4757 Further Pure 3

1 (i)	Putting $x=0$, $-3y+10z=6$, $-4y-2z=8$ $y=-2$, $z=0$ Direction is given by $\begin{pmatrix}8\\-3\\10\end{pmatrix}\times\begin{pmatrix}3\\-4\\-2\end{pmatrix}$ $=\begin{pmatrix}46\\46\\-23\end{pmatrix}$ Equation of L is $\mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$	M1 A1 M1 A1 A1 ft **5**	Finding coords of a point on the line or $(2,\ 0,\ -1)$, $(1,\ -1,\ -\tfrac{1}{2})$ etc or finding a second point *Dependent on M1M1* Accept any form Condone omission of '**r** ='
(ii)	$\overrightarrow{AB}\times\mathbf{d}=\begin{pmatrix}7\\-14\\4\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}=\begin{pmatrix}6\\15\\42\end{pmatrix}$ $\left[\ =3\begin{pmatrix}2\\5\\14\end{pmatrix}\ \right]$ Distance is $\left[\begin{pmatrix}-1\\12\\5\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}\right].\hat{\mathbf{n}}=\dfrac{\begin{pmatrix}-1\\14\\5\end{pmatrix}.\begin{pmatrix}2\\5\\14\end{pmatrix}}{\sqrt{2^2+5^2+14^2}}$ $=\dfrac{138}{15}=\dfrac{46}{5}=9.2$	M1 A2 ft M1 A1 ft A1 **6**	Evaluating $\overrightarrow{AB}\times\mathbf{d}$ Give A1 ft if just one error Appropriate scalar product Fully correct expression
(iii)	$\left\lvert\overrightarrow{AB}\times\mathbf{d}\right\rvert=\left\lvert\begin{pmatrix}6\\15\\42\end{pmatrix}\right\rvert=\sqrt{6^2+15^2+42^2}$ Distance is $\dfrac{\left\lvert\overrightarrow{AB}\times\mathbf{d}\right\rvert}{\lvert\mathbf{d}\rvert}=\dfrac{\sqrt{6^2+15^2+42^2}}{\sqrt{2^2+2^2+1^2}}$ $=\dfrac{45}{3}=15$	M1 M1 M1A1 ft A1 **5**	For $\left\lvert\overrightarrow{AB}\times\mathbf{d}\right\rvert$ Evaluating magnitude *In this part, M marks are dependent on previous M marks*

(iv)	At D, $\begin{pmatrix}-1\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}k+1\\-12\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\\9\end{pmatrix}+\mu\begin{pmatrix}2\\2\\-1\end{pmatrix}$	M1	*Condone use of same parameter on both sides*
	$12-12\lambda=-2+2\mu$ $5-3\lambda=9-\mu$	A1 ft	Two equations for λ and μ
	$\lambda=\frac{1}{3},\quad \mu=5$	M1M1	Obtaining λ and μ (numerically) Give M1 for λ and μ in terms of *k*
	$-1+\frac{1}{3}(k+1)=6+10$ $k=50$	M1 A1	Equation for *k*
	D is $(6+2\mu,\ -2+2\mu,\ 9-\mu)$ i.e. $(16,\ 8,\ 4)$	M1 A1 **8**	Obtaining coordinates of D

