4727 Further Pure Mathematics 3

1		$\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\text{i}\right)^{\frac{1}{3}}=\left(\cos\frac{1}{6}\pi+\text{i}\sin\frac{1}{6}\pi\right)^{\frac{1}{3}}$	B1		For $\arg z=\frac{1}{6}\pi$ seen or implied
		$=\cos\frac{1}{18}\pi+\text{i}\sin\frac{1}{18}\pi$,	M1		For dividing $\arg z$ by 3
		$\cos\frac{13}{18}\pi+\text{i}\sin\frac{13}{18}\pi$,	A1		For any one correct root
		$\cos\frac{25}{18}\pi+\text{i}\sin\frac{25}{18}\pi$	A1	**4**	For 2 other roots and no more in range $0 \leqslant \theta < 2\pi$
			4		
2	**(i)**	$\frac{1}{5}\text{e}^{-\frac{1}{3}\pi\text{i}}$	B1	**1**	For stating correct inverse in the form $r\,\text{e}^{\text{i}\theta}$
	(ii)	$r_1\text{e}^{\text{i}\theta}\times r_2\text{e}^{\text{i}\phi}=r_1r_2\,\text{e}^{\text{i}(\theta+\phi)}$	M1		For stating 2 distinct elements multiplied
			A1	**2**	For showing product of correct form
	(iii)	$Z^2=\text{e}^{2\text{i}\gamma}$	B1		For $\text{e}^{2\text{i}\gamma}$ seen or implied
		$\Rightarrow\ \text{e}^{2\text{i}\gamma-2\pi\text{i}}$	B1	**2**	For correct answer. aef
			5		
3	**(i)**	$[6-4\lambda,\,-7+8\lambda,\,-10+7\lambda]$ on p	B1		For point on l seen or implied
		$\Rightarrow 3(6-4\lambda)-4(-7+8\lambda)-2(-10+7\lambda)=8$	M1		For substituting into equation of p
		$\Rightarrow\ \lambda=1\ \Rightarrow\ (2,1,-3)$	A1	**3**	For correct point. Allow position vector
	(ii)	METHOD 1			
		$\mathbf{n}=[-4,8,7]\times[3,-4,-2]$	M1*		For direction of l and normal of p seen
			M1 (*dep)		For attempting to find $\mathbf{n}_1\times\mathbf{n}_2$
		$\mathbf{n}=k[12,13,-8]$	A1		For correct vector
		$(2,1,-3)$ OR $(6,-7,-10)$	M1		For finding scalar product of their point on l with their attempt at **n**, or equivalent
		$\Rightarrow\ 12x+13y-8z=61$	A1	**5**	For correct equation, aef cartesian
		METHOD 2			
		$\mathbf{r}=[2,1,-3]$ *OR* $[6,-7,-10]$ $+\lambda[-4,8,7]+\mu[3,-4,-2]$	M1		For stating eqtn of plane in parametric form (may be implied by next stage), using $[2,1,-3]$ (ft from
			A1√		(**i**)) Or $[6,-7,-10]$, $\mathbf{n}_1$ and $\mathbf{n}_2$ (as above)
		$x=\ 2-4\lambda+3\mu$	M1		For writing as 3 linear equations
		$y=\ 1+8\lambda-4\mu$	M1		For attempting to eliminate λ and μ
		$z=-3+7\lambda-2\mu$			
		$\Rightarrow\ 12x+13y-8z=61$	A1		For correct equation aef cartesian
		METHOD 3			
		$3(6+3\mu)-4(-7-4\mu)-2(-10-2\mu)=8$	M1		For finding foot of perpendicular from point on l to p
		$\Rightarrow\ \mu=-2\ \Rightarrow\ (0,1,-6)$	A1		For correct point or position vector
		From 3 points $(2,1,-3)$, $(6,-7,-10)$, $(0,1,-6)$,			
		n = vector product of 2 of $[2,0,3]$, $[6,-8,-4]$, $[-4,8,7]$	M1		Use vector product of 2 vectors in plane
		$\Rightarrow\ \mathbf{n}=k[12,13,-8]$			
		$(2,1,-3)$ OR $(6,-7,-10)$	M1		For finding scalar product of their point on l with their attempt at **n**, or equivalent
		$\Rightarrow\ 12x+13y-8z=61$	A1		For correct equation aef cartesian
			8		

