Exercise 4C

1 $\int \frac{1}{a^2+x^2}\,dx$

Let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\,d\theta$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta}\,d\theta$$
$$= \int \frac{a\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\,d\theta$$
$$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$
$$= \frac{1}{a}\int d\theta$$
$$= \frac{1}{a}\theta + c$$

Since $x = a\tan\theta$

$\theta = \arctan\left(\frac{x}{a}\right)$

Therefore:

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$ as required

2 $\int \frac{1}{\sqrt{1-x^2}}\,dx$

Let $x = \cos\theta \Rightarrow dx = -\sin\theta\,d\theta$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{-\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta$$
$$= -\int \frac{\sin\theta}{\sqrt{\sin^2\theta}}\,d\theta$$
$$= -\int d\theta$$
$$= -\theta + c$$

Since $x = \cos\theta$

$\theta = \arccos x$

Therefore:

$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + c$ as required

3 a $\int \frac{3}{\sqrt{4-x^2}}\,dx = 3\int \frac{1}{\sqrt{2^2-x^2}}\,dx$

$$= 3\arcsin\left(\frac{x}{2}\right) + c$$

3 b $\int \frac{1}{\sqrt{x^2-9}}\,\mathrm{d}x = \int \frac{1}{\sqrt{x^2-3^2}}\,\mathrm{d}x$

$= \operatorname{ar\,cosh}\left(\frac{x}{3}\right)+c$

c $\int \frac{4}{5+x^2}\,\mathrm{d}x = 4\int \frac{1}{\left(\sqrt{5}\right)^2+x^2}\,\mathrm{d}x$

$= \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)+c$

d $\int \frac{1}{\sqrt{4x^2+25}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4\left(x^2+\frac{25}{4}\right)}}\,\mathrm{d}x$

$= \frac{1}{2}\int \frac{1}{\sqrt{x^2+\frac{5^2}{2^2}}}\,\mathrm{d}x$

$= \frac{1}{2}\operatorname{arsinh}\left(\frac{x}{5/2}\right)+c$

$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{5}\right)+c$

4 a $\int \frac{1}{\sqrt{25-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{5^2-x^2}}\,\mathrm{d}x$

$= \arcsin\left(\frac{x}{5}\right)+c$

b $\int \frac{3}{\sqrt{x^2+9}}\,\mathrm{d}x = 3\int \frac{1}{\sqrt{x^2+3^2}}\,\mathrm{d}x$

$= 3\operatorname{ar\,sinh}\left(\frac{x}{3}\right)+c$

c $\int \frac{1}{\sqrt{x^2-2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{x^2-\left(\sqrt{2}\right)^2}}\,\mathrm{d}x$

$= \operatorname{arcosh}\left(\frac{x}{\sqrt{2}}\right)+c$

d $\int \frac{1}{16+x^2}\,\mathrm{d}x = \int \frac{1}{4^2+x^2}\,\mathrm{d}x$

$= \frac{1}{4}\arctan\left(\frac{x}{4}\right)+c$

5 a $4x^2-12=4\left(x^2-3\right)$

$$=4\left(x^2-\left(\sqrt{3}\right)^2\right)$$

$$\sqrt{4x^2-12}=2\sqrt{x^2-\left(\sqrt{3}\right)^2}$$

$$\int\frac{1}{\sqrt{4x^2-12}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{x^2-\left(\sqrt{3}\right)^2}}\,dx$$

$$=\frac{1}{2}\operatorname{arcosh}\left(\frac{x}{\sqrt{3}}\right)+c$$

b $\dfrac{1}{4+3x^2}=\dfrac{1}{3\left[\frac{4}{3}+x^2\right]}$

$$=\frac{1}{3\left[\left(\frac{2}{\sqrt{3}}\right)^2+x^2\right]}$$

$$\int\frac{1}{4+3x^2}\,dx=\frac{1}{3}\int\frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2+x^2}\,dx$$

$$=\frac{1}{3}\times\frac{1}{2/\sqrt{3}}\arctan\left(\frac{x}{2/\sqrt{3}}\right)+c$$

