# OCR Maths FP3

# Past Paper Pack

# 2007-2014

**1** **(i)** By writing $z$ in the form $re^{i\theta}$, show that $zz^* = |z|^2$. [1]

**(ii)** Given that $zz^* = 9$, describe the locus of $z$. [2]

**2** A line $l$ has equation $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$ and a plane $\Pi$ has equation $8x - 7y + 10z = 7$. Determine whether $l$ lies in $\Pi$, is parallel to $\Pi$ without intersecting it, or intersects $\Pi$ at one point. [5]

**3** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x}.$$ [6]

**4** Elements of the set $\{p, q, r, s, t\}$ are combined according to the operation table shown below.

| | $p$ | $q$ | $r$ | $s$ | $t$ |
|---|---|---|---|---|---|
| $p$ | $t$ | $s$ | $p$ | $r$ | $q$ |
| $q$ | $s$ | $p$ | $q$ | $t$ | $r$ |
| $r$ | $p$ | $q$ | $r$ | $s$ | $t$ |
| $s$ | $r$ | $t$ | $s$ | $q$ | $p$ |
| $t$ | $q$ | $r$ | $t$ | $p$ | $s$ |

**(i)** Verify that $q(st) = (qs)t$. [2]

**(ii)** Assuming that the associative property holds for all elements, prove that the set $\{p, q, r, s, t\}$, with the operation table shown, forms a group $G$. [4]

**(iii)** A multiplicative group $H$ is isomorphic to the group $G$. The identity element of $H$ is $e$ and another element is $d$. Write down the elements of $H$ in terms of $e$ and $d$. [2]

**5** **(i)** Use de Moivre's theorem to prove that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$ [4]

**(ii)** Hence find the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 3 = 0,$$

giving your answer in trigonometrical form. [4]

**6** Lines $l_1$ and $l_2$ have equations

$$\frac{x-3}{2}=\frac{y-4}{-1}=\frac{z+1}{1} \qquad \text{and} \qquad \frac{x-5}{4}=\frac{y-1}{3}=\frac{z-1}{2}$$

respectively.

**(i)** Find the equation of the plane $\Pi_1$ which contains $l_1$ and is parallel to $l_2$, giving your answer in the form $\mathbf{r.n} = p$. [5]

**(ii)** Find the equation of the plane $\Pi_2$ which contains $l_2$ and is parallel to $l_1$, giving your answer in the form $\mathbf{r.n} = p$. [2]

**(iii)** Find the distance between the planes $\Pi_1$ and $\Pi_2$. [2]

**(iv)** State the relationship between the answer to part **(iii)** and the lines $l_1$ and $l_2$. [1]

**7** **(i)** Show that $(z - e^{i\phi})(z - e^{-i\phi}) \equiv z^2 - (2\cos\phi)z + 1$. [1]

**(ii)** Write down the seven roots of the equation $z^7 = 1$ in the form $e^{i\theta}$ and show their positions in an Argand diagram. [4]

**(iii)** Hence express $z^7 - 1$ as the product of one real linear factor and three real quadratic factors. [5]

**8** **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \cos^3 x,$$

expressing $y$ in terms of $x$ in your answer. [8]

**(ii)** Find the particular solution for which $y = 2$ when $x = \pi$. [2]

**9** The set $S$ consists of the numbers $3^n$, where $n \in \mathbb{Z}$. ($\mathbb{Z}$ denotes the set of integers $\{0, \pm1, \pm2, \ldots\}$.)

**(i)** Prove that the elements of $S$, under multiplication, form a commutative group $G$. (You may assume that **addition** of integers is associative and commutative.) [6]

**(ii)** Determine whether or not each of the following subsets of $S$, under multiplication, forms a subgroup of $G$, justifying your answers.

**(a)** The numbers $3^{2n}$, where $n \in \mathbb{Z}$. [2]

**(b)** The numbers $3^n$, where $n \in \mathbb{Z}$ and $n \geqslant 0$. [2]

**(c)** The numbers $3^{(\pm n^2)}$, where $n \in \mathbb{Z}$. [2]

**1** **(a)** A group $G$ of order 6 has the combination table shown below.

| | $e$ | $a$ | $b$ | $p$ | $q$ | $r$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $p$ | $q$ | $r$ |
| $a$ | $a$ | $b$ | $e$ | $r$ | $p$ | $q$ |
| $b$ | $b$ | $e$ | $a$ | $q$ | $r$ | $p$ |
| $p$ | $p$ | $q$ | $r$ | $e$ | $a$ | $b$ |
| $q$ | $q$ | $r$ | $p$ | $b$ | $e$ | $a$ |
| $r$ | $r$ | $p$ | $q$ | $a$ | $b$ | $e$ |

**(i)** State, with a reason, whether or not $G$ is commutative. [1]

**(ii)** State the number of subgroups of $G$ which are of order 2. [1]

**(iii)** List the elements of the subgroup of $G$ which is of order 3. [1]

**(b)** A multiplicative group $H$ of order 6 has elements $e, c, c^2, c^3, c^4, c^5$, where $e$ is the identity. Write down the order of each of the elements $c^3$, $c^4$ and $c^5$. [3]

**2** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 4x.$$ [7]

**3** Two fixed points, $A$ and $B$, have position vectors $\mathbf{a}$ and $\mathbf{b}$ relative to the origin $O$, and a variable point $P$ has position vector $\mathbf{r}$.

**(i)** Give a geometrical description of the locus of $P$ when $\mathbf{r}$ satisfies the equation $\mathbf{r} = \lambda\mathbf{a}$, where $0 \leqslant \lambda \leqslant 1$. [2]

**(ii)** Given that $P$ is a point on the line $AB$, use a property of the vector product to explain why $(\mathbf{r} - \mathbf{a}) \times (\mathbf{r} - \mathbf{b}) = \mathbf{0}$. [2]

**(iii)** Give a geometrical description of the locus of $P$ when $\mathbf{r}$ satisfies the equation $\mathbf{r} \times (\mathbf{a} - \mathbf{b}) = \mathbf{0}$. [3]

**4** The integrals $C$ and $S$ are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x} \cos 3x \, dx \qquad \text{and} \qquad S = \int_0^{\frac{1}{2}\pi} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\tfrac{1}{13}(2 + 3e^{\pi}),$$

and obtain a similar expression for $S$. [8]

(You may assume that the standard result for $\int e^{kx}\,dx$ remains true when $k$ is a complex constant, so that $\int e^{(a+ib)x}\,dx = \dfrac{1}{a+ib}e^{(a+ib)x}$.)

**5** **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin 2x,$$

expressing $y$ in terms of $x$ in your answer. [6]

In a particular case, it is given that $y = \dfrac{2}{\pi}$ when $x = \frac{1}{4}\pi$.

