Chapter review 3

1

$$y = \cosh 2x$$
$$\frac{dy}{dx} = 2\sinh 2x$$

2 a $y = \text{arsinh}\, 3x$

Let $t = 3x \quad y = \text{arsinh}\, t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 3$$
$$= \frac{3}{\sqrt{9x^2+1}}$$

b $y = \text{arsinh}\, x^2$

Let $t = x^2 \quad y = \text{arsinh}\, t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$
$$\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}} \times 2x$$
$$= \frac{2x}{\sqrt{x^4+1}}$$

c

$$y = \text{arcosh}\, \frac{x}{2}$$

Let $t = \frac{x}{2} \quad y = \text{arcosh}\, t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2-1}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}} \times \frac{1}{2}$$
$$= \frac{1}{2\sqrt{\frac{x^2}{4}-1}} = \frac{1}{\sqrt{x^2-4}}$$

d $y = x^2\, \text{arcosh}\, 2x$

$$\frac{dy}{dx} = 2x\, \text{arcosh}\, 2x + x^2 \times \frac{2}{\sqrt{4x^2-1}}$$
$$= 2x\left(\text{arcosh}\, 2x + \frac{x}{\sqrt{4x^2-1}}\right)$$

3 Let $y = \arctan x$

$\tan y = x$

$\sec^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$= \frac{1}{1 + \tan^2 y}$

$= \frac{1}{1 + x^2}$ as required

4 $y = (\text{arsinh}x)^2$

Let $z = \text{arsinh}x$

$\sinh z = x$

$\cosh z \frac{dz}{dx} = 1$

$\frac{dz}{dx} = \frac{1}{\cosh z}$

$= \frac{1}{\sqrt{1 + \sinh^2 z}}$

$= \frac{1}{\sqrt{1 + x^2}}$

$\frac{dy}{dx} = 2\text{arsinh}x \frac{d}{dx}(\text{arsinh}x)$

$= \frac{2\text{arsinh}x}{\sqrt{1 + x^2}}$

$$\frac{d^2y}{dx^2} = \frac{(1 + x^2)^{\frac{1}{2}} \frac{d}{dx}(2\text{arsinh}x) - 2\text{arsinh}x \frac{d}{dx}(1 + x^2)^{\frac{1}{2}}}{\left(\sqrt{1 + x^2}\right)^2}$$

$$= \frac{(1 + x^2)^{\frac{1}{2}} \left(\frac{2}{(1 + x^2)^{\frac{1}{2}}}\right) - 2x(1 + x^2)^{-\frac{1}{2}} \text{arsinh}x}{1 + x^2}$$

$$= \frac{2 - \frac{2x\text{arsinh}x}{(1 + x^2)^{\frac{1}{2}}}}{1 + x^2}$$

$$= \frac{2(1 + x^2)^{\frac{1}{2}} - 2x\text{arsinh}x}{(1 + x^2)^{\frac{3}{2}}}$$

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}-2$$

$$=\left(1+x^2\right)\left(\frac{2\left(1+x^2\right)^{\frac{1}{2}}-2x\text{ arsinh }x}{\left(1+x^2\right)^{\frac{3}{2}}}\right)+x\left(\frac{2\text{ arsinh }x}{\left(1+x^2\right)^{\frac{1}{2}}}\right)-2$$

$$=\frac{2\left(1+x^2\right)^{\frac{1}{2}}-2x\text{arsinh}x}{\left(1+x^2\right)^{\frac{1}{2}}}+\frac{2x\text{arsinh}x}{\left(1+x^2\right)^{\frac{1}{2}}}-2$$

$$=\frac{2\left(1+x^2\right)^{\frac{1}{2}}-2x\text{arsinh}x+2x\text{arsinh}x}{\left(1+x^2\right)^{\frac{1}{2}}}-2$$

$$=\frac{2\left(1+x^2\right)^{\frac{1}{2}}}{\left(1+x^2\right)^{\frac{1}{2}}}-2$$

$$=2-2=0$$

As required

5 a $y = 5\cosh x - 3\sinh x$

$\frac{dy}{dx} = 5\sinh x - 3\cosh x$

5 b At the turning points $\frac{dy}{dx} = 0$, therefore:

$5\sinh x - 3\cosh x = 0$

$5\sinh x = 3\cosh x$

$\frac{\sinh x}{\cosh x} = \frac{3}{5}$

$\tanh x = \frac{3}{5}$

$\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{3}{5}$

$5e^{2x} - 5 = 3e^{2x} + 3$

$e^{2x} = 4$

$2x = \ln 4$

$x = \frac{1}{2}\ln 4$

$x = \ln 2$

$y = 5\cosh x - 3\sinh x$

$= \frac{5(e^x + e^{-x})}{2} - \frac{3(e^x - e^{-x})}{2}$

$= e^x + 4e^{-x}$

When $x = \ln 2$

$y = e^{\ln 2} + 4e^{-\ln 2}$

$= e^{\ln 2} + 4e^{\ln \frac{1}{2}}$

$= 2 + 2$

$= 4$

Therefore, there is a turning point at (ln 2, 4)