$$=\frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}x}{2}\right)+c$$

c $\dfrac{1}{\sqrt{9x^2+16}}=\dfrac{1}{\sqrt{9\left(x^2+\frac{16}{9}\right)}}$

$$=\frac{1}{3\sqrt{x^2+\left(\frac{4}{3}\right)^2}}$$

$$\int\frac{1}{\sqrt{9x^2+16}}\,dx=\frac{1}{3}\int\frac{1}{\sqrt{x^2+\left(\frac{4}{3}\right)^2}}\,dx$$

$$=\frac{1}{3}\operatorname{arsinh}\left(\frac{x}{4/3}\right)+c$$

$$=\frac{1}{3}\operatorname{arsinh}\left(\frac{3x}{4}\right)+c$$

5 d $$\frac{1}{\sqrt{3-4x^2}} = \frac{1}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}$$
$$= \frac{1}{2\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - x^2}}$$
$$\int \frac{1}{\sqrt{3-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - x^2}}\,dx$$
$$= \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{3}}\right) + c \qquad |x| < \frac{\sqrt{3}}{2}$$

6 a $$\int_1^3 \frac{2}{1+x^2}\,dx = 2\int_1^3 \frac{1}{1+x^2}\,dx$$
$$= 2\left[\arctan x\right]_1^3$$
$$= 2\left[\arctan 3 - \arctan 1\right]$$
$$= 0.927 \text{ rads}$$

b $$\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,dx = \int_1^2 \frac{3}{\sqrt{4\left(\frac{1}{4}+x^2\right)}}\,dx$$
$$= \frac{3}{2}\int_1^2 \frac{1}{\sqrt{\frac{1}{2^2}+x^2}}\,dx$$
$$= \frac{3}{2}\left[\text{arsinh}\left(\frac{x}{1/2}\right)\right]_1^2$$
$$= \frac{3}{2}\left[\text{arsinh}(2x)\right]_1^2$$
$$= \frac{3}{2}(\text{arsinh}4 - \text{arsinh}2)$$
$$= 0.977$$

6 c $\sqrt{21-3x^2} = \sqrt{3(7-x^2)}$

$= \sqrt{3} \times \sqrt{(\sqrt{7})^2 - x^2}$

$\int_{-1}^{2} \frac{1}{\sqrt{21-3x^2}}\,dx = \frac{1}{\sqrt{3}} \int_{-1}^{2} \frac{1}{\sqrt{(\sqrt{7})^2 - x^2}}\,dx$

$= \frac{1}{\sqrt{3}} \left[\arcsin\left(\frac{x}{\sqrt{7}}\right) \right]_{-1}^{2}$

$= \frac{1}{\sqrt{3}} \left[\arcsin\left(\frac{2}{\sqrt{7}}\right) - \arcsin\left(\frac{-1}{\sqrt{7}}\right) \right]$

$= 0.719$

7 a $\int_{0}^{4} \frac{1}{\sqrt{x^2+16}}\,dx = \int_{0}^{4} \frac{1}{\sqrt{x^2+4^2}}\,dx$

$= \left[\operatorname{arsinh}\left(\frac{x}{4}\right) \right]_{0}^{4}$

$= [\operatorname{arsinh} 1 - \operatorname{arsinh} 0]$

$= \ln\left(1+\sqrt{1^2+1}\right) - \ln\left(0+\sqrt{0^2+1}\right)$

(since $\operatorname{arsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$)

$= \ln\left(1+\sqrt{2}\right) - \ln(1)$

$= \ln\left(1+\sqrt{2}\right)$

7 b $\displaystyle\int_{13}^{15}\frac{1}{\sqrt{x^2-144}}\,dx=\int_{13}^{15}\frac{1}{\sqrt{x^2-12^2}}\,dx$

$$=\left[\operatorname{arcosh}\left(\frac{x}{12}\right)\right]_{13}^{15}$$

$$=\ln\left(\frac{15}{12}+\sqrt{\left(\frac{15}{12}\right)^2-1}\right)-\ln\left(\frac{13}{12}+\sqrt{\left(\frac{13}{12}\right)^2-1}\right)$$

