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GCE

Mathematics

Advanced GCE

Unit **4727:** Further Pure Mathematics 3

Mark Scheme for June 2011

Oxford Cambridge and RSA Examinations

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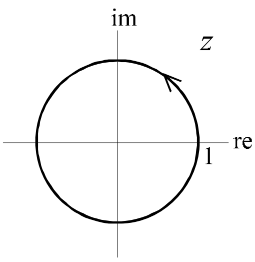
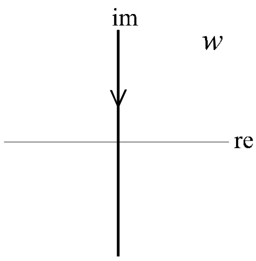
1 (i)	$\theta = \sin^{-1}\dfrac{\lvert[5,\,6,\,-7]\,.\,[1,\,2,\,-1]\rvert}{\sqrt{5^2+6^2+(-7)^2}\sqrt{1^2+2^2+(-1)^2}}$	M1*	For using scalar product of line and plane vectors
		M1 (*dep)	For both moduli seen
	$\theta = \sin^{-1}\dfrac{24}{\sqrt{110}\sqrt{6}} = 69.1^\circ\ (69.099...^\circ,\ 1.206)$	A1	For correct scalar product
		A1 **4**	For correct angle
	$\phi = \sin^{-1}\dfrac{\lvert[5,\,6,\,-7]\times[1,\,2,\,-1]\rvert}{\sqrt{5^2+6^2+(-7)^2}\sqrt{1^2+2^2+(-1)^2}}$	**SR** M1*	For vector product of line and plane vectors AND finding modulus of result
		M1 (*dep)	For moduli of line and plane vectors seen
	$\phi = \sin^{-1}\dfrac{\sqrt{84}}{\sqrt{110}\sqrt{6}} = 20.9^\circ \Rightarrow \theta = 69.1^\circ$	A1	For correct modulus $\sqrt{84}$
		A1	For correct angle
(ii)	METHOD 1		
	$d = \dfrac{\lvert 1+12+3-40\rvert}{\sqrt{1^2+2^2+(-1)^2}} = \dfrac{24}{\sqrt{6}} = 4\sqrt{6} \approx 9.80$	M1	For use of correct formula
		A1 **2**	For correct distance
	METHOD 2		
	$(1+\lambda)+2(6+2\lambda)-(-3-\lambda)=40$	M1	For substituting parametric form into plane
	$\Rightarrow \lambda = 4 \Rightarrow d = 4\sqrt{6}$	A1	For correct distance
	OR distance from $(1,\,6,\,-3)$ to $(5,\,14,\,-7)$		
	$=\sqrt{4^2+8^2+(-4)^2}=\sqrt{96}$		
	METHOD 3		
	Plane through $(1,\,6,\,-3)$ parallel to p is	M1	For finding parallel plane through $(1,\,6,\,-3)$
	$x+2y-z=16 \Rightarrow d = \dfrac{40-16}{\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 4		
	e.g. $(0,\,0,\,-40)$ on p	M1	For using any point on p to find vector and scalar product seen e.g. $[1,\,6,\,37]\,.\,[1,\,2,\,-1]$
	$\Rightarrow$ vector to $(1,\,6,\,-3)$ $= \pm(1,\,6,\,37)$		
	$d = \dfrac{\lvert[1,\,6,\,37]\,.\,[1,\,2,\,-1]\rvert}{\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 5		
	l meets p where $(1+5t)+2(6+6t)-(-3-7t)=40$		For finding t where l meets p
	$\Rightarrow t=1 \Rightarrow d = \lvert[5,\,6,\,-7]\rvert\sin\theta$	M1	and linking d with triangle
	$\Rightarrow d = \sqrt{110}\dfrac{24}{\sqrt{110}\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
		$\boxed{6}$	
2 (i)	METHOD 1		
	EITHER $\dfrac{1+e^{i\theta}}{1-e^{i\theta}} = \dfrac{e^{-\frac{1}{2}i\theta}+e^{\frac{1}{2}i\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}}$	M1	*EITHER* For changing LHS terms to $e^{\pm\frac{1}{2}i\theta}$
			OR in reverse For using $\cot\frac{1}{2}\theta = \dfrac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$
	$= \dfrac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta} = i\cot\frac{1}{2}\theta$	M1	For either of $\begin{matrix}\cos\\ \sin\end{matrix}\frac{1}{2}\theta = \dfrac{e^{\frac{1}{2}i\theta}\pm e^{-\frac{1}{2}i\theta}}{(2)(i)}$ soi
	OR in reverse with similar working	A1 **3**	For fully correct proof to **AG** **SR** If factors of 2 or i are not clearly seen, award M1 M1 A0

