Exercise 3C

1 $y = \arccos x$

$\cos y = x$

$-\sin y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = -\dfrac{1}{\sin y}$

$= -\dfrac{1}{\sqrt{1-\cos^2 y}}$

$= -\dfrac{1}{\sqrt{1-x^2}}$ as required

2 a Let $y = \arccos 2x$

$\cos y = 2x$

$-\sin y \dfrac{dy}{dx} = 2$

$\dfrac{dy}{dx} = -\dfrac{2}{\sin y}$

$= -\dfrac{2}{\sqrt{1-\cos^2 y}}$

$= -\dfrac{2}{\sqrt{1-4x^2}}$

b Let $y = \arctan\left(\dfrac{x}{2}\right)$

$\tan y = \dfrac{x}{2}$

$\sec^2 y \dfrac{dy}{dx} = \dfrac{1}{2}$

$\dfrac{dy}{dx} = \dfrac{1}{2\sec^2 y}$

$= \dfrac{1}{2(1+\tan^2 y)}$

$= \dfrac{1}{2\left(1+\dfrac{x^2}{4}\right)}$

$= \dfrac{2}{4+x^2}$

2 c Let $y = \arcsin 3x$

$\sin y = 3x$

$\cos y \frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{\cos y}$

$= \frac{3}{\sqrt{1-\sin^2 y}}$

$= \frac{3}{\sqrt{1-9x^2}}$

d Let $y = \operatorname{arccot} x$

$\cot y = x$

$\frac{1}{\tan y} = x$

$\tan y = \frac{1}{x}$

$\sec^2 y \frac{dy}{dx} = -\frac{1}{x^2}$

$\frac{dy}{dx} = -\frac{1}{x^2\left(\sec^2 y\right)}$

$= -\frac{1}{x^2\left(1+\tan^2 y\right)}$

$= -\frac{1}{x^2\left(1+\frac{1}{x^2}\right)}$

$= -\frac{1}{x^2+1}$

2 **e** Let $y = \text{arcsec}\, x$

$\sec y = x$

$\frac{1}{\cos y} = x$

$\cos y = \frac{1}{x}$

$-\sin y \frac{dy}{dx} = -\frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{1}{x^2 \sin y}$$

$$= \frac{1}{x^2\sqrt{1-\cos^2 y}}$$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

f Let $y = \text{arccosec}\, x$

$\text{cosec}\, y = x$

$\frac{1}{\sin y} = x$

$\sin y = \frac{1}{x}$

$\cos y \frac{dy}{dx} = -\frac{1}{x^2}$

$$\frac{dy}{dx} = -\frac{1}{x^2 \cos y}$$

$$= -\frac{1}{x^2\sqrt{1-\sin^2 y}}$$

$$= -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$= -\frac{1}{x\sqrt{x^2-1}}$$

2 g Let $y = \arcsin\left(\dfrac{x}{x-1}\right)$

$$\sin y = 1 + \frac{1}{x-1}$$

$$\cos y \frac{dy}{dx} = -\frac{1}{(x-1)^2}$$

$$\frac{dy}{dx} = -\frac{1}{(x-1)^2 \cos y}$$

$$= -\frac{1}{(x-1)^2\sqrt{1-\sin^2 y}}$$

$$= -\frac{1}{(x-1)^2\sqrt{1-\left(\frac{x}{x-1}\right)^2}}$$

$$= -\frac{1}{(x-1)^2\sqrt{1-\frac{x^2}{(x-1)^2}}}$$

$$= -\frac{1}{(x-1)^2\sqrt{\frac{(x-1)^2-x^2}{(x-1)^2}}}$$

$$= -\frac{(x-1)}{(x-1)^2\sqrt{(x-1)^2-x^2}}$$

$$= -\frac{1}{(x-1)\sqrt{(x^2-2x+1)-x^2}}$$

$$= -\frac{1}{(x-1)\sqrt{1-2x}}$$

2 h Let $y = \arccos x^2$

$\cos y = x^2$

$-\sin y \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = -\frac{2x}{\sin y}$

$= -\frac{2x}{\sqrt{1-\cos^2 y}}$

$= -\frac{2x}{\sqrt{1-x^4}}$

i Let $y = e^x \arccos x$

$\frac{dy}{dx} = e^x \frac{d}{dx}(\arccos x) + \arccos x \frac{d}{dx}(e^x)$

From question 1:

$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

Therefore:

$\frac{dy}{dx} = -\frac{e^x}{\sqrt{1-x^2}} + e^x \arccos x$

$= e^x \left(\arccos x - \frac{1}{\sqrt{1-x^2}} \right)$

2 j Let $y = \arcsin x \cos x$

$$\frac{dy}{dx} = \cos x \frac{d}{dx}(\arcsin x) + \arcsin x \frac{d}{dx}(\cos x)$$