(since $\operatorname{arcosh}x=\ln\left(x+\sqrt{x^2-1}\right)x\geq 1$)

$$=\ln\left(\frac{15}{12}+\sqrt{\frac{9}{16}}\right)-\ln\left(\frac{13}{12}+\sqrt{\frac{25}{144}}\right)$$

$$=\ln\left(\frac{15}{12}+\frac{3}{4}\right)-\ln\left(\frac{13}{12}+\frac{5}{12}\right)$$

$$=\ln(2)-\ln\left(\frac{3}{2}\right)$$

$$=\ln\left(\frac{2}{3/2}\right)$$

$$=\ln\left(\frac{4}{3}\right)$$

c $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}}\frac{1}{\sqrt{4-x^2}}\,dx=\int_{\sqrt{2}}^{\sqrt{3}}\frac{1}{\sqrt{2^2-x^2}}\,dx$

$$=\left[\arcsin\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$=\arcsin\left(\frac{\sqrt{3}}{2}\right)-\arcsin\left(\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\pi}{3}-\frac{\pi}{4}$$

$$=\frac{\pi}{12}$$

8 a $R=\int_{-1}^{3}\frac{2}{\sqrt{2x^2+9}}\,\mathrm{d}x$

$$=\int_{-1}^{3}\frac{2}{\sqrt{2\left(x^2+\frac{9}{2}\right)}}\,\mathrm{d}x$$

$$=\frac{2}{\sqrt{2}}\int_{-1}^{3}\frac{1}{\sqrt{x^2+\left(\frac{3}{\sqrt{2}}\right)^2}}\,\mathrm{d}x$$

$$=\frac{2}{\sqrt{2}}\left[\text{arsinh}\left(\frac{x}{3/\sqrt{2}}\right)\right]_{-1}^{3}$$

$$=\sqrt{2}\left[\text{arsinh}\left(\frac{\sqrt{2}x}{3}\right)\right]_{-1}^{3}$$

$$=\sqrt{2}\left[\text{arsinh}\left(\sqrt{2}\right)-\text{arsinh}\left(-\frac{\sqrt{2}}{3}\right)\right]$$

$$=2.27\text{ (3 s.f.)}$$

b $V=\int_{a}^{b}y^2\,\mathrm{d}x$

$$=\pi\int_{-1}^{3}\left(\frac{2}{\sqrt{2x^2+9}}\right)^2\mathrm{d}x$$

$$=4\pi\int_{-1}^{3}\frac{1}{2x^2+9}\,\mathrm{d}x$$

$$=4\pi\int_{-1}^{3}\frac{1}{2\left(x^2+\frac{9}{2}\right)}\,\mathrm{d}x$$

$$=2\pi\int_{-1}^{3}\frac{1}{x^2+\left(\frac{3}{\sqrt{2}}\right)^2}\,\mathrm{d}x$$

$$=2\pi\times\frac{1}{3/\sqrt{2}}\left[\arctan\left(\frac{x}{3/\sqrt{2}}\right)\right]_{-1}^{3}$$

$$=\frac{2\sqrt{2}\pi}{3}\left[\arctan\left(\frac{\sqrt{2}x}{3}\right)\right]_{-1}^{3}$$

$$=\frac{2\sqrt{2}\pi}{3}\left[\arctan\left(\sqrt{2}\right)-\arctan\left(-\frac{\sqrt{2}}{3}\right)\right]$$

$$=4.13\text{ (3 s.f.)}$$

9 **a** $x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$

Area of one-quarter of the circle is given by $\int_0^r y\,dx$

Therefore:

$A = 4\int_0^r y\,dx$

$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx$ as required

9 b $x = r\cos\theta \Rightarrow \mathrm{d}x = -r\sin\theta\,\mathrm{d}\theta$

when $x = r,\ \theta = \cos^{-1}1 = 0$

when $x = 0,\ \theta = \cos^{-1}0 = \frac{\pi}{2}$

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x$$

Therefore:

$$A = 4\int_{\frac{\pi}{2}}^{0} \sqrt{\left(r^2 - r^2\cos^2\theta\right)}\,(-r\sin\theta)\,\mathrm{d}\theta$$

$$= -4r\int_{\frac{\pi}{2}}^{0} \sqrt{r^2\left(1-\cos^2\theta\right)}\,(\sin\theta)\,\mathrm{d}\theta$$

$$= -4r^2\int_{\frac{\pi}{2}}^{0} \sqrt{\left(1-\cos^2\theta\right)}\,(\sin\theta)\,\mathrm{d}\theta$$

$$= -4r^2\int_{\frac{\pi}{2}}^{0} \sin^2\theta\,\mathrm{d}\theta$$

$$= -4r^2\int_{\frac{\pi}{2}}^{0} \left(\frac{1-\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= -4r^2\int_{\frac{\pi}{2}}^{0} \frac{1}{2}\,\mathrm{d}\theta - 4r^2\int_{\frac{\pi}{2}}^{0} \left(\frac{\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= -2r^2\int_{\frac{\pi}{2}}^{0} \mathrm{d}\theta - 2r^2\int_{\frac{\pi}{2}}^{0} \cos 2\theta\,\mathrm{d}\theta$$

$\int_{\frac{\pi}{2}}^{0} \cos 2\theta\,\mathrm{d}\theta = 0$, therefore:

$$A = -2r^2\int_{\frac{\pi}{2}}^{0} \mathrm{d}\theta$$

$$= -2r^2\left[\theta\right]_{\frac{\pi}{2}}^{0}$$

$$= -2r^2\left(0 - \frac{\pi}{2}\right)$$

$= \pi r^2$ as required

10 a $\int \frac{x^2}{9x^2+4}\,dx$

Let $x = \frac{2}{3}\tan\theta \Rightarrow \frac{dx}{d\theta} = \frac{2}{3}\sec^2\theta$

$$\int \frac{x^2}{9x^2+4}\,dx = \int \frac{\frac{4}{9}\tan^2\theta}{9\times\frac{4}{9}\tan^2\theta+4}\times\frac{2}{3}\sec^2\theta\,d\theta$$

$$= \frac{8}{27}\int \frac{\tan^2\theta}{4\tan^2\theta+4}\sec^2\theta\,d\theta$$

$$= \frac{8}{27}\int \frac{\tan^2\theta}{4\left(\tan^2\theta+1\right)}\sec^2\theta\,d\theta$$

$$= \frac{2}{27}\int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \frac{2}{27}\int \tan^2\theta\,d\theta$$

$$= \frac{2}{27}\int \left(\sec^2\theta-1\right)d\theta$$

$$= \frac{2}{27}\left(\tan\theta-\theta\right)+c$$

$x = \frac{2}{3}\tan\theta \Rightarrow \theta = \arctan\left(\frac{3}{2}x\right)$

Therefore:

$$\int \frac{x^2}{9x^2+4}\,dx = \frac{2}{27}\left(\tan\left(\arctan\left(\frac{3}{2}x\right)\right)-\arctan\left(\frac{3}{2}x\right)\right)+c$$

$$= \frac{2}{27}\left(\frac{3}{2}x-\arctan\left(\frac{3}{2}x\right)\right)+c$$

$$= \frac{1}{9}x-\frac{2}{27}\arctan\left(\frac{3}{2}x\right)+c$$

10 b $\int \sqrt{\frac{x}{x+1}}\,dx$

Let $x = \sinh^2 u \Rightarrow dx = 2\sinh u \cosh u\,du$ and $u = \text{arsinh}\left(\sqrt{x}\right)$

$$\int \sqrt{\frac{x}{x+1}}\,dx = \int \sqrt{\frac{\sinh^2 u}{\sinh^2 u+1}} \times 2\sinh u \cosh u\,du$$
$$= 2\int \sqrt{\frac{\sinh^2 u}{\cosh^2 u}}\ \sinh u \cosh u\,du$$
$$= 2\int \sinh^2 u\,du$$
$$= 2\int \frac{\cosh 2u - 1}{2}\,du$$
$$= \int \left(\cosh 2u - 1\right)du$$
$$= \frac{1}{2}\sinh 2u - u + c$$
$$= \sinh u \cosh u - u + c$$
$$= (\sinh u)\sqrt{1+\sinh^2 u} - u + c$$