4 (i)	IF $e^{\int \frac{1}{1-x^2}\,dx} = e^{\frac{1}{2}\ln\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$	M1 A1	**2**	For IF stated or implied. Allow $\pm\int$ and omission of dx For integration and simplification to **AG** (intermediate step must be seen)
(ii)	$\frac{d}{dx}\left(y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right) = (1+x)^{\frac{1}{2}}$	M1*		For multiplying both sides by IF
	$y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{2}{3}(1+x)^{\frac{3}{2}} + c$	M1 A1		For integrating RHS to $k(1+x)^n$ For correct equation (including + c)
	$(0, 2) \Rightarrow 2 = \frac{2}{3} + c \Rightarrow c = \frac{4}{3}$	M1 (*dep) M1 (*dep)		In either order: For substituting $(0, 2)$ into their GS (including $+c$) For dividing solution through by IF, including dividing c or their numerical value for c
	$y = \frac{2}{3}(1+x)(1-x)^{\frac{1}{2}} + \frac{4}{3}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$	A1	**6**	For correct solution aef (even unsimplified) in form $y = f(x)$
		8		
5 (i)	$m^2 - 6m + 9 \ (=0) \Rightarrow m = 3$	M1 A1		For attempting to solve correct auxiliary equation For correct m
	CF = $(A + Bx)e^{3x}$	A1	**3**	For correct CF
(ii)	ke^{3x} and kxe^{3x} both appear in CF	B1	**1**	For correct statement
(iii)	$y = kx^2e^{3x} \Rightarrow y' = 2kxe^{3x} + 3kx^2e^{3x}$	M1 A1		For differentiating kx^2e^{3x} twice For correct y' aef
	$\Rightarrow y'' = 2ke^{3x} + 12kxe^{3x} + 9kx^2e^{3x}$	A1		For correct y'' aef
	$\Rightarrow ke^{3x}\left(2 + 12x + 9x^2 - 12x - 18x^2 + 9x^2\right) = e^{3x}$	M1		For substituting y'', y', y into DE
	$\Rightarrow k = \frac{1}{2}$	A1	**5**	For correct k
		9		

6 (i)	METHOD 1		
	$\mathbf{n}_1 = [1, 1, 0] \times [1, -5, -2]$	M1	For attempting to find vector product of the pair of direction vectors
	$= [-2, 2, -6] = k[1, -1, 3]$	A1	For correct $\mathbf{n}_1$
	Use $(2, 2, 1)$	M1	For substituting a point into equation
	$\Rightarrow \mathbf{r}.[-2, 2, -6] = -6 \Rightarrow \mathbf{r}.[1, -1, 3] = 3$	A1 **4**	For correct equation. aef in this form
	METHOD 2		
	$x = 2 + \lambda + \mu$	M1	For writing as 3 linear equations
	$y = 2 + \lambda - 5\mu$ $z = 1 \quad - 2\mu$	M1	For attempting to eliminate λ and μ
	$\Rightarrow x - y + 3z = 3$	A1	For correct cartesian equation
	$\Rightarrow \mathbf{r}.[1, -1, 3] = 3$	A1	For correct equation. aef in this form
(ii)	For $\mathbf{r} = \mathbf{a} + t\mathbf{b}$		
	METHOD 1		
	$\mathbf{b} = [1, -1, 3] \times [7, 17, -3]$	M1	For attempting to find $\mathbf{n}_1 \times \mathbf{n}_2$
	$= k[2, -1, -1]$	A1√	For a correct vector. ft from $\mathbf{n}_1$ in **(i)**
	e.g. x, y or $z = 0$ in $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$	M1	For attempting to find a point on the line
	$\Rightarrow \mathbf{a} = \left[0, \frac{3}{2}, \frac{3}{2}\right]$ OR $[3, 0, 0]$ OR $[1, 1, 1]$	A1√	For a correct vector. ft from equation in **(i)** **SR** a correct vector may be stated without working
	Line is (e.g.) $\mathbf{r} = [1, 1, 1] + t[2, -1, -1]$	A1√ **5**	For stating equation of line ft from **a** and **b** **SR** for $\mathbf{a} = [2, 2, 1]$ stated award M0
	METHOD 2		
	Solve $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$ by eliminating one variable (e.g. z)	M1	In either order: For attempting to solve equations
	Use parameter for another variable (e.g. x) to find other variables in terms of t	M1	For attempting to find parametric solution
	(eg) $y = \frac{3}{2} - \frac{1}{2}t$, $z = \frac{3}{2} - \frac{1}{2}t$	A1√ A1√	For correct expression for one variable For correct expression for the other variable ft from equation in **(i)** for both
	Line is (eg) $\mathbf{r} = \left[0, \frac{3}{2}, \frac{3}{2}\right] + t[2, -1, -1]$	A1√	For stating equation of line. ft from parametric solutions
	METHOD 3		
	eg x, y or $z = 0$ in $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$	M1	For attempting to find a point on the line
	$\Rightarrow \mathbf{a} = \left[0, \frac{3}{2}, \frac{3}{2}\right]$ *OR* $[3, 0, 0]$ *OR* $[1, 1, 1]$	A1√	For a correct vector. ft from equation in **(i)** **SR** a correct vector may be stated without working **SR** for $\mathbf{a} = [2, 2, 1]$ stated award M0
	eg $[3, 0, 0] - [1, 1, 1]$	M1	For finding another point on the line and using it with the one already found to find **b**
	$\mathbf{b} = k[2, -1, -1]$	A1√	For a correct vector. ft from equation in **(i)**
	Line is (eg) $\mathbf{r} = [1, 1, 1] + t[2, -1, -1]$	A1√	For stating equation of line. ft from **a** and **b**