**(ii)** Find the solution of the differential equation in this case. [2]

**(iii)** Write down a function to which $y$ approximates when $x$ is large and positive. [1]

**6** A tetrahedron $ABCD$ is such that $AB$ is perpendicular to the base $BCD$. The coordinates of the points $A$, $C$ and $D$ are $(-1, -7, 2)$, $(5, 0, 3)$ and $(-1, 3, 3)$ respectively, and the equation of the plane $BCD$ is $x + 2y - 2z = -1$.

**(i)** Find, in either order, the coordinates of $B$ and the length of $AB$. [5]

**(ii)** Find the acute angle between the planes $ACD$ and $BCD$. [6]

**7** **(i)** **(a)** Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation $\sin 6\theta = \sin 2\theta$. [1]

**(b)** By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, or otherwise, find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$. [2]

**(ii)** Use de Moivre's theorem to prove that

$$\sin 6\theta \equiv \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3).$$ [5]

**(iii)** Hence show that one of the solutions obtained in part **(i)** satisfies $\cos^2\theta = \frac{1}{4}(2 - \sqrt{2})$, and justify which solution it is. [3]

Jan 2008

**8** Groups $A$, $B$, $C$ and $D$ are defined as follows:

$A$: the set of numbers $\{2, 4, 6, 8\}$ under multiplication modulo 10,

$B$: the set of numbers $\{1, 5, 7, 11\}$ under multiplication modulo 12,

$C$: the set of numbers $\{2^0, 2^1, 2^2, 2^3\}$ under multiplication modulo 15,

$D$: the set of numbers $\left\{\dfrac{1+2m}{1+2n}, \text{where } m \text{ and } n \text{ are integers}\right\}$ under multiplication.

**(i)** Write down the identity element for each of groups $A$, $B$, $C$ and $D$. [2]

**(ii)** Determine in each case whether the groups

$A$ and $B$,

$B$ and $C$,

$A$ and $C$

are isomorphic or non-isomorphic. Give sufficient reasons for your answers. [5]

**(iii)** Prove the closure property for group $D$. [4]

**(iv)** Elements of the set $\left\{\dfrac{1+2m}{1+2n}, \text{where } m \text{ and } n \text{ are integers}\right\}$ are combined under **addition**. State which of the four basic group properties are **not** satisfied. (Justification is not required.) [2]

June 2008

**1** **(a)** A cyclic multiplicative group $G$ has order 12. The identity element of $G$ is $e$ and another element is $r$, with order 12.

**(i)** Write down, in terms of $e$ and $r$, the elements of the subgroup of $G$ which is of order 4. [2]

**(ii)** Explain briefly why there is no proper subgroup of $G$ in which two of the elements are $e$ and $r$. [1]

**(b)** A group $H$ has order $mnp$, where $m$, $n$ and $p$ are prime. State the possible orders of proper subgroups of $H$. [2]

**2** Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and the plane with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$. [7]

**3** **(i)** Use the substitution $z = x + y$ to show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y+3}{x+y-1} \qquad \text{(A)}$$

may be written in the form $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{2(z+1)}{z-1}$. [3]

**(ii)** Hence find the general solution of the differential equation (A). [4]

**4** **(i)** By expressing $\cos\theta$ in terms of $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, show that

$$\cos^5\theta \equiv \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta). \qquad [5]$$

**(ii)** Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \leqslant \theta \leqslant \pi$. [4]

**5** Two lines have equations

$$\frac{x-k}{2} = \frac{y+1}{-5} = \frac{z-1}{-3} \qquad \text{and} \qquad \frac{x-k}{1} = \frac{y+4}{-4} = \frac{z}{-2},$$

where $k$ is a constant.

**(i)** Show that, for all values of $k$, the lines intersect, and find their point of intersection in terms of $k$. [6]

**(ii)** For the case $k = 1$, find the equation of the plane in which the lines lie, giving your answer in the form $ax + by + cz = d$. [4]

**6** The operation $\circ$ on real numbers is defined by $a \circ b = a|b|$.

**(i)** Show that $\circ$ is not commutative. [2]

**(ii)** Prove that $\circ$ is associative. [4]

**(iii)** Determine whether the set of real numbers, under the operation $\circ$, forms a group. [4]

**7** The roots of the equation $z^3 - 1 = 0$ are denoted by 1, $\omega$ and $\omega^2$.

**(i)** Sketch an Argand diagram to show these roots. [1]

**(ii)** Show that $1 + \omega + \omega^2 = 0$. [2]

**(iii)** Hence evaluate

**(a)** $(2 + \omega)(2 + \omega^2)$, [2]

**(b)** $\dfrac{1}{2 + \omega} + \dfrac{1}{2 + \omega^2}$. [2]

**(iv)** Hence find a cubic equation, with integer coefficients, which has roots 2, $\dfrac{1}{2 + \omega}$ and $\dfrac{1}{2 + \omega^2}$. [4]

**8** **(i)** Find the complementary function of the differential equation

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$ [2]

**(ii)** It is given that $y = p(\ln \sin x)\sin x + qx\cos x$, where $p$ and $q$ are constants, is a particular integral of this differential equation.

**(a)** Show that $p - 2(p + q)\sin^2 x \equiv 1$. [6]

**(b)** Deduce the values of $p$ and $q$. [2]

**(iii)** Write down the general solution of the differential equation. State the set of values of $x$, in the interval $0 \leqslant x \leqslant 2\pi$, for which the solution is valid, justifying your answer. [3]

1 In this question $G$ is a group of order $n$, where $3 \leqslant n < 8$.

**(i)** In each case, write down the smallest possible value of $n$:

**(a)** if $G$ is cyclic, **[1]**

**(b)** if $G$ has a proper subgroup of order 3, **[1]**

**(c)** if $G$ has at least two elements of order 2. **[1]**

**(ii)** Another group has the same order as $G$, but is not isomorphic to $G$. Write down the possible value(s) of $n$. **[2]**

2 **(i)** Express $\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[3]**

**(ii)** Hence find the smallest positive value of $n$ for which $\left(\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)^n$ is real and positive. **[2]**

3 Two skew lines have equations

$$\frac{x}{2} = \frac{y+3}{1} = \frac{z-6}{3} \qquad \text{and} \qquad \frac{x-5}{3} = \frac{y+1}{1} = \frac{z-7}{5}.$$