$x = \sinh^2 u$,
Therefore:

$$\int \sqrt{\frac{x}{x+1}}\,dx = \sqrt{x} \times \sqrt{1+x} - \text{arsinh}\left(\sqrt{x}\right) + c$$

11 a $\int \frac{x-2}{\sqrt{x^2-4}}\,dx = \int \frac{x}{\sqrt{x^2-4}}\,dx - 2\int \frac{1}{\sqrt{x^2-4}}\,dx$

$$= \int \frac{x}{\sqrt{x^2-4}}\,dx - 2\int \frac{1}{\sqrt{x^2-2^2}}\,dx$$
$$= \sqrt{x^2-4} - 2\text{arcosh}\left(\frac{x}{2}\right)dx + c$$

b $\int \frac{2x-1}{\sqrt{2-x^2}}\,dx = \int \frac{2x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{2-x^2}}\,dx$

$$= -2\int \frac{-x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{\left(\sqrt{2}\right)^2 - x^2}}\,dx$$
$$= -2\sqrt{2-x^2} - \arcsin\left(\frac{x}{\sqrt{2}}\right) + c \qquad |x| < \sqrt{2}$$

Note that this can equivalently be written as

$$= -2\sqrt{2-x^2} + \arccos\left(\frac{x}{\sqrt{2}}\right) + c \qquad |x| < \sqrt{2}$$

11 c $\int \frac{2+3x}{1+3x^2}\,dx = \int \frac{2}{1+3x^2}\,dx + \int \frac{3x}{1+3x^2}\,dx$

$$= \int \frac{2}{3\left(\frac{1}{3}+x^2\right)}\,dx + \frac{1}{2}\int \frac{6x}{1+3x^2}\,dx$$

$$= \frac{2}{3}\int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 + x^2}\,dx + \frac{1}{2}\int \frac{6x}{1+3x^2}\,dx$$

$$= \frac{2}{3}\left[\frac{1}{1/\sqrt{3}}\arctan\left(\frac{x}{1/\sqrt{3}}\right)\right] + \frac{1}{2}\ln\left(1+3x^2\right) + c$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right) + \ln\sqrt{1+3x^2} + c$$

12 $\frac{x^2+4x+10}{x^3+5x} = \frac{x^2+4x+10}{x\left(x^2+5\right)}$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+5}$$

$x^2+4x+10 = A\left(x^2+5\right) + x\left(Bx+C\right)$

Comparing coefficients:

For constant terms:

$5A = 10$

$A = 2$

For x:

$C = 4$

For x^2:

$A + B = 1$

$B = -1$

Therefore:

$$\frac{x^2+4x+10}{x^3+5x} = \frac{2}{x} + \frac{-x+4}{x^2+5}$$

$$= \frac{2}{x} - \frac{x}{x^2+5} + \frac{4}{x^2+5}$$

Therefore:

$$\int \frac{x^2+4x+10}{x^3+5x}\,dx = \int \frac{2}{x}\,dx - \int \frac{x}{x^2+5}\,dx + \int \frac{4}{x^2+5}\,dx$$

$$= 2\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+5}\,dx + 4\int \frac{1}{x^2+\left(\sqrt{5}\right)^2}\,dx$$

$$= 2\ln x - \frac{1}{2}\ln\left(x^2+5\right) + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + c$$

$$= \ln x^2 - \ln\sqrt{x^2+5} + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + c$$

$$= \ln\left(\frac{x^2}{\sqrt{x^2+5}}\right) + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + c$$

13 $\dfrac{2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$$2 = A(x^2+1) + (Bx+C)(x+1)$$
$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$
$$= x^2(A+B) + x(B+C) + (A+C)$$