Alternative solutions for Q1

1 (i)	e.g. $23x-23y=46$ $x=t,\ \ y=t-2$ $3t-4(t-2)-2z=8$ $x=t,\ \ y=t-2,\ \ z=-\frac{1}{2}t$	M1A1 M1A1 ft A1 **5**	Eliminating one of *x, y, z*
(ii)	$\overrightarrow{PQ}=\begin{pmatrix}-1+7\mu\\12-14\mu\\5+4\mu\end{pmatrix}-\begin{pmatrix}2\lambda\\-2+2\lambda\\-\lambda\end{pmatrix}\quad \overrightarrow{PQ}.\mathbf{d}=\overrightarrow{PQ}.\overrightarrow{AB}=0$ $2(-1+7\mu-2\lambda)+2(14-14\mu-2\lambda)-(5+4\mu+\lambda)=0$ $7(-1+7\mu-2\lambda)-14(14-14\mu-2\lambda)+4(5+4\mu+\lambda)=0\ \ \lambda=27/25,\ \ \mu=47/75$ $\left\|\overrightarrow{PQ}\right\|=\sqrt{(92/75)^2+(230/75)^2+(644/75)^2}=9.2$	M1 A1 ft A1 ft M1A1 ft A1 **6**	Two equations for λ and μ Expression for shortest distance
(iii)	$\overrightarrow{AX}\cdot\mathbf{d}=\begin{pmatrix}6+2\lambda+1\\-2+2\lambda-12\\9-\lambda-5\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-1\end{pmatrix}=0$ $2(7+2\lambda)+2(2\lambda-14)-(4-\lambda)=0$ $\lambda=2$ $\overrightarrow{AX}=\begin{pmatrix}11\\-10\\2\end{pmatrix}$ $AX=\sqrt{11^2+10^2+2^2}$ $=15$	M1 A1 ft M1 M1 A1 **5**	

(iv)	$\begin{pmatrix}7\\-14\\4\end{pmatrix}\cdot\left[\begin{pmatrix}k+1\\-12\\-3\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}\right]=0$ $\begin{pmatrix}7\\-14\\4\end{pmatrix}\cdot\begin{pmatrix}18\\k-5\\2k+26\end{pmatrix}=0$ $126-14k+70+8k+104=0$ $k=50$ At D, $\begin{pmatrix}-1\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}51\\-12\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\\9\end{pmatrix}+\mu\begin{pmatrix}2\\2\\-1\end{pmatrix}$ $-1+51\lambda=6+2\mu$ $12-12\lambda=-2+2\mu$ $5-3\lambda=9-\mu$ $\lambda=\frac{1}{3},\quad \mu=5$ D is $(6+2\mu,\ -2+2\mu,\ 9-\mu)$ i.e. $(16,\ 8,\ 4)$	M1 M1 A1 M1 A1 ft M1 M1 A1 **8**	Appropriate scalar triple product equated to zero Equation for *k* *Condone use of same parameter on both sides* Two equations for λ and μ Obtaining λ or μ Obtaining coordinates of D

2 (i)	$\frac{\partial z}{\partial x} = 3(x+y)^3 + 9x(x+y)^2 - 6x^2 + 24$ $\frac{\partial z}{\partial y} = 9x(x+y)^2$	M1 A2 A1 **4**	Partial differentiation Give A1 if just one minor error
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 0$ $y = -2$; one stationary point is $(0, -2, 0)$ If $y = -x$ then $-6x^2 + 24 = 0$ $x = \pm 2$; stationary points are $(2, -2, 32)$ and $(-2, 2, -32)$	M1 M1 A1A1 M1 A1 A1 **7**	If A0A0, give A1 for $x = \pm 2$
(iii)	At P $(1, -2, 19)$, $\frac{\partial z}{\partial x} = 24$, $\frac{\partial z}{\partial y} = 9$ Normal line is $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 19 \end{pmatrix} + \lambda \begin{pmatrix} 24 \\ 9 \\ -1 \end{pmatrix}$	B1 M1 A1 ft **3**	For normal vector (allow sign error) Condone omission of '**r** ='
(iv)	$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$ $= 24\delta x + 9\delta y$ $3h \approx 24k + 9h$ $k \approx -\frac{1}{4}h$	M1 A1 ft M1 A1 **4**	
	OR Tangent plane is $24x + 9y - z = -13$ $24(1+k) + 9(-2+h) - (19+3h) \approx -13$ M2A1 ft $k \approx -\frac{1}{4}h$ A1		
(v)	$\frac{\partial z}{\partial x} = 27$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 27$ $y = 1,\ z = 0$; point is $(0, 1, 0)$ $d = 0$ If $y = -x$ then $-6x^2 + 24 = 27$ $x^2 = -\frac{1}{2}$; there are no other points	M1 M1 A1 A1 M1 A1 **6**	(Allow M1 for $\frac{\partial z}{\partial x} = -27$)