Let $z = \arcsin x$

$\sin z = x$

$$\cos z \frac{dz}{dx} = 1$$

$$\frac{dz}{dx} = \frac{1}{\cos z}$$

$$= \frac{1}{\sqrt{1-\sin^2 z}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Therefore:

$$\frac{dy}{dx} = \frac{\cos x}{\sqrt{1-x^2}} - \arcsin x \sin x$$

k Let $y = x^2 \arccos x$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\arccos x) + \arccos x \frac{d}{dx}(x^2)$$

From question 1:

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

Therefore:

$$\frac{dy}{dx} = -\frac{x^2}{\sqrt{1-x^2}} + 2x \arccos x$$

$$= x\left(2 \arccos x - \frac{x}{\sqrt{1-x^2}}\right)$$

2 l Let $y = e^{\arctan x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\arctan x)e^{\arctan x}$$

Let $z = \arctan x$

$\tan z = x$

$$\sec^2 z \frac{dz}{dx} = 1$$

$$\frac{dz}{dx} = \frac{1}{\sec^2 z}$$

$$= \frac{1}{1+\tan^2 z}$$

$$= \frac{1}{1+x^2}$$

Therefore:

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

3 $\tan y = x \arctan x$

Let $z = \arctan x$

$\tan z = x$

$$\sec^2 z \frac{dz}{dx} = 1$$

$$\frac{dz}{dx} = \frac{1}{\sec^2 z}$$

$$= \frac{1}{1+\tan^2 z}$$

$$= \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x \arctan x)$$

$$\sec^2 y \frac{dy}{dx} = \arctan x + \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(\arctan x + \frac{x}{1+x^2}\right)$$

$$= \frac{1}{1+\tan^2 y}\left(\arctan x + \frac{x}{1+x^2}\right)$$

$\tan y = x \arctan x \Rightarrow \tan^2 y = x^2 (\arctan x)^2$

Therefore:

$$\frac{dy}{dx} = \frac{1}{1+x^2(\arctan x)^2}\left(\arctan x + \frac{x}{1+x^2}\right)$$

4 $y = \arcsin x$

$\sin y = x$

$\cos y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$= \dfrac{1}{\sqrt{1-\sin^2 y}}$

$= \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{dy}{dx} = (1-x^2)^{-\frac{1}{2}}$

$\dfrac{d^2y}{dx^2} = -\dfrac{1}{2}(-2x)(1-x^2)^{-\frac{3}{2}}$

$= \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$

$(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx}$ **(1)**

Substituting into **(1)** gives:

$(1-x^2)\left(\dfrac{x}{(1-x^2)^{\frac{3}{2}}}\right) - x\left(\dfrac{1}{(1-x^2)^{\frac{1}{2}}}\right)$

$= \dfrac{x}{(1-x^2)^{\frac{1}{2}}} - \dfrac{x}{(1-x^2)^{\frac{1}{2}}} = 0$, as required

5 $y = \arcsin 2x$

$\sin y = 2x$

$\cos y \frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{2}{\cos y}$

$= \frac{2}{\sqrt{1-\sin^2 y}}$

$= \frac{2}{\sqrt{1-4x^2}}$

When $x = \frac{1}{4}$

$y = \arcsin\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

and

$\frac{dy}{dx} = \frac{2}{\sqrt{1-4\left(\frac{1}{4}\right)^2}}$

$= \frac{4}{\sqrt{3}}$

Using $y - y_1 = m(x - x_1)$ at $\left(\frac{1}{4}, \frac{\pi}{6}\right)$ with $m = \frac{4}{\sqrt{3}}$ gives:

$y - \frac{\pi}{6} = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$

$\sqrt{3}y - \frac{\sqrt{3}\pi}{6} = 4x - 1$

Challenge

$y = \cos x \cosh x$

$\frac{dy}{dx} = \cos x \sinh x - \sin x \cosh x$

$\frac{d^2y}{dx^2} = \cos x \cosh x - \sin x \sinh x - \sin x \sinh x - \cos x \cosh x$

$= -2\sin x \sinh x$

$\frac{d^3y}{dx^3} = -2\sin x \cosh x - 2\cos x \sinh x$

$\frac{d^4y}{dx^4} = -2\sin x \sinh x - 2\cos x \cosh x - 2\cos x \cosh x + 2\sin x \sinh x$

$= -4\cos x \cosh x$

Since $y = \cos x \cosh x$

$\frac{d^4y}{dx^4} = -4y$ as required