$\frac{d^2y}{dx^2} = 5\cosh x - 3\sinh x$

$= \frac{5(e^x + e^{-x})}{2} - \frac{3(e^x - e^{-x})}{2}$

$= e^x + 4e^{-x}$

when $x = \ln 2$

$\frac{d^2y}{dx^2} = 4$

Therefore, there is a minimum at (ln 2, 4)

6 $y = (\arcsin x)^2$

Let $z = \arcsin x$

$\sin z = x$

$\cos z \dfrac{dz}{dx} = 2$

$\dfrac{dz}{dx} = \dfrac{1}{\cos z}$

$= \dfrac{1}{\sqrt{1-\sin^2 z}}$

$= \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{dy}{dx} = 2\arcsin x \dfrac{d}{dx}(\arcsin x)$

$= \dfrac{2\arcsin x}{\sqrt{1-x^2}}$

$\dfrac{dy}{dx} = 2\arcsin x\left(1-x^2\right)^{-\frac{1}{2}}$

$\dfrac{d^2y}{dx^2} = 2\arcsin x \dfrac{d}{dx}\left(1-x^2\right)^{-\frac{1}{2}} + \left(1-x^2\right)^{-\frac{1}{2}} \dfrac{d}{dx}\left(2\arcsin x\right)$

$= 2x\arcsin x\left(1-x^2\right)^{-\frac{3}{2}} + \left(1-x^2\right)^{-\frac{1}{2}} \dfrac{2}{\left(1-x^2\right)^{\frac{1}{2}}}$

$= \dfrac{2x\arcsin x}{\left(1-x^2\right)^{\frac{3}{2}}} + \dfrac{2}{1-x^2}$

$= \dfrac{2x\arcsin x + 2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^{\frac{3}{2}}}$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2$$

$$=\left(1-x^2\right)\left(\frac{2x\arcsin x+2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^{\frac{3}{2}}}\right)-x\left(\frac{2\arcsin x}{\left(1-x^2\right)^{\frac{1}{2}}}\right)-2$$

$$=\frac{2x\arcsin x+2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^{\frac{1}{2}}}-\frac{2x\arcsin x}{\left(1-x^2\right)^{\frac{1}{2}}}-2$$

$$=\frac{2x\arcsin x+2\left(1-x^2\right)^{\frac{1}{2}}-2x\arcsin x}{\left(1-x^2\right)^{\frac{1}{2}}}-2$$

$$=\frac{2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^{\frac{1}{2}}}-2=0\text{, as required}$$

7

$y=\operatorname{arcosh}(\sinh 2x)$

Let $t=\sinh 2x \quad y=\operatorname{arcosh} t$

$$\frac{dt}{dx}=2\cosh 2x \quad \frac{dy}{dt}=\frac{1}{\sqrt{t^2-1}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{t^2-1}}\times 2\cosh 2x$$

$$=\frac{2\cosh 2x}{\sqrt{\sinh^2 2x-1}}$$

8 $y = x - \arctan x$

Let $z = \arctan x$

$\tan z = x$

$\sec^2 y \dfrac{dz}{dx} = 1$

$$\frac{dz}{dx} = \frac{1}{\sec^2 z}$$

$$= \frac{1}{1+\tan^2 z}$$

$$= \frac{1}{1+x^2}$$

$y = x - \arctan x$

$$\frac{dy}{dx} = 1 - \frac{1}{1+x^2}$$

$$= 1 - \left(1+x^2\right)^{-1}$$

$$\frac{d^2y}{dx^2} = 2x\left(1+x^2\right)^{-2}$$

$$= \frac{2x}{\left(1+x^2\right)^2}$$

$$2x\left(1-\frac{dy}{dx}\right)^2 = 2x\left(1-\left(1-\frac{1}{1+x^2}\right)\right)^2$$

$$= 2x\left(\frac{1}{1+x^2}\right)^2$$

$$= \frac{2x}{\left(1+x^2\right)^2}$$

$$= \frac{d^2y}{dx^2}$$

As required

9 Let $y = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$\sin y = \frac{x}{\sqrt{1+x^2}}$$

$$\cos y \frac{dy}{dx} = \frac{(1+x^2)^{\frac{1}{2}}\frac{d}{dx}(x) - x\frac{d}{dx}(1+x^2)^{\frac{1}{2}}}{\left((1+x^2)^{\frac{1}{2}}\right)^2}$$

$$\cos y \frac{dy}{dx} = \frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2}$$

$$= \frac{(1+x^2)^{\frac{1}{2}} - \frac{x^2}{(1+x^2)^{\frac{1}{2}}}}{1+x^2}$$

$$= \frac{(1+x^2) - x^2}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{\cos y(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{(1-\sin^2 y)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{\left(1-\frac{x^2}{1+x^2}\right)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{\left(\frac{1+x^2-x^2}{1+x^2}\right)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{\left(\frac{1}{1+x^2}\right)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{(1+x^2)^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{1+x^2}$$

