x + \tan x\right| + c$$

c $\displaystyle\int_0^{\frac{\pi}{4}} \sec^5 x \, \mathrm{d}x = \left[\frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln\left|\sec x + \tan x\right|\right]_0^{\frac{\pi}{4}}$

$$= \left[\frac{1}{4}\left(\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\right)^3 \times 1 + \frac{3}{8}\left(\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\right) \times 1 + \frac{3}{8}\ln\left|\frac{1}{\left(\frac{1}{\sqrt{2}}\right)} + 1\right|\right] - 0$$

$$= \frac{\left(\sqrt{2}\right)^3}{4} + \frac{3}{8}\sqrt{2} + \frac{3}{8}\ln\left(\sqrt{2}+1\right)$$

$$= \frac{2\sqrt{2}}{4} + \frac{3}{8}\sqrt{2} + \frac{3}{8}\ln\left(\sqrt{2}+1\right)$$

$$= \frac{4}{8}\sqrt{2} + \frac{3}{8}\sqrt{2} + \frac{3}{8}\ln\left(\sqrt{2}+1\right)$$

$$= \frac{7}{8}\sqrt{2} + \frac{3}{8}\ln\left(\sqrt{2}+1\right)$$

$$= \frac{1}{8}\left(7\sqrt{2} + 3\ln\left(\sqrt{2}+1\right)\right) \text{ as required}$$

20 a $\int \sqrt{a^2 - x^2}\,\mathrm{d}x$

Let $x = a\sin\theta \Rightarrow \mathrm{d}x = a\cos\theta$

$$\begin{aligned}
\int \sqrt{a^2 - x^2}\,\mathrm{d}x &= \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta\,\mathrm{d}\theta \\
&= \int \sqrt{a^2\left(1-\sin^2\theta\right)} \times a\cos\theta\,\mathrm{d}\theta \\
&= a\int \sqrt{\cos^2\theta} \times a\cos\theta\,\mathrm{d}\theta \\
&= a^2\int \cos^2\theta\,\mathrm{d}\theta \\
&= \frac{a^2}{2}\int \left(\cos 2\theta + 1\right)\mathrm{d}\theta \\
&= \frac{a^2}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right] + c \\
&= \frac{a^2}{2}\left[\frac{1}{2}\times 2\sin\theta\cos\theta + \theta\right] + c \\
&= \frac{a^2}{2}\left(\sin\theta\cos\theta + \theta\right) + c \\
&= \frac{a^2}{2}\left(\sin\theta\sqrt{1-\sin^2\theta} + \theta\right) + c \\
&= \frac{a^2}{2}\left(\sin\left(\arcsin\left(\frac{x}{a}\right)\right)\sqrt{1-\sin^2\left(\arcsin\left(\frac{x}{a}\right)\right)} + \arcsin\left(\frac{x}{a}\right)\right) + c \\
&= \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} + \arcsin\left(\frac{x}{a}\right)\right) + c \\
&= \frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}} + \arcsin\left(\frac{x}{a}\right)\right) + c \\
&= \frac{a^2}{2}\left(\frac{x}{a^2}\sqrt{a^2-x^2} + \arcsin\left(\frac{x}{a}\right)\right) + c \\
&= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + c \text{ as required}
\end{aligned}$$

20 b $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

The area, A, of one-quarter of the ellipse is:

$$A=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,\mathrm{d}x$$

$$=\frac{b}{a}\left[\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}\right]_0^a$$

$$=\frac{b}{a}\left[\left(\frac{a^2}{2}\arcsin\left(\frac{a}{a}\right)+\frac{a}{2}\sqrt{a^2-a^2}\right)-\left(\frac{a^2}{2}\arcsin\left(\frac{0}{a}\right)+\frac{0}{2}\sqrt{a^2-0^2}\right)\right]$$

$$=\frac{b}{a}\left(\frac{a^2}{2}\arcsin(1)\right)$$

$$=\frac{\pi ab}{4}$$

Therefore, the total area of the ellipse is πab

Challenge

$$x = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) \mathrm{d}u$$

The integrand is the derivative of $x(t)$ with respect to t expressed in terms of the dummy variable u.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos\left(\frac{\pi t^2}{2}\right)$$

$$y = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) \mathrm{d}u$$

The integrand is the derivative of $y(t)$ with respect to t expressed in terms of the dummy variable u.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sin\left(\frac{\pi t^2}{2}\right)$$

Thus:

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \cos^2\left(\frac{\pi t^2}{2}\right) + \sin^2\left(\frac{\pi t^2}{2}\right) \\ &= 1\end{aligned}$$

$$\begin{aligned}\int_0^a \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t &= \int_0^a 1\, \mathrm{d}t \\ &= a\end{aligned}$$