**(i)** Find the direction of the common perpendicular to the lines. **[2]**

**(ii)** Find the shortest distance between the lines. **[4]**

4 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 65\sin 2x.$$ **[9]**

5 The variables $x$ and $y$ are related by the differential equation

$$x^3\frac{\mathrm{d}y}{\mathrm{d}x} = xy + x + 1. \qquad \text{(A)}$$

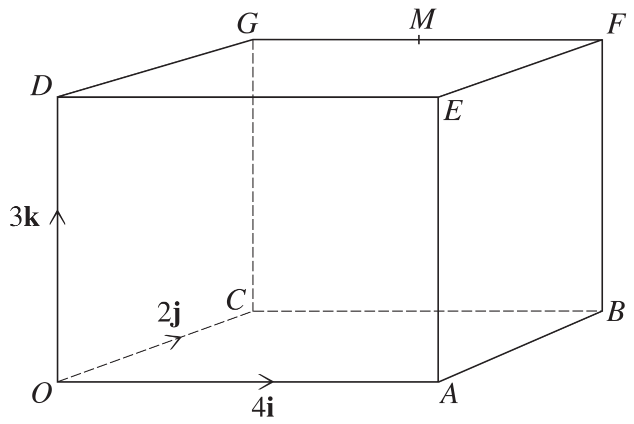
**(i)** Use the substitution $y = u - \frac{1}{x}$, where $u$ is a function of $x$, to show that the differential equation may be written as

$$x^2\frac{\mathrm{d}u}{\mathrm{d}x} = u.$$ **[4]**

**(ii)** Hence find the general solution of the differential equation (A), giving your answer in the form $y = \mathrm{f}(x)$. **[5]**

**6**




The cuboid $OABCDEFG$ shown in the diagram has $\overrightarrow{OA} = 4\mathbf{i}$, $\overrightarrow{OC} = 2\mathbf{j}$, $\overrightarrow{OD} = 3\mathbf{k}$, and $M$ is the mid-point of $GF$.

**(i)** Find the equation of the plane $ACGE$, giving your answer in the form $\mathbf{r.n} = p$. **[4]**

**(ii)** The plane $OEFC$ has equation $\mathbf{r.}(3\mathbf{i} - 4\mathbf{k}) = 0$. Find the acute angle between the planes $OEFC$ and $ACGE$. **[4]**

**(iii)** The line $AM$ meets the plane $OEFC$ at the point $W$. Find the ratio $AW : WM$. **[5]**

**7** **(i)** The operation $*$ is defined by $x * y = x + y - a$, where $x$ and $y$ are real numbers and $a$ is a real constant.

**(a)** Prove that the set of real numbers, together with the operation $*$, forms a group. **[6]**

**(b)** State, with a reason, whether the group is commutative. **[1]**

**(c)** Prove that there are no elements of order 2. **[2]**

**(ii)** The operation $\circ$ is defined by $x \circ y = x + y - 5$, where $x$ and $y$ are **positive** real numbers. By giving a numerical example in each case, show that two of the basic group properties are not necessarily satisfied. **[4]**

**8** **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$ **[5]**

**(ii)** Replace $\theta$ by $\left(\frac{1}{2}\pi - \theta\right)$ in the identity in part **(i)** to obtain a similar identity for $\cos^6\theta$. **[3]**

**(iii)** Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \left(\sin^6\theta - \cos^6\theta\right) d\theta$. **[4]**

1 Find the cube roots of $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$, giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$, where $0 \leqslant \theta < 2\pi$. **[4]**

2 It is given that the set of complex numbers of the form $r\mathrm{e}^{\mathrm{i}\theta}$ for $-\pi < \theta \leqslant \pi$ and $r > 0$, under multiplication, forms a group.

**(i)** Write down the inverse of $5\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$. **[1]**

**(ii)** Prove the closure property for the group. **[2]**

**(iii)** $Z$ denotes the element $\mathrm{e}^{\mathrm{i}\gamma}$, where $\frac{1}{2}\pi < \gamma < \pi$. Express $Z^2$ in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta < 0$. **[2]**

3 A line $l$ has equation $\dfrac{x-6}{-4} = \dfrac{y+7}{8} = \dfrac{z+10}{7}$ and a plane $p$ has equation $3x - 4y - 2z = 8$.

**(i)** Find the point of intersection of $l$ and $p$. **[3]**

**(ii)** Find the equation of the plane which contains $l$ and is perpendicular to $p$, giving your answer in the form $ax + by + cz = d$. **[5]**

4 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1-x^2}\,y = (1-x)^{\frac{1}{2}}, \quad \text{where } |x| < 1,$$

can be solved by the integrating factor method.

**(i)** Use an appropriate result given in the List of Formulae (MF1) to show that the integrating factor can be written as $\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}$. **[2]**

**(ii)** Hence find the solution of the differential equation for which $y = 2$ when $x = 0$, giving your answer in the form $y = \mathrm{f}(x)$. **[6]**

5 The variables $x$ and $y$ satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{3x}.$$

**(i)** Find the complementary function. **[3]**

**(ii)** Explain briefly why there is no particular integral of either of the forms $y = k\mathrm{e}^{3x}$ or $y = kx\mathrm{e}^{3x}$. **[1]**

**(iii)** Given that there is a particular integral of the form $y = kx^2\mathrm{e}^{3x}$, find the value of $k$. **[5]**

6 The plane $\Pi_1$ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$.

**(i)** Express the equation of $\Pi_1$ in the form $\mathbf{r.n} = p$. **[4]**

The plane $\Pi_2$ has equation $\mathbf{r.}\begin{pmatrix} 7 \\ 17 \\ -3 \end{pmatrix} = 21$.

**(ii)** Find an equation of the line of intersection of $\Pi_1$ and $\Pi_2$, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[5]**

7 **(i)** Use de Moivre's theorem to prove that

$$\tan 3\theta \equiv \frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta}.$$ **[4]**

**(ii) (a)** By putting $\theta = \frac{1}{12}\pi$ in the identity in part **(i)**, show that $\tan\frac{1}{12}\pi$ is a solution of the equation

$$t^3 - 3t^2 - 3t + 1 = 0.$$ **[1]**

**(b)** Hence show that $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$. **[4]**

**(iii)** Use the substitution $t = \tan\theta$ to show that

$$\int_0^{2-\sqrt{3}} \frac{t(3 - t^2)}{(1 - 3t^2)(1 + t^2)}\,\mathrm{d}t = a\ln b,$$

where $a$ and $b$ are positive constants to be determined. **[5]**

8 A multiplicative group $Q$ of order 8 has elements $\{e, p, p^2, p^3, a, ap, ap^2, ap^3\}$, where $e$ is the identity. The elements have the properties $p^4 = e$ and $a^2 = p^2 = (ap)^2$.

**(i)** Prove that $a = pap$ and that $p = apa$. **[2]**

**(ii)** Find the order of each of the elements $p^2, a, ap, ap^2$. **[5]**

**(iii)** Prove that $\{e, a, p^2, ap^2\}$ is a subgroup of $Q$. **[4]**

**(iv)** Determine whether $Q$ is a commutative group. **[4]**

1 Determine whether the lines

$$\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z+4}{2} \qquad \text{and} \qquad \frac{x+3}{2} = \frac{y-1}{3} = \frac{z-5}{4}$$

intersect or are skew. **[5]**

2 $H$ denotes the set of numbers of the form $a + b\sqrt{5}$, where $a$ and $b$ are rational. The numbers are combined under multiplication.