Comparing coefficients:

For x^2:

$A+B=0 \Rightarrow A=-B$

For x:

$B+C=0 \Rightarrow B=-C \Rightarrow A=C$

For constant term:

$A+C=2 \Rightarrow 2A=2 \Rightarrow A=C=1$

and

$B=-1$

Therefore:

$$\frac{2}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{-x+1}{x^2+1}$$
$$= \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$$

$$\int_0^1 \frac{2}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{1}{x+1}\,dx - \int_0^1 \frac{x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx$$
$$= \int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx$$
$$= \left[\ln(x+1)\right]_0^1 - \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 + \left[\arctan x\right]_0^1$$
$$= \left[\ln(x+1)\right]_0^1 - \left[\ln\sqrt{x^2+1}\right]_0^1 + \left[\arctan x\right]_0^1$$
$$= \left[\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right)\right]_0^1 + \left[\arctan x\right]_0^1$$
$$= \ln\left(\frac{2}{\sqrt{2}}\right) - \ln\left(\frac{1}{\sqrt{1}}\right) + \arctan 1 - \arctan 0$$
$$= \ln\left(\frac{2}{\sqrt{2}}\right) + \arctan 1$$
$$= \ln 2 - \frac{1}{2}\ln 2 + \arctan 1$$
$$= \frac{1}{2}\ln 2 + \arctan 1$$
$$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$$
$$= \frac{1}{4}(\pi + \ln 2) \text{ as required}$$

14 $\int_2^3 \frac{2x}{\sqrt{x^4 - 1}} \, dx$

Let $u = x^2$

$\frac{du}{dx} = 2x \Rightarrow dx = 2x du$

When $x = 2$, $u = 4$

When $x = 3$, $u = 9$

$$\int_2^3 \frac{2x}{\sqrt{x^4 - 1}} \, dx = \int_4^9 \frac{1}{\sqrt{u^2 - 1}} \, du$$

$$= [\text{arcosh } u]_4^9$$

$$= \text{arcosh } 9 - \text{arcosh } 4$$

$$= 0.824 \text{ (3 s.f.)}$$

15 $\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx$

Let $x = \frac{1}{2}\sin\theta \Rightarrow dx = \frac{1}{2}\cos\theta\,d\theta$ and $\theta = \arcsin 2x$

When $x = 0$, $\theta = 0$

When $x = \frac{1}{4}$, $\theta = \frac{\pi}{6}$

$$\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\frac{1}{4}\sin^2\theta}{\sqrt{1-4\times\frac{1}{4}\sin^2\theta}} \times \frac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{16}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{16}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{16}\left(\frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi}{96} - \frac{\sqrt{3}}{64}$$

$$= \frac{1}{192}\left(2\pi - 3\sqrt{3}\right) \text{ as required}$$

16 a $\int \sqrt{x^2 - 4}\, dx$

Let $x = 2\cosh u \Rightarrow dx = 2\sinh u\ du$

$$\int \sqrt{x^2 - 4}\, dx = \int \sqrt{4\cosh^2 u - 4} \times 2\sinh u\ du$$
$$= 2\int \sqrt{4\left(\cosh^2 u - 1\right)} \times \sinh u\ du$$
$$= 4\int \sqrt{\sinh^2 u} \times \sinh u\ du$$
$$= 4\int \sinh^2 u\, du$$
$$= 4\int \frac{\cosh 2u - 1}{2} du$$
$$= 2\int (\cosh 2u - 1)\, du$$
$$= 2\int \cosh 2u\, du - 2\int du$$
$$= \sinh 2u - 2u + c$$
$$= 2\sinh u \cosh u - 2u + c$$
$$= 2\cosh u \sqrt{\left(\cosh^2 u - 1\right)} - 2u + c$$