6 (ii) contd	METHOD 4			
	A point on Π_1 is $[2+\lambda+\mu,\ 2+\lambda-5\mu,\ 1-2\mu]$	M1		For using parametric form for Π_1 and substituting into Π_2
	On $\Pi_2 \Rightarrow$ $[2+\lambda+\mu,\ 2+\lambda-5\mu,\ 1-2\mu].[7, 17, -3] = 21$	A1		For correct unsimplified equation
	$\Rightarrow \lambda - 3\mu = -1$	A1		For correct equation
	Line is (e.g.) $\mathbf{r} = [2, 2, 1] + (3\mu-1)[1, 1, 0] + \mu[1, -5, -2]$	M1		For substituting into Π_1 for λ *or* μ
	$\Rightarrow \mathbf{r} = [1, 1, 1]$ *or* $\left[\frac{7}{3}, \frac{1}{3}, \frac{1}{3}\right] + t\,[2, -1, -1]$	A1		For stating equation of line
			9	
7 (i)	$\cos 3\theta + \mathrm{i}\sin 3\theta = c^3 + 3\mathrm{i}\,c^2 s - 3cs^2 - \mathrm{i}\,s^3$	M1		For using de Moivre with $n = 3$
	$\Rightarrow \cos 3\theta = c^3 - 3cs^2$ and $\sin 3\theta = 3c^2 s - s^3$	A1		For both expressions in this form (seen or implied) **SR** For expressions found without de Moivre M0 A0
	$\Rightarrow \tan 3\theta = \dfrac{3c^2 s - s^3}{c^3 - 3cs^2}$	M1		For expressing $\dfrac{\sin 3\theta}{\cos 3\theta}$ in terms of c and s
	$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \dfrac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta}$	A1	**4**	For simplifying to **AG**
(ii) (a)	$\theta = \frac{1}{12}\pi \Rightarrow \tan 3\theta = 1$ $\Rightarrow 1 - 3t^2 = t(3 - t^2) \Rightarrow$ $t^3 - 3t^2 - 3t + 1 = 0$	B1	**1**	For both stages correct **AG**
(b)	$(t+1)(t^2 - 4t + 1) = 0$	M1 A1		For attempt to factorise cubic For correct factors
	$\Rightarrow (t = -1),\ t = 2 \pm \sqrt{3}$	A1		For correct roots of quadratic
	– sign for smaller root $\Rightarrow$ $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$	A1	**4**	For choice of – sign and correct root **AG**
(iii)	$\mathrm{d}t = (1 + t^2)\,\mathrm{d}\theta$	B1		For differentiation of substitution and use of $\sec^2\theta = 1 + \tan^2\theta$
	$\Rightarrow \int_0^{\frac{1}{12}\pi} \tan 3\theta\,\mathrm{d}\theta$	B1		For integral with correct θ limits seen
	$= \left[\frac{1}{3}\ln(\sec 3\theta)\right]_0^{\frac{1}{12}\pi} = \frac{1}{3}\ln\left(\sec\frac{1}{4}\pi\right)$	M1		For integrating to $k\ln(\sec 3\theta)$ OR $k\ln(\cos 3\theta)$
	$= \frac{1}{3}\ln\sqrt{2} = \frac{1}{6}\ln 2$	M1		For substituting limits and $\sec\frac{1}{4}\pi = \sqrt{2}$ OR $\cos\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$ seen
		A1	**5**	For correct answer aef
			14	

