Exercise 4B

1 a $\int \sinh^3 x \cosh x \, dx$

Let $f(x) = \sinh x \Rightarrow f'(x) = \cosh x$

Using $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ gives:

$\int \sinh^3 x \cosh x \, dx = \frac{1}{4}\sinh^4 x + c$

b $\int \tanh 4x \, dx$

Let $f(x) = 4x \Rightarrow f'(x) = 4$

Using the chain rule,

$\int \tanh 4x \, dx = \frac{1}{4}\int 4\tanh 4x \, dx = \frac{1}{4}\ln(\cosh 4x)$

c $\int \tanh^5 x \operatorname{sech}^2 x \, dx$

Let $f(x) = \tanh x \Rightarrow f'(x) = \operatorname{sech}^2 x$

Using $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ gives:

$\int \tanh^5 x \operatorname{sech}^2 x \, dx = \frac{1}{6}\tanh^6 x + c$

d $\int \operatorname{cosech}^7 x \coth x \, dx$

Let $f(x) = \operatorname{cosech} x \Rightarrow f'(x) = -\operatorname{cosech} x \coth x$

Using $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ gives:

$\int \operatorname{cosech}^7 x \coth x \, dx = -\int \operatorname{cosech}^6 x(-\operatorname{cosech} x \coth x) \, dx$

$= -\frac{1}{7}\operatorname{cosech}^7 x + c$

e $\int \sqrt{\cosh 2x} \, \sinh 2x \, dx = \int \cosh^{\frac{1}{2}} 2x \, (\sinh 2x) \, dx$

$= \frac{1}{2}\int \cosh^{\frac{1}{2}} 2x(2\sinh 2x) \, dx$

Let $f(x) = \cosh 2x \Rightarrow f'(x) = 2\sinh 2x$

Using $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ gives:

$\int \sqrt{\cosh 2x} \, \sinh 2x \, dx = \frac{1}{2}\left(\frac{\cosh^{\frac{3}{2}} x}{\frac{3}{2}}\right) + c$

$= \frac{1}{3}\cosh^{\frac{3}{2}} x + c$

1 f $\int \operatorname{sech}^{10} 3x \tanh 3x \, dx = -\int \operatorname{sech}^9 3x \left(-\operatorname{sech} 3x \tanh 3x\right) dx$

$= -\frac{1}{3}\int \operatorname{sech}^9 3x \left(-3\operatorname{sech} 3x \tanh 3x\right) dx$

Let $f(x) = \operatorname{sech}^9 3x \Rightarrow f'(x) = -3\operatorname{sech} 3x \tanh 3x$

Using $\int \left[f(x)\right]^n f'(x) \, dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$ gives:

$\int \operatorname{sech}^{10} 3x \tanh 3x \, dx = -\frac{1}{3}\left(\frac{1}{10}\operatorname{sech}^{10} 3x\right) + c$

$= -\frac{1}{30}\operatorname{sech}^{10} 3x + c$

2 a $\int \frac{\sinh x}{2 + 3\cosh x} dx = \frac{1}{3}\int \frac{3\sinh x}{2 + 3\cosh x} dx$

Let $f(x) = 2 + 3\cosh x \Rightarrow f'(x) = 3\sinh x$

Using $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$ gives:

$\int \frac{\sinh x}{2 + 3\cosh x} dx = \frac{1}{3}\ln\left(2 + 3\cosh x\right) + c$

b $\int \frac{1 + \tanh x}{\cosh^2 x} dx = \int \frac{1}{\cosh^2 x} dx + \int \frac{\tanh x}{\cosh^2 x} dx$

$= \int \operatorname{sech}^2 x \, dx + \int \tanh x \operatorname{sech}^2 x \, dx$

$= \tanh x + \int \tanh x \operatorname{sech}^2 x \, dx$

Let $f(x) = \tanh x \Rightarrow f'(x) = \operatorname{sech}^2 x$

Using $\int \left[f(x)\right]^n f'(x) \, dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$ on the remaining integral gives:

$\int \frac{1 + \tanh x}{\cosh^2 x} dx = \tanh x - \frac{1}{2}\tanh^2 x + c$

Alternative Solution

$\int \frac{1 + \tanh x}{\cosh^2 x} dx = \int \frac{1}{\cosh^2 x} dx + \int \frac{\tanh x}{\cosh^2 x} dx$