Since $x = \cosh u$

$$\int \sqrt{x^2 - 4}\, dx = 2\left(\frac{x}{2}\right)\sqrt{\left(\left(\frac{x}{2}\right)^2 - 1\right)} - 2\text{arcosh}\left(\frac{x}{2}\right) + c$$
$$= x\sqrt{\left(\frac{x^2}{4} - 1\right)} - 2\text{arcosh}\left(\frac{x}{2}\right) + c$$
$$= x\sqrt{\left(\frac{x^2 - 4}{4}\right)} - 2\text{arcosh}\left(\frac{x}{2}\right) + c$$
$$= \frac{x}{2}\sqrt{x^2 - 4} - 2\text{arcosh}\left(\frac{x}{2}\right) + c \text{ as required}$$

16 b $\frac{x^2}{4}-\frac{y^2}{9}=1$

$\frac{y^2}{9}=\frac{x^2}{4}-1$

$y^2=9\left(\frac{x^2}{4}-1\right)$

$=9\left(\frac{x^2-4}{4}\right)$

$=\frac{9}{4}\left(x^2-4\right)$

$y=\frac{3}{2}\sqrt{x^2-4}$

At $y=0$, $x=\pm 2$

As the hyperbola has a line of symmetry at the x-axis, the area between the hyperbola, $x=2$ and $x=4$ is given by:

$$A=2\int_2^4 y\,\mathrm{d}x$$

$$=3\int_2^4 \sqrt{x^2-4}\,\mathrm{d}x$$

$$=3\left[\frac{x}{2}\sqrt{x^2-4}-2\operatorname{arcosh}\left(\frac{x}{2}\right)\right]_2^4$$

$$=3\left[\left(\frac{4}{2}\sqrt{4^2-4}-2\operatorname{arcosh}\left(\frac{4}{2}\right)\right)-\left(\frac{2}{2}\sqrt{2^2-4}-2\operatorname{arcosh}\left(\frac{2}{2}\right)\right)\right]$$

$$=3\left[\left(2\sqrt{12}-2\operatorname{arcosh}(2)\right)-\left(\sqrt{4-4}-2\operatorname{arcosh}(1)\right)\right]$$

$$=3\left(2\sqrt{12}+2\left(\operatorname{arcosh}(1)-\operatorname{arcosh}(2)\right)\right)$$

$$=6\sqrt{12}+6\left(\operatorname{arcosh}(1)-\operatorname{arcosh}(2)\right)$$

$$=6\sqrt{12}+6\operatorname{arcosh}(2)$$

$$=12.9\ (3\text{ s.f.})$$

17 $\int \frac{1}{2\cosh x - \sinh x}\,dx$

$$\frac{1}{2\cosh x - \sinh x} = \frac{1}{2\left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) - \frac{1}{2}\left(e^x - e^{-x}\right)}$$

$$= \frac{1}{e^x + e^{-x} - \frac{1}{2}e^x + \frac{1}{2}e^{-x}}$$

$$= \frac{1}{\frac{1}{2}e^x + \frac{3}{2}e^{-x}}$$

$$= \frac{2}{e^x + 3e^{-x}}$$

$$= \frac{2e^x}{e^{2x} + 3}$$

Therefore:

$$\int \frac{1}{2\cosh x - \sinh x}\,dx = \int \frac{2e^x}{e^{2x} + 3}\,dx$$

Let $u = e^x \Rightarrow du = e^x dx$

$$\int \frac{2e^x}{2e^x + 3}\,dx = \int \frac{2}{u^2 + 3}\,du$$

$$= 2\int \frac{1}{u^2 + \left(\sqrt{3}\right)^2}\,du$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + c$$

18 $$\int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x+9}}\,dx = \int_0^1 \frac{1}{\sqrt{9\left(\frac{4}{9}\sinh^2 x+1\right)}}\cosh x\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{\sqrt{\left(\frac{2}{3}\sinh x\right)^2+1}}\cosh x\,dx$$

Let $u = \frac{2}{3}\sinh x \Rightarrow du = \frac{2}{3}\cosh x\,dx$

When $x = 0$, $u = 0$

When $x = 1$, $u = \frac{2}{3}\sinh 1$

$$\int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x+9}}\,dx = \frac{1}{2}\int_0^{\frac{2}{3}\sinh 1} \frac{1}{\sqrt{u^2+1}}\,du$$

$$= \frac{1}{2}\left[\operatorname{arsinh} u\right]_0^{\frac{2}{3}\sinh 1}$$

$$= \frac{1}{2}\left[\operatorname{arsinh}\left(\frac{2}{3}\sinh 1\right) - \operatorname{arsinh}(0)\right]$$

$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2}{3}\sinh 1\right)$$

$= 0.360$ (3 s.f.)