10 $y = \operatorname{sech} x$

$$\frac{dy}{dx} = -\tanh x \operatorname{sech} x$$

$$\frac{d^2y}{dx^2} = -\tanh x \frac{d}{dx}(\operatorname{sech} x) + \operatorname{sech} x \frac{d}{dx}(-\tanh x)$$

$$= \tanh^2 x \operatorname{sech} x - \operatorname{sech}^3 x$$

$$= \operatorname{sech} x\left(\tanh^2 x - \operatorname{sech}^2 x\right)$$

At $\frac{d^2y}{dx^2} = 0$

$$\operatorname{sech} x\left(\tanh^2 x - \operatorname{sech}^2 x\right) = 0$$

$\operatorname{sech} x = 0$ or $\tanh^2 x - \operatorname{sech}^2 x = 0$

$\operatorname{sech} x \neq 0$

$$\tanh^2 x - \operatorname{sech}^2 x = 0$$

$$\tanh^2 x = \operatorname{sech}^2 x$$

$$\frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\sinh^2 x = 1$$

$$\sinh x = \pm 1$$

$$\frac{e^x - e^{-x}}{2} = \pm 1$$

$$e^x - e^{-x} = \pm 2$$

$$e^{2x} - 1 = \pm 2e^x$$

Therefore:

$e^{2x} + 2e^x - 1 = 0$ **(1)**

and

$e^{2x} - 2e^x - 1 = 0$ **(2)**

From **(1)**

$$e^x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= -1 \pm \sqrt{2}$$

$$e^x \neq -1 - \sqrt{2}$$

Therefore:

$$e^x = -1 + \sqrt{2}$$

$$x = \ln\left(-1 + \sqrt{2}\right)$$

From (**2**)

$$e^x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= 1 \pm \sqrt{2}$$

$$e^x \neq 1 - \sqrt{2}$$

Therefore:

$$e^x = 1 + \sqrt{2}$$

$$x = \ln\left(1 + \sqrt{2}\right)$$

Now $-1+\sqrt{2} = \dfrac{(-1+\sqrt{2})(1+\sqrt{2})}{1+\sqrt{2}} = \dfrac{1}{1+\sqrt{2}}$,

So $x = \ln(-1+\sqrt{2}) = \ln\left(\dfrac{1}{1+\sqrt{2}}\right) = -\ln(1+\sqrt{2})$

And $x = \pm\ln(p)$ as required, for $p + 1 + \sqrt{2}$

11 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$b^2x^2-a^2y^2=a^2b^2$

$2xb^2-2ya^2\frac{dy}{dx}=0$

$ya^2\frac{dy}{dx}=xb^2$

$\frac{dy}{dx}=\frac{xb^2}{ya^2}$

At $(a\cosh q, b\sinh q)$

The tangent has gradient:

$m_T=\frac{ab^2\cosh q}{a^2b\sinh q}$

$=\frac{b\cosh q}{a\sinh q}$

Using $y-y_1=m(x-x_1)$ at $(a\cosh q, b\sinh q)$ with $m_T=\frac{b\cosh q}{a\sinh q}$ gives:

$y-b\sinh q=\frac{b\cosh q}{a\sinh q}(x-a\cosh q)$

$ay\sinh q-ab\sinh^2 q=bx\cosh q-ab\cosh^2 q$

$ay\sinh q-bx\cosh q+ab\cosh^2 q-ab\sinh^2 q=0$

So the tangent has equation:

$ay\sinh q-bx\cosh q+ab=0$

The normal has gradient:

$m_N=-\frac{a\sinh q}{b\cosh q}$

Using $y-y_1=m(x-x_1)$ at $(a\cosh q, b\sinh q)$ with $m_N=-\frac{a\sinh q}{b\cosh q}$ gives:

$y-b\sinh q=-\frac{a\sinh q}{b\cosh q}(x-a\cosh q)$

$by\cosh q-b^2\sinh q\cosh q=-ax\sinh q+a^2\sinh q\cosh q$

$by\cosh q+ax\sinh q-b^2\sinh q\cosh q-a^2\sinh q\cosh q=0$

So the normal has equation:

$by\cosh q+ax\sinh q-(a^2+b^2)\sinh q\cosh q=0$