3 (i)	$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2=[a(1+\cos\theta)]^2+(a\sin\theta)^2$ $=a^2(2+2\cos\theta)$ $=4a^2\cos^2\frac{1}{2}\theta$ $s=\int 2a\cos\frac{1}{2}\theta\,\mathrm{d}\theta$ $=4a\sin\frac{1}{2}\theta+C$ $s=0$ when $\theta=0\Rightarrow C=0$	M1 A1 M1 M1 A1 A1 ag **6**	Forming $\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2$ Using half-angle formula Integrating to obtain $k\sin\frac{1}{2}\theta$ Correctly obtained (+C not needed) *Dependent on all previous marks*
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a\sin\theta}{a(1+\cos\theta)}$ $=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2a\cos^2\frac{1}{2}\theta}=\tan\frac{1}{2}\theta$ $\psi=\frac{1}{2}\theta$, and so $s=4a\sin\psi$	M1 M1 A1 A1 **4**	 Using half-angle formulae
(iii)	$\rho=\frac{\mathrm{d}s}{\mathrm{d}\psi}=4a\cos\psi$ $=4a\cos\frac{1}{2}\theta$	M1 A1 ft A1 ag **3**	Differentiating intrinsic equation
	OR $\rho=\frac{\left(4a^2\cos^2\frac{1}{2}\theta\right)^{3/2}}{a(1+\cos\theta)(a\cos\theta)-(-a\sin\theta)(a\sin\theta)}$ M1A1 ft $=\frac{8a^3\cos^3\frac{1}{2}\theta}{a^2(1+\cos\theta)}=\frac{8a^3\cos^3\frac{1}{2}\theta}{2a^2\cos^2\frac{1}{2}\theta}=4a\cos\frac{1}{2}\theta$ A1 ag		Correct expression for ρ or κ
(iv)	When $\theta=\frac{2}{3}\pi,\ \psi=\frac{1}{3}\pi,\ x=a(\frac{2}{3}\pi+\frac{1}{2}\sqrt{3}),\ y=\frac{3}{2}a$ $\rho=2a$ $\hat{\mathbf{n}}=\begin{pmatrix}-\sin\psi\\ \cos\psi\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}a(\frac{2}{3}\pi+\frac{1}{2}\sqrt{3})\\ \frac{3}{2}a\end{pmatrix}+2a\begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ Centre of curvature is $(\ a(\frac{2}{3}\pi-\frac{1}{2}\sqrt{3}),\ \frac{5}{2}a\ )$	B1 M1 A1 M1 A1A1 **6**	 Obtaining a normal vector Correct unit normal *(possibly in terms of θ)* Accept $(1.23a,\ 2.5a)$

(v)	Curved surface area is $\int 2\pi y \,\mathrm{d}s$ $= \int_0^{\pi} 2\pi a(1-\cos\theta)\, 2a\cos\frac{1}{2}\theta \,\mathrm{d}\theta$ $= \int_0^{\pi} 8\pi a^2 \sin^2\frac{1}{2}\theta \cos\frac{1}{2}\theta \,\mathrm{d}\theta$ $= \left[\frac{16}{3}\pi a^2 \sin^3\frac{1}{2}\theta \right]_0^{\pi}$ $= \frac{16}{3}\pi a^2$	M1 A1 ft M1 M1 A1 **5**	 Correct integral expression in any form *(including limits; may be implied by later working)* Obtaining an integrable form Obtaining $k\sin^3\frac{1}{2}\theta$ *or equivalent*