19 a i $\frac{1}{a^2-x^2}=\frac{1}{(a-x)(a+x)}$

$=\frac{A}{(a-x)}+\frac{B}{(a+x)}$

$1=A(a+x)+B(a-x)$

Comparing coefficients:

For x:

$A-B=0\Rightarrow A=B$

For constant terms:

$aA-aB=1\Rightarrow A=\frac{1}{2a}$ and $B=\frac{1}{2a}$

$\frac{1}{a^2-x^2}=\frac{1}{2a(a-x)}+\frac{1}{2a(a+x)}$

$$\int\frac{1}{a^2-x^2}\,\mathrm{d}x=\int\frac{1}{2a(a-x)}\,\mathrm{d}x+\int\frac{1}{2a(a+x)}\,\mathrm{d}x$$

$$=-\frac{1}{2a}\int\frac{-1}{(a-x)}\,\mathrm{d}x+\frac{1}{2a}\int\frac{1}{(a+x)}\,\mathrm{d}x$$

$$=-\frac{1}{2a}\ln(a-x)+\frac{1}{2a}\ln(a+x)+c$$

$$=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c$$

ii $\int\frac{1}{a^2-x^2}\,\mathrm{d}x$

Let $x=a\tanh\theta\Rightarrow \mathrm{d}x=a\,\mathrm{sech}^2\theta\,\mathrm{d}\theta$

$$\int\frac{1}{a^2-x^2}\,\mathrm{d}x=\int\frac{1}{a^2-a^2\tanh^2\theta}\times a\,\mathrm{sech}^2\theta\,\mathrm{d}\theta$$

$$=\int\frac{a\,\mathrm{sech}^2\theta}{a^2(1-\tanh^2\theta)}\,\mathrm{d}\theta$$

$$=\frac{1}{a}\int\frac{\mathrm{sech}^2\theta}{\mathrm{sech}^2\theta}\,\mathrm{d}\theta$$

$$=\frac{1}{a}\int\mathrm{d}\theta$$

$$=\frac{1}{a}\theta+c$$

$$=\frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)+c$$

19 b Equating the answers from a parts **i** and **ii** gives:

$$\frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)+c=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c$$

$$\frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$$

$$\operatorname{artanh}\left(\frac{x}{a}\right)=\frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)\qquad |x|<a$$

20 a $\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$

Let $x=\sec\theta \Rightarrow \mathrm{d}x=\sec\theta\tan\theta\,\mathrm{d}\theta$

$$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x=\int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}\times\sec\theta\tan\theta\,\mathrm{d}\theta$$
$$=\int \frac{1}{\sec\theta\sqrt{\tan^2\theta}}\times\sec\theta\tan\theta\,\mathrm{d}\theta$$
$$=\int \mathrm{d}\theta$$
$$=\theta+c$$

Since $x=\sec\theta \Rightarrow \theta=\operatorname{arcsec} x$

Therefore:

$$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x=\operatorname{arcsec} x+c$$

b $\int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$

Let $x=\sec\theta \Rightarrow \mathrm{d}x=\sec\theta\tan\theta\,\mathrm{d}\theta$

$$\int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x=\int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\,\tan\theta\,\mathrm{d}\theta$$
$$=\int \tan^2\theta\,\mathrm{d}\theta$$
$$=\tan\theta-\theta+c$$
$$=\sqrt{\sec^2\theta-1}-\theta+c$$
$$=\sqrt{x^2-1}-\operatorname{arcsec} x+c$$