$= \int \operatorname{sech}^2 x \, dx + \int \tanh x \operatorname{sech}^2 x \, dx$

$= \tanh x - \int \operatorname{sech} x \left(-\operatorname{sech} x \tanh x\right) dx$

Let $f(x) = \operatorname{sech} x \Rightarrow f'(x) = -\operatorname{sech} x \tanh x$

Using $\int \left[f(x)\right]^n f'(x) \, dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$ on the remaining integral gives:

$\int \frac{1 + \tanh x}{\cosh^2 x} dx = \tanh x - \frac{1}{2}\operatorname{sech}^2 x + c$

Note that the two solutions are equivalent, as $\tanh^2 x + \operatorname{sech}^2 x + c$ and c is an arbitrary constant.

c $\int \frac{5\cosh x + 2\sinh x}{\cosh x} dx = \int 5 \, dx + 2\int \tanh x \, dx$

$= 5x + 2\ln\cosh x + c$

3 a $\int \coth x \, dx = \int \frac{1}{\tanh x} dx$

$= \int \frac{\cosh x}{\sinh x} dx$

Let $f(x) = \sinh x \Rightarrow f'(x) = \cosh x$

Using $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ gives:

$\int \frac{\cosh x}{\sinh x} dx = \ln(\sinh x) + c$ as required.

3 b $\int \coth 2x \, dx = \int \frac{1}{\tanh 2x} dx$

$= \int \frac{\cosh 2x}{\sinh 2x} dx$

$= \frac{1}{2} \int \frac{2\cosh 2x}{\sinh 2x} dx$

Let $f(x) = \sinh 2x \Rightarrow f'(x) = 2\cosh 2x$

Using $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ gives:

$\frac{1}{2} \int \frac{2\cosh 2x}{\sinh 2x} dx = \frac{1}{2} \ln(\sinh 2x) + c$

$= \ln \sqrt{\sinh 2x} + c$

$\int_1^2 \coth 2x \, dx = \left[\ln \sqrt{\sinh 2x} \right]_1^2$

$= \left[\ln \sqrt{\frac{1}{2}(e^{2x} - e^{-2x})} \right]_1^2$

$= \left[\ln \sqrt{\frac{1}{2}(e^4 - e^{-4})} - \ln \sqrt{\frac{1}{2}(e^2 - e^{-2})} \right]$

$= \ln \sqrt{\frac{(e^4 - e^{-4})}{(e^2 - e^{-2})}}$

$= \ln \sqrt{\frac{(e^2 + e^{-2})(e^2 - e^{-2})}{(e^2 - e^{-2})}}$

$= \ln \sqrt{(e^2 + e^{-2})}$ as required

4 a $\int x\sinh 3x\,dx$

Use integration by parts with:

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sinh 3x \Rightarrow v = \frac{1}{3}\cosh 3x$

$$\int x\sinh 3x\,dx = \frac{1}{3}x\cosh 3x - \frac{1}{3}\int \cosh 3x\,dx$$
$$= \frac{1}{3}x\cosh 3x - \frac{1}{9}\sinh 3x\,dx + c$$

b $\int x\,\text{sech}^2 x\,dx$

Use integration by parts with:

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \text{sech}^2 x \Rightarrow v = \tanh x$

$$\int x\,\text{sech}^2 x\,dx = x\tanh x - \int \tanh x\,dx$$
$$= x\tanh x - \ln(\cosh x) + c$$

5 a $\int e^x \cosh x\,dx = \int e^x\left(\frac{e^x + e^{-x}}{2}\right)dx$

$$= \frac{1}{2}\int (e^{2x} + 1)\,dx$$
$$= \frac{1}{2}\int e^{2x}dx + \frac{1}{2}\int dx$$
$$= \frac{1}{4}e^{2x} + \frac{1}{2}x + c$$

b $\int e^{-2x}\sinh 3x\,dx = \int e^{-2x}\left(\frac{e^{3x} - e^{-3x}}{2}\right)dx$

$$= \frac{1}{2}\int e^{-2x}(e^{3x} - e^{-3x})\,dx$$
$$= \frac{1}{2}\int (e^x - e^{-5x})\,dx$$
$$= \frac{1}{2}\int e^x\,dx - \frac{1}{2}\int e^{-5x}\,dx$$
$$= \frac{1}{2}e^x + \frac{1}{10}e^{-5x} + c$$