**(i)** Show that the product of any two members of $H$ is a member of $H$. **[2]**

It is now given that, for $a$ and $b$ not both zero, $H$ forms a group under multiplication.

**(ii)** State the identity element of the group. **[1]**

**(iii)** Find the inverse of $a + b\sqrt{5}$. **[2]**

**(iv)** With reference to your answer to part **(iii)**, state a property of the number 5 which ensures that every number in the group has an inverse. **[1]**

3 Use the integrating factor method to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^{-3x}$$

for which $y = 1$ when $x = 0$. Express your answer in the form $y = \mathrm{f}(x)$. **[6]**

4 **(i)** Write down, in cartesian form, the roots of the equation $z^4 = 16$. **[2]**

**(ii)** Hence solve the equation $w^4 = 16(1-w)^4$, giving your answers in cartesian form. **[5]**

5 A regular tetrahedron has vertices at the points

$$A\left(0,\ 0,\ \tfrac{2}{3}\sqrt{6}\right), \quad B\left(\tfrac{2}{3}\sqrt{3},\ 0,\ 0\right), \quad C\left(-\tfrac{1}{3}\sqrt{3},\ 1,\ 0\right), \quad D\left(-\tfrac{1}{3}\sqrt{3},\ -1,\ 0\right).$$

**(i)** Obtain the equation of the face $ABC$ in the form

$$x + \sqrt{3}y + \left(\tfrac{1}{2}\sqrt{2}\right)z = \tfrac{2}{3}\sqrt{3}.$$ **[5]**

(Answers which only verify the given equation will not receive full credit.)

**(ii)** Give a geometrical reason why the equation of the face $ABD$ can be expressed as

$$x - \sqrt{3}y + \left(\tfrac{1}{2}\sqrt{2}\right)z = \tfrac{2}{3}\sqrt{3}.$$ **[2]**

**(iii)** Hence find the cosine of the angle between two faces of the tetrahedron. **[4]**

**6** The variables $x$ and $y$ satisfy the differential equation

$$\frac{d^2y}{dx^2} + 16y = 8\cos 4x.$$

**(i)** Find the complementary function of the differential equation. **[2]**

**(ii)** Given that there is a particular integral of the form $y = px\sin 4x$, where $p$ is a constant, find the general solution of the equation. **[6]**

**(iii)** Find the solution of the equation for which $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[4]**

**7** **(i)** Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

**(ii)** By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1).$$ **[5]**

**(iii)** Hence find the exact value of

$$\cos\left(\tfrac{1}{12}\pi\right)\cos\left(\tfrac{5}{12}\pi\right)\cos\left(\tfrac{7}{12}\pi\right)\cos\left(\tfrac{11}{12}\pi\right),$$

justifying your answer. **[5]**

**8** The function f is defined by $\mathrm{f} : x \mapsto \frac{1}{2 - 2x}$ for $x \in \mathbb{R}$, $x \neq 0$, $x \neq \frac{1}{2}$, $x \neq 1$. The function g is defined by $\mathrm{g}(x) = \mathrm{ff}(x)$.

**(i)** Show that $\mathrm{g}(x) = \frac{1 - x}{1 - 2x}$ and that $\mathrm{gg}(x) = x$. **[4]**

It is given that f and g are elements of a group $K$ under the operation of composition of functions. The element e is the identity, where $\mathrm{e} : x \mapsto x$ for $x \in \mathbb{R}$, $x \neq 0$, $x \neq \frac{1}{2}$, $x \neq 1$.

**(ii)** State the orders of the elements f and g. **[2]**

**(iii)** The inverse of the element f is denoted by h. Find $\mathrm{h}(x)$. **[2]**

**(iv)** Construct the operation table for the elements e, f, g, h of the group $K$. **[4]**

**1** The line $l_1$ passes through the points (0, 0, 10) and (7, 0, 0) and the line $l_2$ passes through the points (4, 6, 0) and (3, 3, 1). Find the shortest distance between $l_1$ and $l_2$. **[7]**

**2** A multiplicative group with identity $e$ contains distinct elements $a$ and $r$, with the properties $r^6 = e$ and $ar = r^5a$.

**(i)** Prove that $rar = a$. **[2]**

**(ii)** Prove, by induction or otherwise, that $r^n ar^n = a$ for all positive integers $n$. **[4]**

**3** In this question, $w$ denotes the complex number $\cos\frac{2}{5}\pi + \mathrm{i}\sin\frac{2}{5}\pi$.

**(i)** Express $w^2$, $w^3$ and $w^*$ in polar form, with arguments in the interval $0 \leqslant \theta < 2\pi$. **[4]**

**(ii)** The points in an Argand diagram which represent the numbers

$$1, \quad 1+w, \quad 1+w+w^2, \quad 1+w+w^2+w^3, \quad 1+w+w^2+w^3+w^4$$

are denoted by $A$, $B$, $C$, $D$, $E$ respectively. Sketch the Argand diagram to show these points and join them in the order stated. (Your diagram need not be exactly to scale, but it should show the important features.) **[4]**

**(iii)** Write down a polynomial equation of degree 5 which is satisfied by $w$. **[1]**

**4** **(i)** Use the substitution $y = xz$ to find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x\cos\left(\frac{y}{x}\right),$$

giving your answer in a form without logarithms. (You may quote an appropriate result given in the List of Formulae (MF1).) **[6]**

**(ii)** Find the solution of the differential equation for which $y = \pi$ when $x = 4$. **[2]**

**5** Convergent infinite series $C$ and $S$ are defined by

$$C = 1 + \tfrac{1}{2}\cos\theta + \tfrac{1}{4}\cos 2\theta + \tfrac{1}{8}\cos 3\theta + \dots ,$$
$$S = \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{8}\sin 3\theta + \dots .$$

**(i)** Show that $C + \mathrm{i}S = \dfrac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$. **[4]**

**(ii)** Hence show that $C = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$, and find a similar expression for $S$. **[4]**

**6** **(i)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 17x + 36.$$ **[7]**

**(ii)** Show that, when $x$ is large and positive, the solution approximates to a linear function, and state its equation. **[2]**

**7** A line $l$ has equation $\mathbf{r} = \begin{pmatrix} -7 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$. A plane $\Pi$ passes through the points (1, 3, 5) and (5, 2, 5), and is parallel to $l$.

**(i)** Find an equation of $\Pi$, giving your answer in the form $\mathbf{r.n} = p$. **[4]**

**(ii)** Find the distance between $l$ and $\Pi$. **[4]**

**(iii)** Find an equation of the line which is the reflection of $l$ in $\Pi$, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[4]**

**8** A set of matrices $M$ is defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad C = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix},$$

where $\omega$ and $\omega^2$ are the complex cube roots of 1. It is given that $M$ is a group under matrix multiplication.