8 (i)	$a^2 = (ap)^2 = apap \Rightarrow a = pap$	B1		For use of given properties to obtain **AG**
	$p^2 = (ap)^2 = apap \Rightarrow p = apa$	B1	**2**	For use of given properties to obtain **AG** **SR** allow working from **AG** to obtain relevant properties
(ii)	$\left(p^2\right)^2 = p^4 = e \Rightarrow$ order $p^2 = 2$	B1		For correct order with no incorrect working seen
	$\left(a^2\right)^2 = \left(p^2\right)^2 = e \Rightarrow$ order $a = 4$	B1		For correct order with no incorrect working seen
	$(ap)^4 = a^4 = e \Rightarrow$ order $ap = 4$	B1		For correct order with no incorrect working seen
	$\left(ap^2\right)^2 = ap^2ap^2 = ap\,.\,a\,.\,p = a^2$	M1		For relevant use of **(i)** or given properties
	OR $ap^2 = a.a^2 = a^3 \Rightarrow$ $\left(ap^2\right)^2 = a^6 = a^2$ $\Rightarrow$ order $ap^2 = 4$	A1	**5**	For correct order with no incorrect working seen
(iii)	METHOD 1 $p^2 = a^2,\ ap^2 = a^3$	M2		For use of the given properties to simplify p^2 and $a\,p^2$
	$\Rightarrow \{e, a, p^2, ap^2\} = \{e, a, a^2, a^3\}$	A1		For obtaining a^2 and a^3
	which is a cyclic group	A1	**4**	For justifying that the set is a group

METHOD 2

	e	a	p^2	ap^2
e	e	a	p^2	ap^2
a	a	p^2	ap^2	e
p^2	p^2	ap^2	e	a
ap^2	ap^2	e	a	p^2

Answer	Marks	Guidance
(table above)	M1	For attempting closure with all 9 non-trivial products seen
	A1	For all 16 products correct
Completed table is a cyclic group	B2	For justifying that the set is a group

METHOD 3

	e	a	p^2	ap^2
e	e	a	p^2	ap^2
a	a	p^2	ap^2	e
p^2	p^2	ap^2	e	a
ap^2	ap^2	e	a	p^2

Answer	Marks	Guidance
(table above)	M1	For attempting closure with all 9 non-trivial products seen
	A1	For all 16 products correct
Identity = e	B1	For stating identity
Inverses exist since EITHER: e is in each row/column OR: p^2 is self-inverse; a, ap^2 form an inverse pair	B1	For justifying inverses ($e^{-1} = e$ may be assumed)

(iv)	METHOD 1 e.g. $\left.\begin{array}{l} a\,.\,ap = a^2 p = p^3 \\ ap\,.\,a = p \end{array}\right\} \Rightarrow$ not commutative	M1 M1 B1 A1	 **4**	For attempting to find a non-commutative pair of elements, at least one involving a (may be embedded in a full or partial table) For simplifying elements both ways round For a correct pair of non-commutative elements For stating Q non-commutative, with a clear argument
	METHOD 2 Assume commutativity, so (eg) $ap = pa$ **(i)** $\Rightarrow$ $p = ap\,.\,a \Rightarrow p = pa\,.\,a = pa^2 = pp^2 = p^3$ But p and p^3 are distinct $\Rightarrow Q$ is non-commutative	 M1 M1 B1 A1		 For setting up proof by contradiction For using **(i)** and/or given properties For obtaining and stating a contradiction For stating Q non-commutative, with a clear argument
			15	