2 (i)	METHOD 2			
	EITHER $\dfrac{1+e^{i\theta}}{1-e^{i\theta}}\times\dfrac{1-e^{-i\theta}}{1-e^{-i\theta}}=\dfrac{e^{i\theta}-e^{-i\theta}}{2-\left(e^{i\theta}+e^{-i\theta}\right)}$ *OR* $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\times\dfrac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$	M1		For multiplying top and bottom by complex conjugate in exp or trig form
	$=\dfrac{2i\sin\theta}{2-2\cos\theta}=\dfrac{2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}=i\cot\frac{1}{2}\theta$	M1 A1		For using both double angle formulae correctly For fully correct proof to **AG**
	METHOD 3			
	$\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}=\dfrac{2\cos^2\frac{1}{2}\theta+2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta-2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$	M1		For using both double angle formulae correctly
	$=\dfrac{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}$	M1		For appropriate factorisation
	$=i\cot\frac{1}{2}\theta\dfrac{\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}{\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}=i\cot\frac{1}{2}\theta$	A1		For fully correct proof to **AG**
	METHOD 4			
	$\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}=\dfrac{1+\dfrac{1-t^2}{1+t^2}+i\dfrac{2t}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}-i\dfrac{2t}{1+t^2}}$	M1		For substituting both t formulae correctly
	$=\dfrac{2+2it}{2t^2-2it}=\dfrac{1}{t}\dfrac{1+it}{t-i}=\dfrac{i}{t}\dfrac{t-i}{t-i}=i\cot\frac{1}{2}\theta$	M1 A1		For appropriate factorisation For fully correct proof to **AG**
	METHOD 5			
	$\dfrac{1+e^{i\theta}}{1-e^{i\theta}}\times\dfrac{1+e^{i\theta}}{1+e^{i\theta}}=\dfrac{1+2e^{i\theta}+e^{2i\theta}}{1-e^{2i\theta}}$ $=\dfrac{2+e^{i\theta}+e^{-i\theta}}{e^{-i\theta}-e^{i\theta}}$	M1		For multiplying top and bottom by $1+e^{i\theta}$ and attempting to divide by $e^{i\theta}$ *OR* multiplying top and bottom by $1+e^{-i\theta}$
	$=\dfrac{2(1+\cos\theta)}{-2i\sin\theta}=\dfrac{2\cos^2\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=\dfrac{\cos\frac{1}{2}\theta}{-i\sin\frac{1}{2}\theta}$	M1		For using both double angle formulae correctly
	$=i\cot\frac{1}{2}\theta$	A1	**3**	For fully correct proof to **AG**
(ii)	 	M1 A1 B1	 **3**	For a circle centre O For indication of radius = 1 and anticlockwise arrow shown For locus of w shown as imaginary axis described downwards
			6	