**(i)** Write down the elements of a subgroup of order 2. **[1]**

**(ii)** Explain why there is no element $X$ of the group, other than $A$, which satisfies the equation $X^5 = A$. **[2]**

**(iii)** By finding $BE$ and $EB$, verify the closure property for the pair of elements $B$ and $E$. **[4]**

**(iv)** Find the inverses of $B$ and $E$. **[3]**

**(v)** Determine whether the group $M$ is isomorphic to the group $N$ which is defined as the set of numbers {1, 2, 4, 8, 7, 5} under multiplication modulo 9. Justify your answer clearly. **[3]**

**1** **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + xy = xe^{\frac{1}{2}x^2},$$

giving your answer in the form $y = \mathrm{f}(x)$. **[4]**

**(ii)** Find the particular solution for which $y = 1$ when $x = 0$. **[2]**

**2** Two intersecting lines, lying in a plane $p$, have equations

$$\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-4}{-3} \qquad \text{and} \qquad \frac{x-1}{-1} = \frac{y-3}{2} = \frac{z-4}{4}.$$

**(i)** Obtain the equation of $p$ in the form $2x - y + z = 3$. **[3]**

**(ii)** Plane $q$ has equation $2x - y + z = 21$. Find the distance between $p$ and $q$. **[3]**

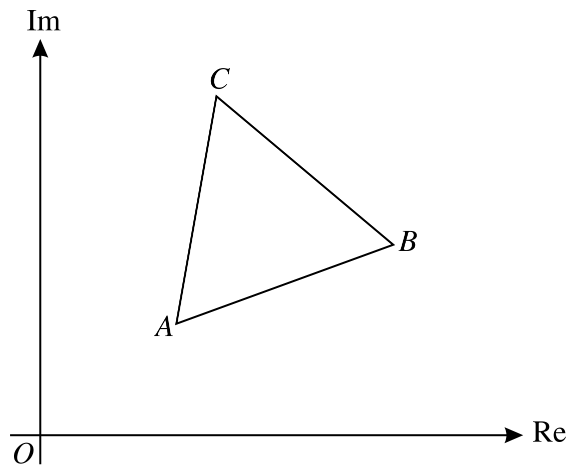
**3** **(i)** Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and show that

$$\sin^4\theta \equiv \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$ **[4]**

**(ii)** Hence find the exact value of $\int_0^{\frac{1}{6}\pi} \sin^4\theta \, \mathrm{d}\theta$. **[4]**

**4** The cube roots of 1 are denoted by 1, $\omega$ and $\omega^2$, where the imaginary part of $\omega$ is positive.

**(i)** Show that $1 + \omega + \omega^2 = 0$. **[2]**




In the diagram, $ABC$ is an equilateral triangle, labelled anticlockwise. The points $A$, $B$ and $C$ represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively.

**(ii)** State the geometrical effect of multiplication by $\omega$ and hence explain why $z_1 - z_3 = \omega(z_3 - z_2)$. **[4]**

**(iii)** Hence show that $z_1 + \omega z_2 + \omega^2 z_3 = 0$. **[2]**

**5** **(i)** Find the general solution of the differential equation

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = -2x + 13. \qquad \textbf{[7]}$$

**(ii)** Find the particular solution for which $y = -\frac{7}{2}$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[5]**

**(iii)** Write down the function to which $y$ approximates when $x$ is large and positive. **[1]**

**6** $Q$ is a multiplicative group of order 12.

**(i)** Two elements of $Q$ are $a$ and $r$. It is given that $r$ has order 6 and that $a^2 = r^3$. Find the orders of the elements $a$, $a^2$, $a^3$ and $r^2$. **[4]**

The table below shows the number of elements of $Q$ with each possible order.

| Order of element | 1 | 2 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| Number of elements | 1 | 1 | 2 | 6 | 2 |

$G$ and $H$ are the non-cyclic groups of order 4 and 6 respectively.

**(ii)** Construct two tables, similar to the one above, to show the number of elements with each possible order for the groups $G$ and $H$. Hence explain why there are no non-cyclic proper subgroups of $Q$. **[5]**

**7** Three planes $\Pi_1$, $\Pi_2$ and $\Pi_3$ have equations

$$\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 5, \qquad \mathbf{r}.(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 6, \qquad \mathbf{r}.(\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}) = 12,$$

respectively. Planes $\Pi_1$ and $\Pi_2$ intersect in a line $l$; planes $\Pi_2$ and $\Pi_3$ intersect in a line $m$.

**(i)** Show that $l$ and $m$ are in the same direction. **[5]**

**(ii)** Write down what you can deduce about the line of intersection of planes $\Pi_1$ and $\Pi_3$. **[1]**

**(iii)** By considering the cartesian equations of $\Pi_1$, $\Pi_2$ and $\Pi_3$, or otherwise, determine whether or not the three planes have a common line of intersection. **[4]**

**[Question 8 is printed overleaf.]**

**8** The operation $*$ is defined on the elements $(x, y)$, where $x, y \in \mathbb{R}$, by

$$(a, b) * (c, d) = (ac, ad + b).$$

It is given that the identity element is $(1, 0)$.

**(i)** Prove that $*$ is associative. **[3]**

**(ii)** Find all the elements which commute with $(1, 1)$. **[3]**

**(iii)** It is given that the particular element $(m, n)$ has an inverse denoted by $(p, q)$, where

$$(m, n) * (p, q) = (p, q) * (m, n) = (1, 0).$$

Find $(p, q)$ in terms of $m$ and $n$. **[2]**

**(iv)** Find all self-inverse elements. **[3]**

**(v)** Give a reason why the elements $(x, y)$, under the operation $*$, do not form a group. **[1]**

**1** A line $l$ has equation $\frac{x-1}{5} = \frac{y-6}{6} = \frac{z+3}{-7}$ and a plane $p$ has equation $x + 2y - z = 40$.

**(i)** Find the acute angle between $l$ and $p$. **[4]**

**(ii)** Find the perpendicular distance from the point $(1, 6, -3)$ to $p$. **[2]**

**2** It is given that $z = \mathrm{e}^{\mathrm{i}\theta}$, where $0 < \theta < 2\pi$, and $w = \frac{1+z}{1-z}$.

**(i)** Prove that $w = \mathrm{i}\cot\frac{1}{2}\theta$. **[3]**

**(ii)** Sketch separate Argand diagrams to show the locus of $z$ and the locus of $w$. You should show the direction in which each locus is described when $\theta$ increases in the interval $0 < \theta < 2\pi$. **[3]**

**3** The variables $x$ and $y$ satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 5\cos 3x.$$

**(i)** Find the complementary function. **[2]**

**(ii)** Hence, or otherwise, find the general solution. **[7]**

**(iii)** Find the approximate range of values of $y$ when $x$ is large and positive. **[2]**

**4** A group $G$, of order 8, is generated by the elements $a$, $b$, $c$. $G$ has the properties

$$a^2 = b^2 = c^2 = e, \quad ab = ba, \quad bc = cb, \quad ca = ac,$$

where $e$ is the identity.

