OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4757**

Further Applications of Advanced Mathematics (FP3)

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Monday 13 June 2011
Morning

Duration: 1 hour 30 minutes

* 4 7 5 7 *

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **8** pages. Any blank pages are indicated.

Option 1: Vectors

1 The points A (2, −1, 3), B (−2, −7, 7) and C (7, 5, 1) are three vertices of a tetrahedron ABCD.

The plane ABD has equation $x + 4y + 7z = 19$.

The plane ACD has equation $2x - y + 2z = 11$.

(i) Find the shortest distance from B to the plane ACD. **[3]**

(ii) Find an equation for the line AD. **[3]**

(iii) Find the shortest distance from C to the line AD. **[6]**

(iv) Find the shortest distance between the lines AD and BC. **[6]**

(v) Given that the tetrahedron ABCD has volume 20, find the coordinates of the two possible positions for the vertex D. **[6]**

Option 2: Multi-variable calculus

2 A surface S has equation $z = 8y^3 - 6x^2y - 15x^2 + 36x$.

(i) Sketch the section of S given by $y = -3$, and sketch the section of S given by $x = -6$. Your sketches should include the coordinates of any stationary points but need not include the coordinates of the points where the sections cross the axes. **[7]**

(ii) From your sketches in part **(i)**, deduce that (−6, −3, −324) is a stationary point on S, and state the nature of this stationary point. **[2]**

(iii) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and hence find the coordinates of the other three stationary points on S. **[8]**

(iv) Show that there are exactly two values of k for which the plane with equation

$$120x - z = k$$

is a tangent plane to S, and find these values of k. **[7]**

Option 3: Differential geometry

3 **(a)** **(i)** Given that $y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$, show that $1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x}\right)^2$. **[3]**

The arc of the curve $y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$ for $0 \leqslant x \leqslant \ln a$ (where $a > 1$) is denoted by C.

(ii) Show that the length of C is $\frac{a-1}{\sqrt{a}}$. **[3]**

(iii) Find the area of the surface formed when C is rotated through 2π radians about the x-axis. **[5]**

(b) An ellipse has parametric equations $x = 2\cos\theta$, $y = \sin\theta$ for $0 \leqslant \theta < 2\pi$.

(i) Show that the normal to the ellipse at the point with parameter θ has equation

$$y = 2x\tan\theta - 3\sin\theta.$$ **[3]**

(ii) Find parametric equations for the evolute of the ellipse, and show that the evolute has cartesian equation

$$(2x)^{\frac{2}{3}} + y^{\frac{2}{3}} = 3^{\frac{2}{3}}.$$ **[6]**

(iii) Using the evolute found in part **(ii)**, or otherwise, find the radius of curvature of the ellipse

(A) at the point (2, 0),

(B) at the point (0, 1). **[4]**

Option 4: Groups

4 **(i)** Show that the set $G = \{1, 3, 4, 5, 9\}$, under the binary operation of multiplication modulo 11, is a group. You may assume associativity. **[6]**

(ii) Explain why any two groups of order 5 must be isomorphic to each other. **[3]**

The set $H = \left\{1, e^{\frac{2}{5}\pi j}, e^{\frac{4}{5}\pi j}, e^{\frac{6}{5}\pi j}, e^{\frac{8}{5}\pi j}\right\}$ is a group under the binary operation of multiplication of complex numbers.

(iii) Specify an isomorphism between the groups G and H. **[3]**

The set K consists of the 25 ordered pairs (x, y), where x and y are elements of G. The set K is a group under the binary operation defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$$

where the multiplications are carried out modulo 11; for example, $(3, 5)(4, 4) = (1, 9)$.

(iv) Write down the identity element of K, and find the inverse of the element $(9, 3)$. **[2]**

(v) Explain why $(x, y)^5 = (1, 1)$ for every element (x, y) in K. **[3]**

(vi) Deduce that all the elements of K, except for one, have order 5. State which is the exceptional element. **[3]**

(vii) A subgroup of K has order 5 and contains the element $(9, 3)$. List the elements of this subgroup. **[2]**

(viii) Determine how many subgroups of K there are with order 5. **[2]**

Option 5: Markov chains

This question requires the use of a calculator with the ability to handle matrices.

5 In this question, give probabilities correct to 4 decimal places.

Alpha and Delta are two companies which compete for the ownership of insurance bonds. Boyles and Cayleys are companies which trade in these bonds. When a new bond becomes available, it is first acquired by either Boyles or Cayleys. After a certain amount of trading it is eventually owned by either Alpha or Delta. Change of ownership always takes place overnight, so that on any particular day the bond is owned by one of the four companies. The trading process is modelled as a Markov chain with four states, as follows.

On the first day, the bond is owned by Boyles or Cayleys, with probabilities 0.4, 0.6 respectively.

If the bond is owned by Boyles, then on the next day it could be owned by Alpha, Boyles or Cayleys, with probabilities 0.07, 0.8, 0.13 respectively.

If the bond is owned by Cayleys, then on the next day it could be owned by Boyles, Cayleys or Delta, with probabilities 0.15, 0.75, 0.1 respectively.

If the bond is owned by Alpha or Delta, then no further trading takes place, so on the next day it is owned by the same company.

(i) Write down the transition matrix **P**. **[2]**

(ii) Explain what is meant by an absorbing state of a Markov chain. Identify any absorbing states in this situation. **[2]**

(iii) Find the probability that the bond is owned by Boyles on the 10th day. **[3]**

(iv) Find the probability that on the 14th day the bond is owned by the same company as on the 10th day. **[3]**

(v) Find the day on which the probability that the bond is owned by Alpha or Delta exceeds 0.9 for the first time. **[4]**

(vi) Find the limit of $\mathbf{P}^n$ as n tends to infinity. **[2]**

(vii) Find the probability that the bond is eventually owned by Alpha. **[3]**

The probabilities that Boyles and Cayleys own the bond on the first day are changed (but all the transition probabilities remain the same as before). The bond is now equally likely to be owned by Alpha or Delta at the end of the trading process.

(viii) Find the new probabilities for the ownership of the bond on the first day. **[5]**