5 c $\int \cosh x \cosh 3x \, dx = \int \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^{3x} + e^{-3x}}{2}\right) dx$

$= \frac{1}{4}\int \left(e^x + e^{-x}\right)\left(e^{3x} + e^{-3x}\right) dx$

$= \frac{1}{4}\int \left(e^{4x} + e^{-2x} + e^{2x} + e^{-4x}\right) dx$

$= \frac{1}{4}\int e^{4x}\, dx + \frac{1}{4}\int e^{-2x}\, dx + \frac{1}{4}\int e^{2x}\, dx + \frac{1}{4}\int e^{-4x}\, dx$

$= \frac{1}{16}e^{4x} - \frac{1}{8}e^{-2x} + \frac{1}{8}e^{2x} - \frac{1}{16}e^{-4x} + c$

$= \frac{1}{16}\left(e^{4x} - e^{-4x}\right) + \frac{1}{8}\left(e^{2x} - e^{-2x}\right) + c$

$= \frac{1}{8}\left(\frac{e^{4x} - e^{-4x}}{2}\right) + \frac{1}{4}\left(\frac{e^{2x} - e^{-2x}}{2}\right) + c$

$= \frac{1}{8}\sinh 4x + \frac{1}{4}\sinh 2x + c$

6 $\int \cosh^2 3x \, dx = \int \left(\frac{e^{3x} + e^{-3x}}{2}\right)^2 dx$

$= \frac{1}{4}\int \left(e^{6x} + e^{-6x} + 2\right) dx$

$= \frac{1}{4}\int e^{6x} dx + \frac{1}{4}\int e^{-6x} dx + \frac{1}{2}\int dx$

$= \frac{1}{24}e^{6x} - \frac{1}{24}e^{-6x} + \frac{1}{2}x + c$

$= \frac{1}{24}\left(e^{6x} - e^{-6x}\right) + \frac{1}{2}x + c$

$= \frac{1}{12}\left(\frac{e^{6x} - e^{-6x}}{2}\right) + \frac{1}{2}x + c$

$= \frac{1}{12}\sinh 6x + \frac{1}{2}x + c$

Which is the same answer as found in **Example 5b**

7 $\int \frac{1}{\sinh x + \cosh x} \mathrm{d}x = \int \frac{\cosh x - \sinh}{(\sinh x + \cosh x)(\cosh x - \sinh)} \mathrm{d}x$

$$= \int \frac{\cosh x - \sinh}{\sinh x \cosh x - \sinh^2 x + \cosh^2 x - \sinh x \cosh x} \mathrm{d}x$$

$$= \int \frac{\cosh x - \sinh}{\cosh^2 x - \sinh^2 x} \mathrm{d}x$$

$$= \int (\cosh x - \sinh) \mathrm{d}x$$

Therefore:

$$\int_0^1 \frac{1}{\sinh x + \cosh x} \mathrm{d}x = \int_0^1 (\cosh x - \sinh x) \mathrm{d}x$$

$$= \int_0^1 \cosh x \, \mathrm{d}x - \int_0^1 \sinh x \, \mathrm{d}x$$

$$= [\sinh x]_0^1 - [\cosh x]_0^1$$

$$= \left[\frac{e^x - e^{-x}}{2}\right]_0^1 - \left[\frac{e^x + e^{-x}}{2}\right]_0^1$$

$$= \frac{1}{2}\left[e^x - e^{-x} - e^x - e^{-x}\right]_0^1$$

$$= -\left[e^{-x}\right]_0^1$$

$$= -\left(e^{-1} - e^0\right)$$

$$= 1 - \frac{1}{e}$$

8 **a** $\int \sinh^2 x \, \mathrm{d}x = \frac{1}{2}\int (\cosh 2x - 1) \, \mathrm{d}x$

$$= \frac{1}{2}\int \cosh 2x \, \mathrm{d}x - \frac{1}{2}\int \mathrm{d}x$$

$$= \frac{1}{4}\sinh 2x - \frac{1}{2}x + c$$

b $\int (\operatorname{sech} x - \tanh x)^2 \, \mathrm{d}x = \int \left(\operatorname{sech}^2 x + \tanh^2 x - 2\operatorname{sech} x \tanh x\right) \mathrm{d}x$

$$= \int \left(1 - \tanh^2 x + \tanh^2 x - 2\operatorname{sech} x \tanh x\right) \mathrm{d}x$$

$$= \int (1 - 2\operatorname{sech} x \tanh x) \, \mathrm{d}x$$

$$= \int \mathrm{d}x - 2\int \operatorname{sech} x \tanh x \, \mathrm{d}x$$

$$= x + 2\operatorname{sech} x + c$$

8 c $\int \frac{\cosh^2 3x}{\sinh^2 3x}\,dx = \int \frac{1+\sinh^2 3x}{\sinh^2 3x}\,dx$