**(i)** Using these properties and basic group properties as necessary, prove that $abc = cba$. **[2]**

The operation table for $G$ is shown below.

| | $e$ | $a$ | $b$ | $c$ | $bc$ | $ca$ | $ab$ | $abc$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ | $bc$ | $ca$ | $ab$ | $abc$ |
| $a$ | $a$ | $e$ | $ab$ | $ca$ | $abc$ | $c$ | $b$ | $bc$ |
| $b$ | $b$ | $ab$ | $e$ | $bc$ | $c$ | $abc$ | $a$ | $ca$ |
| $c$ | $c$ | $ca$ | $bc$ | $e$ | $b$ | $a$ | $abc$ | $ab$ |
| $bc$ | $bc$ | $abc$ | $c$ | $b$ | $e$ | $ab$ | $ca$ | $a$ |
| $ca$ | $ca$ | $c$ | $abc$ | $a$ | $ab$ | $e$ | $bc$ | $b$ |
| $ab$ | $ab$ | $b$ | $a$ | $abc$ | $ca$ | $bc$ | $e$ | $c$ |
| $abc$ | $abc$ | $bc$ | $ca$ | $ab$ | $a$ | $b$ | $c$ | $e$ |

**(ii)** List all the subgroups of order 2. **[2]**

**(iii)** List five subgroups of order 4. **[3]**

**(iv)** Determine whether all the subgroups of $G$ which are of order 4 are isomorphic. **[2]**

5 The substitution $y = u^k$, where $k$ is an integer, is to be used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x^2y^2 \qquad \text{(A)}$$

by changing it into an equation (B) in the variables $u$ and $x$.

**(i)** Show that equation (B) may be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3}{kx}u = \frac{1}{k}xu^{k+1}.$$ **[4]**

**(ii)** Write down the value of $k$ for which the integrating factor method may be used to solve equation (B). **[1]**

**(iii)** Using this value of $k$, solve equation (B) and hence find the general solution of equation (A), giving your answer in the form $y = \mathrm{f}(x)$. **[4]**

6 **(a)** The set of polynomials $\{ax + b\}$, where $a, b \in \mathbb{R}$, is denoted by $P$. Assuming that the associativity property holds, prove that $P$, under addition, is a group. **[4]**

**(b)** The set of polynomials $\{ax + b\}$, where $a, b \in \{0, 1, 2\}$, is denoted by $Q$. It is given that $Q$, under addition modulo 3, is a group, denoted by $\big(Q, +(\mathrm{mod}3)\big)$.

**(i)** State the order of the group. **[1]**

**(ii)** Write down the inverse of the element $2x + 1$. **[1]**

**(iii)** $\mathrm{q}(x) = ax + b$ is any element of $Q$ other than the identity. Find the order of $\mathrm{q}(x)$ and hence determine whether $\big(Q, +(\mathrm{mod}3)\big)$ is a cyclic group. **[4]**

7 (In this question, the notation $\Delta ABC$ denotes the area of the triangle $ABC$.)

The points $P$, $Q$ and $R$ have position vectors $p\mathbf{i}$, $q\mathbf{j}$ and $r\mathbf{k}$ respectively, relative to the origin $O$, where $p$, $q$ and $r$ are positive. The points $O$, $P$, $Q$ and $R$ are joined to form a tetrahedron.

**(i)** Draw a sketch of the tetrahedron and write down the values of $\Delta OPQ$, $\Delta OQR$ and $\Delta ORP$. **[3]**

**(ii)** Use the definition of the vector product to show that $\frac{1}{2}\left|\overrightarrow{RP} \times \overrightarrow{RQ}\right| = \Delta PQR$. **[1]**

**(iii)** Show that $(\Delta OPQ)^2 + (\Delta OQR)^2 + (\Delta ORP)^2 = (\Delta PQR)^2$. **[6]**

8 **(i)** Use de Moivre's theorem to express $\cos 4\theta$ as a polynomial in $\cos\theta$. **[4]**

**(ii)** Hence prove that $\cos 4\theta \cos 2\theta \equiv 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$. **[1]**

**(iii)** Use part **(ii)** to show that the only roots of the equation $\cos 4\theta \cos 2\theta = 1$ are $\theta = n\pi$, where $n$ is an integer. **[3]**

**(iv)** Show that $\cos 4\theta \cos 2\theta = -1$ only when $\cos\theta = 0$. **[3]**

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**1** The variables $x$ and $y$ are related by the differential equation

$$\frac{dy}{dx} = \frac{2x^2 + y^2}{xy}. \qquad \text{(A)}$$

**(i)** Use the substitution $y = ux$, where $u$ is a function of $x$, to obtain the differential equation

$$x\frac{du}{dx} = \frac{2}{u}.$$ **[3]**

**(ii)** Hence find the general solution of the differential equation (A), giving your answer in the form $y^2 = f(x)$. **[4]**

**2** **(i)** Show that $\left(z^n - e^{i\theta}\right)\left(z^n - e^{-i\theta}\right) \equiv z^{2n} - (2\cos\theta)\, z^n + 1$. **[1]**

**(ii)** Express $z^4 - z^2 + 1$ as the product of four factors of the form $\left(z - e^{i\alpha}\right)$, where $0 \leqslant \alpha < 2\pi$. **[6]**

**3** A multiplicative group contains the distinct elements $e$, $x$ and $y$, where $e$ is the identity.

**(i)** Prove that $x^{-1}y^{-1} = (yx)^{-1}$. **[2]**

**(ii)** Given that $x^n y^n = (xy)^n$ for some integer $n \geqslant 2$, prove that $x^{n-1}y^{n-1} = (yx)^{n-1}$. **[3]**

**(iii)** If $x^{n-1}y^{n-1} = (yx)^{n-1}$, does it follow that $x^n y^n = (xy)^n$? Give a reason for your answer. **[2]**

**4** The line $l$ has equations $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{2}$ and the point $A$ is (7, 3, 7). $M$ is the point where the perpendicular from $A$ meets $l$.

**(i)** Find, in either order, the coordinates of $M$ and the perpendicular distance from $A$ to $l$. **[7]**

**(ii)** Find the coordinates of the point $B$ on $AM$ such that $\overrightarrow{AB} = 3\overrightarrow{BM}$. **[3]**

**5** The variables $x$ and $y$ satisfy the differential equation

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 5e^{-2x}.$$

**(i)** Find the complementary function of the differential equation. **[2]**

**(ii)** Given that there is a particular integral of the form $y = pxe^{-2x}$, find the constant $p$. **[4]**

**(iii)** Find the solution of the equation for which $y = 0$ and $\frac{dy}{dx} = 4$ when $x = 0$. **[5]**

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**6** The plane $\Pi$ has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$ and the line $l$ has equation $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

**(i)** Express the equation of $\Pi$ in the form $\mathbf{r}.\mathbf{n} = p$. **[4]**

**(ii)** Find the point of intersection of $l$ and $\Pi$. **[2]**

**(iii)** The equation of $\Pi$ may be expressed in the form $\mathbf{r} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu\mathbf{c}$, where $\mathbf{c}$ is perpendicular to $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$. Find $\mathbf{c}$. **[3]**

**7** The set $M$ consists of the six matrices $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, where $n \in \{0, 1, 2, 3, 4, 5\}$. It is given that $M$ forms a group $(M, \times)$ under matrix multiplication, with numerical addition and multiplication both being carried out modulo 6.