3	**(i)**	METHOD 1 $m+4\ (=0) \Rightarrow$ CF $(y=)Ae^{-4x}$	M1 A1 **2**	For correct auxiliary equation (soi) For correct CF
		METHOD 2 Separating variables on $\frac{dy}{dx}+4y=0$ $\Rightarrow \ln y=-4x$ $\Rightarrow$ CF $(y=)Ae^{-4x}$	M1 A1	For integration to this stage For correct CF
	(ii)	PI $(y=)\ p\cos 3x+q\sin 3x$ $y'=-3p\sin 3x+3q\cos 3x$ $\Rightarrow (-3p+4q)\sin 3x+(4p+3q)\cos 3x=5\cos 3x$ $\Rightarrow \left.\begin{array}{r}-3p+4q=0\\ 4p+3q=5\end{array}\right\} \Rightarrow p=\frac{4}{5},\ q=\frac{3}{5}$ GS $(y=)\ Ae^{-4x}+\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x$	B1 M1 A1 M1 A1 A1 B1√ **7**	For stating PI of correct form For substituting y and y' into DE For correct equation For equating coeffs and solving For correct value of p, and of q For GS f.t. from their CF+PI with 1 arbitrary constant in CF and none in PI
		SR Integrating factor method may be used, followed by 2-stage integration by parts or $C+\mathrm{i}S$ method Marks for **(i)** are awarded only if CF is clearly identified		
	(iii)	$e^{-4x}\to 0$, $\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x=\frac{\sin}{\cos}(3x+\alpha)$ $\Rightarrow\ -1\leqslant y\leqslant 1\quad OR\quad -1\lessapprox y\lessapprox 1$	M1 A1√ **2**	For considering either term For correct range (allow <) **CWO** f.t. as $-\sqrt{p^2+q^2}\leqslant y\leqslant\sqrt{p^2+q^2}$ from **(ii)**
			11	
4	**(i)**	$abc=(ab)c=(ba)c=b(ac)=$ $b(ca)=(bc)a=(cb)a=cba$ Minimum working: $abc=bac=bca=cba$ $OR\ abc=acb=cab=cba$ $OR\ abc=bac=bca=cba$	M1 A1 **2**	For using commutativity correctly For correct proof (use of associativity may be implied)
	(ii)	$\{e,a\},\{e,b\},\{e,c\},\{e,bc\},\{e,ca\},\{e,ab\},\{e,abc\}$	B1 B1 **2**	For any 5 subgroups For the other 2 subgroups and none incorrect
	(iii)	$\{e,a,b,ab\},\{e,a,c,ca\},\{e,b,c,bc\}$ $\{e,a,bc,abc\},\{e,b,ca,abc\},\{e,c,ab,abc\}$ $\{e,bc,ca,ab\}$	B1 B1 B1 **3**	For any 3 subgroups For 1 more subgroup For 1 more subgroup (5 in total) and none incorrect
	(iv)	All elements $(\neq e)$ have order 2 *OR* all are self-inverse *OR* no element of G has order 4 *OR* no order 4 subgroup has a generator *or* is cyclic *OR* subgroups are of the form $\{e,a,b,ab\}$ (the Klein group) $\Rightarrow$ all order 4 subgroups are isomorphic	B1* B1 (*dep)**2**	For appropriate reference to order of elements in G For correct conclusion
			9	

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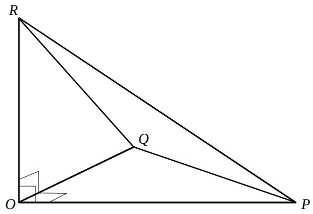
5	**(i)**	$\frac{dy}{dx} = k u^{k-1} \frac{du}{dx}$	M1 A1		For using chain rule For correct $\frac{dy}{dx}$
		$\Rightarrow x k u^{k-1} \frac{du}{dx} + 3u^k = x^2 u^{2k}$	M1		For substituting for y and $\frac{dy}{dx}$
		$\Rightarrow \frac{du}{dx} + \frac{3}{kx} u = \frac{1}{k} x u^{k+1}$	A1	**4**	For correct equation **AG**
	(ii)	$k = -1$	B1	**1**	For correct k
	(iii)	$\frac{du}{dx} - \frac{3}{x} u = -x \Rightarrow$ IF $e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = \frac{1}{x^3}$	B1√		For correct IF f.t. for IF = $x^{\frac{3}{k}}$ using k or their numerical value for k
		$\Rightarrow \frac{d}{dx}\left(u.\frac{1}{x^3}\right) = -\frac{1}{x^2}$	M1		For $\frac{d}{dx}(u.\text{their IF}) = -x.\text{their IF}$
		$\Rightarrow u.\frac{1}{x^3} = \frac{1}{x} (+c) \Rightarrow y = \frac{1}{cx^3 + x^2}$	A1 A1	 **4**	For correct integration both sides For correct solution for y
			$\boxed{9}$		
6	**(a)**	Closure $(ax+b)+(cx+d)=(a+c)x+(b+d)$	B1		For obtaining correct sum from 2 distinct elements
		$\in P$	B1		For stating result is in P *OR* is of the correct form **SR** award this mark if any of the closure result, the identity or the inverse element is stated to be in P *OR* of the correct form
		Identity $0x+0$	B1		For stating identity (allow 0)
		Inverse $-ax-b$	B1	**4**	For stating inverse
	(b) (i)	Order 9	B1*	**1**	For correct order
	(ii)	$x+2$	B1	**1**	For correct inverse element
	(iii)	$(ax+b)+(ax+b)+(ax+b)=3ax+3b$	M1		For considering sums of $ax+b$ and obtaining $3ax+3b$
		$= 0x+0$ $\Rightarrow ax+b$ has order 3 $\forall\, a, b$ (except $a=b=0$)	A1		For equating to $0x+0$ *OR* 0 and obtaining order 3 **SR** For order 3 stated only *OR* found from incomplete consideration of numerical cases award B1
		Cyclic group of order 9 has element(s) of order 9	M1 (*dep)		For reference to element(s) of order 9
		$\Rightarrow (Q, +(\text{mod } 3))$ is not cyclic	A1	**4**	For correct conclusion
			$\boxed{10}$		