$= \int \frac{1}{\sinh^2 3x}\,dx + \int dx$

$= \int \text{cosech}^2 3x\,dx + \int dx$

$= -\frac{1}{3}\coth 3x + x + c$

d $\int \sinh^2 x \cosh^2 x\,dx = \int \sinh x \cosh x \sinh x \cosh x\,dx$

$= \frac{1}{4}\int (2\sinh x\cosh x)(2\sinh x\cosh x)\,dx$

$= \frac{1}{4}\int (\sinh 2x)^2\,dx$

$= \frac{1}{4}\int \left(\frac{\cosh 4x - 1}{2}\right)dx$

$= \frac{1}{8}\int \cosh 4x\,dx - \frac{1}{8}\int dx$

$= \frac{1}{32}\sinh 4x - \frac{1}{8}x + c$

e $\int \cosh^5 x\,dx = \int (1+\sinh^2 x)(1+\sinh^2 x)\cosh x\,dx$

$= \int (1 + 2\sinh^2 x + \sinh^4 x)\cosh x\,dx$

$= \int (\cosh x + 2\sinh^2 x\cosh x + \sinh^4 x\cosh x)\,dx$

$= \int \cosh x\,dx + 2\int (\sinh^2 x\cosh x)\,dx + \int (\sinh^4 x\cosh x)\,dx$

Let $f(x) = \sinh x \Rightarrow f'(x) = \cosh x$

Using $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ gives:

$\int \cosh^5 x\,dx = \sinh x + \frac{2}{3}\sinh^3 x + \frac{1}{5}\sinh^5 x + c$

8 f $\int \tanh^3 2x\,dx = \int \tanh^2 2x \tanh 2x\,dx$

$= \int (1-\text{sech}^2 2x)\tanh 2x\,dx$

$= \int (\tanh 2x - \tanh 2x\ \text{sech}^2 2x)\ dx$

$= \int \left(\tanh 2x - \frac{1}{2}(\tanh 2x)(2\text{sech}^2 2x)\right)\ dx$

$= \int \tanh 2x\,dx - \frac{1}{2}\int (\tanh 2x)(2\text{sech}^2 2x)\,dx$

Let $f(x) = \tanh 2x \Rightarrow f'(x) = 2\,\text{sech}^2 2x$

Using $\int [f(x)]^n f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$ gives:

$\int \tanh^3 2x\,dx = \frac{1}{2}\ln(\cosh 2x) - \frac{1}{4}\tanh^2 2x + c$

Alternative solutions may give $\int \tanh^3 2x\,dx = \frac{1}{2}\ln(\cosh 2x) + \frac{1}{4}\text{sech}^2 2x + c$

Note that the two solutions are equivalent, as $\tanh^2 x + \text{sech}^2 x + c$ and c is an arbitrary constant.

9 $\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right) dx$

Let $u = \frac{x}{2}$

$\cosh^2 u = \left(\frac{e^u + e^u}{2}\right)^2$

$= \frac{1}{4}(e^{2u} + e^{-2u} + 2)$

$= \frac{1}{4}(e^x + e^{-x} + 2)$

$= \frac{1}{4}e^x + \frac{1}{4}e^{-x} + \frac{1}{2}$

$\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right) dx = \frac{1}{4}\int_0^{\ln 2} e^x\ dx + \frac{1}{4}\int_0^{\ln 2} e^{-x}\ dx + \frac{1}{2}\int_0^{\ln 2} dx$

$= \left[\frac{1}{4}e^x - \frac{1}{4}e^{-x} + \frac{1}{2}x\right]_0^{\ln 2}$

$= \left(\frac{1}{4}e^{\ln 2} - \frac{1}{4}e^{-\ln 2} + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4}e^0 - \frac{1}{4}e^0 + \frac{1}{2}(0)\right)$