**(i)** Determine whether $(M, \times)$ is a commutative group, justifying your answer. **[2]**

**(ii)** Write down the identity element of the group and find the inverse of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. **[3]**

**(iii)** State the order of $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and give a reason why $(M, \times)$ has no subgroup of order 4. **[2]**

**(iv)** The multiplicative group $G$ has order 6. All the elements of $G$, apart from the identity, have order 2 or 3. Determine whether $G$ is isomorphic to $(M, \times)$, justifying your answer. **[2]**

**8** **(i)** Use de Moivre's theorem to prove that

$$\tan 5\theta \equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$ **[4]**

**(ii)** Solve the equation $\tan 5\theta = 1$, for $0 \leqslant \theta < \pi$. **[3]**

**(iii)** Show that the roots of the equation

$$t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$$

may be expressed in the form $\tan\alpha$, stating the exact values of $\alpha$, where $0 \leqslant \alpha < \pi$. **[5]**

1 The plane $p$ has equation $\mathbf{r}.(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 4$ and the line $l_1$ has equation $\mathbf{r} = 2\mathbf{j} - \mathbf{k} + t(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The line $l_2$ is parallel to $p$ and perpendicular to $l_1$, and passes through the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Find the equation of $l_2$, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[4]**

2 **(i)** Solve the equation $z^4 = 2(1 + i\sqrt{3})$, giving the roots exactly in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[5]**

**(ii)** Sketch an Argand diagram to show the lines from the origin to the point representing $2(1 + i\sqrt{3})$ and from the origin to the points which represent the roots of the equation in part **(i)**. **[3]**

3 Find the solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x$$

for which $y = 2$ when $x = \frac{1}{6}\pi$. Give your answer in the form $y = f(x)$. **[9]**

4 The elements $a, b, c, d$ are combined according to the operation table below, to form a group $G$ of order 4.

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $b$ | $a$ | $d$ | $c$ |
| $b$ | $a$ | $b$ | $c$ | $d$ |
| $c$ | $d$ | $c$ | $a$ | $b$ |
| $d$ | $c$ | $d$ | $b$ | $a$ |

Group $G$ is isomorphic **either** to the multiplicative group $H = \{e, r, r^2, r^3\}$ **or** to the multiplicative group $K = \{e, p, q, pq\}$. It is given that $r^4 = e$ in group $H$ and that $p^2 = q^2 = e$ in group $K$, where $e$ denotes the identity in each group.

**(i)** Write down the operation tables for $H$ and $K$. **[4]**

**(ii)** State the identity element of $G$. **[1]**

**(iii)** Demonstrate the isomorphism between $G$ and either $H$ or $K$ by listing how the elements of $G$ correspond to the elements of the other group. If the correspondence can be shown in more than one way, list the alternative correspondence(s). **[4]**

5 **(i)** By expressing $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, prove that

$$\sin^3\theta\cos^2\theta \equiv -\tfrac{1}{16}(\sin 5\theta - \sin 3\theta - 2\sin\theta).$$ **[6]**

**(ii)** Hence show that all the roots of the equation

$$\sin 5\theta = \sin 3\theta + 2\sin\theta$$

are of the form $\theta = \dfrac{n\pi}{k}$, where $n$ is any integer and $k$ is to be determined. **[3]**

6 The variables $x$ and $y$ satisfy the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 12e^{2x}.$$

**(i)** Find the general solution of the differential equation. **[6]**

**(ii)** It is given that the curve which represents a particular solution of the differential equation has gradient 6 when $x = 0$, and approximates to $y = e^{2x}$ when $x$ is large and positive. Find the equation of the curve. **[4]**

7 With respect to the origin $O$, the position vectors of the points $U$, $V$ and $W$ are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively. The mid-points of the sides $VW$, $WU$ and $UV$ of the triangle $UVW$ are $M$, $N$ and $P$ respectively.

**(i)** Show that $\overrightarrow{UM} = \tfrac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u})$. **[2]**

**(ii)** Verify that the point $G$ with position vector $\tfrac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ lies on $UM$, and deduce that the lines $UM$, $VN$ and $WP$ intersect at $G$. **[5]**

**(iii)** Write down, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, an equation of the line through $G$ which is perpendicular to the plane $UVW$. (It is not necessary to simplify the expression for $\mathbf{b}$.) **[2]**

**(iv)** It is now given that $\mathbf{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Find the perpendicular distance from $O$ to the plane $UVW$. **[3]**

8 The set $M$ of matrices $\begin{pmatrix}a & b\\c & d\end{pmatrix}$, where $a$, $b$, $c$ and $d$ are real and $ad - bc = 1$, forms a group $(M, \times)$ under matrix multiplication. $R$ denotes the set of all matrices $\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$.

**(i)** Prove that $(R, \times)$ is a subgroup of $(M, \times)$. **[6]**

**(ii)** By considering geometrical transformations in the $x$-$y$ plane, find a subgroup of $(R, \times)$ of order 6. Give the elements of this subgroup in exact numerical form. **[5]**

**1** Two planes have equations

$$x + 2y + 5z = 12 \text{ and } 2x - y + 3z = 5.$$

**(i)** Find the acute angle between the planes. **[3]**

**(ii)** Find a vector equation of the line of intersection of the planes. **[4]**

**2** The elements of a group $G$ are the complex numbers $a + b\mathrm{i}$ where $a, b \in \{0, 1, 2, 3, 4\}$. These elements are combined under the operation of addition modulo 5.

**(i)** State the identity element and the order of $G$. **[2]**

**(ii)** Write down the inverse of $2 + 4\mathrm{i}$. **[1]**

**(iii)** Show that every non-zero element of $G$ has order 5. **[3]**

**3** Solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^4 \mathrm{e}^{2x}$ for $y$ in terms of $x$, given that $y = 0$ when $x = 1$. **[8]**

**4** The lines $l_1$ and $l_2$ have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$$

respectively.