7 (i)		B1		For sketch of tetrahedron labelled in some way At least one right angle at *O* must be indicated or clearly implied
		M1		For using $\Delta = \frac{1}{2}\text{base} \times \text{height}$
	$\Delta\, OPQ = \frac{1}{2}pq$, $\Delta\, OQR = \frac{1}{2}qr$, $\Delta\, ORP = \frac{1}{2}rp$	A1	3	For all areas correct **CAO**
(ii)	$\frac{1}{2}\left\|\overrightarrow{RP} \times \overrightarrow{RQ}\right\| = \frac{1}{2}\|\overrightarrow{RP}\|\|\overrightarrow{RQ}\|\sin R = \Delta\, PQR$	B1	1	For correct justification
(iii)	LHS = $\left(\frac{1}{2}pq\right)^2 + \left(\frac{1}{2}qr\right)^2 + \left(\frac{1}{2}rp\right)^2$	B1		For correct expression
	$\Delta\, PQR = \frac{1}{2}\left\|(p\mathbf{i} - q\mathbf{j}) \times (p\mathbf{i} - r\mathbf{k})\right\|$ *OR* $\frac{1}{2}\left\|(p\mathbf{i} - r\mathbf{k}) \times (q\mathbf{j} - r\mathbf{k})\right\|$ *OR* $\frac{1}{2}\left\|(p\mathbf{i} - q\mathbf{j}) \times (q\mathbf{j} - r\mathbf{k})\right\|$	B1		For ΔPQR in vector form
	$\Delta\, PQR = \frac{1}{2}\left\|qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}\right\|$	M1		For finding vector product of their attempt at ΔPQR
		A1		For correct expression
	$\text{RHS} = \frac{1}{4}\left((pq)^2 + (qr)^2 + (rp)^2\right)$	M1		For using $\left\|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\right\| = \sqrt{a^2 + b^2 + c^2}$
		A1	6	For completing proof of **AG WWW**
			$\boxed{10}$	

8 (i)	$\text{Re}(c+\text{i}\,s)^4 = \cos 4\theta = c^4 - 6c^2s^2 + s^4$	M1*		For expanding $(c+\text{i}\,s)^4$: at least 2 terms and 1 binomial coefficient needed
		A1		For 3 correct terms
	$\cos 4\theta = c^4 - 6c^2(1-c^2) + (1-c^2)^2$	M1 (*dep)		For using $s^2 = 1 - c^2$
	$\Rightarrow \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$	A1	**4**	For correct expression for $\cos 4\theta$ **CAO**
(ii)	$\cos 4\theta \cos 2\theta = \left(8c^4 - 8c^2 + 1\right)\left(2c^2 - 1\right)$			For multiplying by $\left(2c^2 - 1\right)$
	$= 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$	B1	**1**	to obtain **AG WWW**
(iii)	$16c^6 - 24c^4 + 10c^2 - 2 = 0$	M1		For factorising sextic
	$\Rightarrow \left(c^2 - 1\right)\left(8c^4 - 4c^2 + 1\right) = 0$			with $(c-1)$, $(c+1)$ *or* $(c^2 - 1)$
	For quartic, $b^2 - 4ac = 16 - 32 < 0$	A1		For justifying no other roots **CWO**
	$\Rightarrow c = \pm 1$ only $\Rightarrow \theta = n\pi$	A1	**3**	For obtaining $\theta = n\pi$ **AG**
				Note that M1 A0 A1 is possible
			SR	For verifying $\theta = n\pi$ by substituting $c = \pm 1$ into $16c^6 - 24c^4 + 10c^2 - 2 = 0$ B1
(iv)	$16c^6 - 24c^4 + 10c^2 = 0$			
	$\Rightarrow c^2\left(8c^4 - 12c^2 + 5\right) = 0$	M1		For factorising sextic with c^2
	For quartic, $b^2 - 4ac = 144 - 160 < 0$	A1		For justifying no other roots **CWO**
	$\Rightarrow \cos\theta = 0$ only	A1	**3**	For correct condition obtained **AG**
				Note that M1 A0 A1 is possible
			SR	For verifying $\cos\theta = 0$ by substituting $c = 0$ into $16c^6 - 24c^4 + 10c^2 = 0$ B1
			SR	For verifying $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$ satisfy $\cos 4\theta \cos 2\theta = -1$ B1
			$\boxed{11}$	

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