$= \frac{1}{4}\times 2 - \frac{1}{4}\times\frac{1}{2} + \frac{1}{2}\ln 2$

$= \frac{3}{8} + \frac{1}{2}\times\frac{1}{4}\times 4\ln 2$

$= \frac{3}{8} + \frac{1}{8}\ln 16$

$= \frac{1}{8}(3 + \ln 16)$ as required

10 $y = \sinh x$

$$= \frac{e^x - e^{-x}}{2}$$

$$V = \pi\int_a^b y^2 \, dx$$

$$= \pi\int_0^1 \left(\frac{e^x - e^{-x}}{2}\right)^2 dx$$

$$= \frac{\pi}{4}\int_0^1 \left(e^{2x} + e^{-2x} - 2\right) dx$$

$$= \frac{\pi}{4}\left[\frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} - 2x\right]_0^1$$

$$= \frac{\pi}{4}\left[\left(\frac{1}{2}e^2 - \frac{1}{2}e^{-2} - 2(1)^2\right) - \left(\frac{1}{2}e^0 - \frac{1}{2}e^0 - 2(0)^2\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{1}{2}e^2 - \frac{1}{2}e^{-2} - 2\right)$$

$$= \frac{\pi}{8}\left(e^2 - e^{-2} - 4\right)$$

$$= \frac{\pi}{8e^2}\left(e^4 - 4e^2 - 1\right) \text{ as required}$$

11 a $\int \frac{2}{\cosh x} dx = 2\int \operatorname{sech} x \, dx$

$$= 2\left[2\arctan\left(e^x\right)\right] + c$$

$$= 4\arctan\left(e^x\right) + c$$

b $\int \operatorname{sech} 2x \, dx = \frac{1}{2}\left[2\arctan\left(e^{2x}\right)\right] + c$

$$= \arctan\left(e^{2x}\right) + c$$

c $\int \sqrt{1 - \tanh^2\left(\frac{x}{2}\right)} \, dx = \int \sqrt{\operatorname{sech}^2\left(\frac{x}{2}\right)} \, dx$

$$= \int \operatorname{sech}\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{\left(\frac{1}{2}\right)} \times 2\arctan\left(e^{\frac{x}{2}}\right) + c$$

$$= 4\arctan\left(e^{\frac{x}{2}}\right) + c$$

12 a $\int x\cosh^2(x^2)\,\mathrm{d}x$

Let $u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}u = 2x\ \mathrm{d}x$

$$\int x\cosh^2(x^2)\,\mathrm{d}x = \frac{1}{2}\int 2x\cosh^2(x^2)\,\mathrm{d}x$$
$$= \frac{1}{2}\int \cosh^2 u\,\mathrm{d}u$$
$$= \frac{1}{2}\int\left(\frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}\right)^2\mathrm{d}u$$
$$= \frac{1}{8}\int\left(\mathrm{e}^{2u} + \mathrm{e}^{-2u} + 2\right)\mathrm{d}u$$
$$= \frac{1}{16}\mathrm{e}^{2u} - \frac{1}{16}\mathrm{e}^{-2u} + \frac{1}{4}u + c$$
$$= \frac{1}{16}\left(\mathrm{e}^{2u} - \mathrm{e}^{-2u}\right) + \frac{1}{4}u + c$$
$$= \frac{1}{8}\left(\frac{\mathrm{e}^{2u} - \mathrm{e}^{-2u}}{2}\right) + \frac{1}{4}u + c$$
$$= \frac{1}{8}\sinh 2u + \frac{1}{4}u + c$$
$$= \frac{1}{8}\sinh 2x^2 + \frac{1}{4}x^2 + c$$

b $\int \frac{x}{\cosh^2(x^2)}\,\mathrm{d}x = \int x\,\mathrm{sech}^2(x^2)\,\mathrm{d}x$

Let $u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$

$$\int x\,\mathrm{sech}^2(x^2)\,\mathrm{d}x = \frac{1}{2}\int 2x\,\mathrm{sech}^2(x^2)\,\mathrm{d}x$$
$$= \frac{1}{2}\int \mathrm{sech}^2 u\,\mathrm{d}u$$
$$= \frac{1}{2}\tanh u + c$$
$$= \frac{1}{2}\tanh x^2 + c$$