**(i)** Find the shortest distance between the lines. **[5]**

**(ii)** Find a cartesian equation of the plane which contains $l_1$ and which is parallel to $l_2$. **[2]**

**5** **(i)** Solve the equation $z^5 = 1$, giving your answers in polar form. **[2]**

**(ii)** Hence, by considering the equation $(z + 1)^5 = z^5$, show that the roots of

$$5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$$

can be expressed in the form $\frac{1}{\mathrm{e}^{\mathrm{i}\theta} - 1}$, stating the values of $\theta$. **[5]**

**6** The differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin kx$ is to be solved, where $k$ is a constant.

**(i)** In the case $k = 2$, by using a particular integral of the form $ax\cos 2x + bx\sin 2x$, find the general solution. **[7]**

**(ii)** Describe briefly the behaviour of $y$ when $x \to \infty$. **[2]**

**(iii)** In the case $k \neq 2$, explain whether $y$ would exhibit the same behaviour as in part **(ii)** when $x \to \infty$. **[2]**

 

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**7** Let $S = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{e}^{3\mathrm{i}\theta} + \ldots + \mathrm{e}^{10\mathrm{i}\theta}$.

**(i) (a)** Show that, for $\theta \neq 2n\pi$, where $n$ is an integer,

$$S = \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}\left(\mathrm{e}^{10\mathrm{i}\theta} - 1\right)}{2\mathrm{i}\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

**(b)** State the value of $S$ for $\theta = 2n\pi$, where $n$ is an integer. **[1]**

**(ii)** Hence show that, for $\theta \neq 2n\pi$, where $n$ is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

**(iii)** Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

**8** A multiplicative group $H$ has the elements $\{e, a, a^2, a^3, w, aw, a^2w, a^3w\}$ where $e$ is the identity, elements $a$ and $w$ have orders 4 and 2 respectively and $wa = a^3w$.

**(i)** Show that $wa^2 = a^2w$ and also that $wa^3 = aw$. **[6]**

**(ii)** Hence show that each of $aw$, $a^2w$ and $a^3w$ has order 2. **[4]**

**(iii)** Find two non-cyclic subgroups of $H$ of order 4, and show that they are not cyclic. **[4]**

1 The plane $\Pi$ passes through the points with coordinates (1, 6, 2), (5, 2, 1) and (1, 0, –2).

**(i)** Find a vector equation of $\Pi$ in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. **[2]**

**(ii)** Find a cartesian equation of $\Pi$. **[4]**

2 $G$ consists of the set $\{1, 3, 5, 7\}$ with the operation of multiplication modulo 8.

**(i)** Write down the operation table and, assuming associativity, show that $G$ is a group. **[5]**

**(ii)** State the order of each element. **[1]**

**(iii)** Find all the proper subgroups of $G$. **[1]**

The group $H$ consists of the set $\{1, 3, 7, 9\}$ with the operation of multiplication modulo 10.

**(iv)** Explaining your reasoning, determine whether $H$ is isomorphic to $G$. **[2]**

3 The differential equation

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^3 = \frac{\cos x}{x}$$

is to be solved for $x > 0$. Use the substitution $u = y^3$ to find the general solution for $y$ in terms of $x$. **[8]**

4 The complex numbers 0, 3 and $3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ are represented in an Argand diagram by the points $O$, $A$ and $B$ respectively.

**(i)** Sketch the triangle $OAB$ and show that it is equilateral. **[3]**

**(ii)** Hence express $3 - 3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ in polar form. **[2]**

**(iii)** Hence find $(3 - 3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})^5$, giving your answer in the form $a + b\sqrt{3}\,\mathrm{i}$ where $a$ and $b$ are rational numbers. **[3]**

5 Find the solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \mathrm{e}^{-x}$ for which $y = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. **[11]**

6 The plane $\Pi$ has equation $x + 2y - 2z = 5$. The line $l$ has equation $\frac{x-1}{2} = \frac{y+1}{5} = \frac{z-2}{1}$.

**(i)** Find the coordinates of the point of intersection of $l$ with the plane $\Pi$. **[3]**

**(ii)** Calculate the acute angle between $l$ and $\Pi$. **[3]**

**(iii)** Find the coordinates of the two points on the line $l$ such that the distance of each point from the plane $\Pi$ is 2. **[5]**

7 A commutative group $G$ has order 18. The elements $a$, $b$ and $c$ have orders 2, 3 and 9 respectively.

**(i)** Prove that $ab$ has order 6. **[4]**

**(ii)** Show that $G$ is cyclic. **[3]**

8 **(i)** Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **[5]**

**(ii)** Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leqslant \alpha \leqslant \pi$. **[4]**

**(iii)** Hence find the exact value of $\cos\frac{1}{10}\pi$. **[3]**

**1** **(i)** Find a vector equation of the line of intersection of the planes $2x + y - z = 4$ and $3x + 5y + 2z = 13$. **[4]**

**(ii)** Find the exact distance of the point $(2, 5, -2)$ from the plane $2x + y - z = 4$. **[2]**

**2** Use the substitution $u = y^2$ to find the general solution of the differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

for $y$ in terms of $x$. **[8]**

**3** **(i)** Solve the equation $z^6 = 1$, giving your answers in the form $re^{i\theta}$, and sketch an Argand diagram showing the positions of the roots. **[4]**

**(ii)** Show that $(1 + i)^6 = -8i$. **[3]**

**(iii)** Hence, or otherwise, solve the equation $z^6 + 8i = 0$, giving your answers in the form $re^{i\theta}$. **[3]**

**4** The group $G$ consists of the set $\{1, 3, 7, 9, 11, 13, 17, 19\}$ combined under multiplication modulo 20.

**(i)** Find the inverse of each element. **[3]**

**(ii)** Show that $G$ is not cyclic. **[3]**

**(iii)** Find two isomorphic subgroups of order 4 and state an isomorphism between them. **[5]**

**5** Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}$$

subject to the conditions $y = \frac{dy}{dx} = 0$ when $x = 0$. **[10]**

**6** The line $l$ has equations $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-7}{5}$. The plane $\Pi$ has equation $4x - y - z = 8$.

**(i)** Show that $l$ is parallel to $\Pi$ but does not lie in $\Pi$. **[3]**

**(ii)** The point $A\ (1, -2, 7)$ is on $l$. Write down a vector equation of the line through $A$ which is perpendicular to $\Pi$. Hence find the position vector of the point on $\Pi$ which is closest to $A$. **[4]**

**(iii)** Hence write down a vector equation of the line in $\Pi$ which is parallel to $l$ and closest to it. **[1]**

**7** **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5\theta \equiv \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$ **[4]**

**(ii)** Hence solve the equation

$$\sin 5\theta + 4\sin\theta = 5\sin 3\theta$$

for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. **[4]**

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**8** $G$ consists of the set of matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a$ and $b$ are real and $a^2 + b^2 \neq 0$, combined under the operation of matrix multiplication.

**(i)** Prove that $G$ is a group. You may assume that matrix multiplication is associative. **[6]**

**(ii)** Determine whether $G$ is commutative. **[2]**

**(iii)** Find the order of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. **[3]**

**END OF QUESTION PAPER**