4 (i)	In *G*, $3^2=2,\ 3^3=6,\ 3^4=4,\ 3^5=5,\ 3^6=1$ [or $5^2=4,\ 5^3=6,\ 5^4=2,\ 5^5=3,\ 5^6=1$] In *H*, $5^2=7,\ 5^3=17,\ 5^4=13,\ 5^5=11,\ 5^6=1$ [or $11^2=13,\ 11^3=17,\ 11^4=7,\ 11^5=5,\ 11^6=1$] *G* has an element 3 (or 5) of order 6 *H* has an element 5 (or 11) of order 6	M1 A1 B1 B1 **4**	All powers of an element of order 6 All powers correct in both groups
(ii)	$\{1,\ 6\}$ $\{1,\ 2,\ 4\}$	B1 B2 **3**	*Ignore* $\{1\}$ *and G* *Deduct 1 mark (from B1B2) for each proper subgroup in excess of two*
(iii)	$G \quad H$ $1 \leftrightarrow 1$ $2 \leftrightarrow 7$ $3 \leftrightarrow 5$ $4 \leftrightarrow 13$ $5 \leftrightarrow 11$ $6 \leftrightarrow 17$ OR $G \quad H$ $1 \leftrightarrow 1$ $2 \leftrightarrow 13$ $3 \leftrightarrow 11$ $4 \leftrightarrow 7$ $5 \leftrightarrow 5$ $6 \leftrightarrow 17$	B4 **4**	Give B3 for 4 correct, B2 for 3 correct, B1 for 2 correct
(iv)	$\text{ad}(1)=\text{a}(3)=1$ $\text{ad}(2)=\text{a}(2)=3$ $\text{ad}(3)=\text{a}(1)=2$, so $\text{ad}=\text{c}$ $\text{da}(1)=\text{d}(2)=2$ $\text{da}(2)=\text{d}(3)=1$ $\text{da}(3)=\text{d}(1)=3$, so $\text{da}=\text{f}$	M1 A1 M1 A1 **4**	Evaluating e.g. $\text{ad}(1)$ *(one case sufficient; intermediate value must be shown)* For $\text{ad}=\text{c}$ correctly shown Evaluating e.g. $\text{da}(1)$ *(one case sufficient; no need for any working)*
(v)	*S* is not abelian; *G* is abelian	B1 **1**	*or S* has 3 elements of order 2; *G* has 1 element of order 2 *or S* is not cyclic *etc*
(vi)	Element: a, b, c, d, e, f Order: 3, 3, 2, 2, 1, 2	B4 **4**	Give B3 for 5 correct, B2 for 3 correct, B1 for 1 correct
(vii)	$\{\text{e},\ \text{c}\},\ \{\text{e},\ \text{d}\},\ \{\text{e},\ \text{f}\}$ $\{\text{e},\ \text{a},\ \text{b}\}$	B1B1B1 B1 **4**	Ignore { e } and *S* If more than 4 proper subgroups are given, deduct 1 mark for each proper subgroup in excess of 4

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0 & 0.1 & 0 & 0.3 \\ 0.7 & 0.8 & 0 & 0.6 \\ 0.1 & 0 & 1 & 0.1 \\ 0.2 & 0.1 & 0 & 0 \end{pmatrix}$	B2	**2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^{13} \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.0810 \\ 0.5684 \\ 0.2760 \\ 0.0746 \end{pmatrix}$	M1 A2	**3**	Using $\mathbf{P}^{13}$ (or $\mathbf{P}^{14}$) Give A1 for 2 probabilities correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001
(iii)	$0.5684 \times 0.8 + 0.2760$ $= 0.731$	M1M1 A1 ft	**3**	For 0.5684×0.8 and 0.2760 *Accept 0.73 to 0.7312*
(iv)	$\mathbf{P}^{30} \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} . \\ . \\ 0.4996 \\ . \end{pmatrix}, \quad \mathbf{P}^{31} \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} . \\ . \\ 0.5103 \\ . \end{pmatrix}$ Level 32	M1 A1 A1	**3**	Finding P(C) for some powers of **P** For identifying $\mathbf{P}^{31}$
(v)	Expected number of levels including the next change of location is $\frac{1}{0.2} = 5$ Expected number of further levels in B is 4	M1 A1 A1	**3**	For $1/(1-0.8)$ or $0.8/(1-0.8)$ For 5 or 4 For 4 as final answer
(vi)	$\mathbf{Q} = \begin{pmatrix} 0 & 0.1 & 0 & 0.3 \\ 0.7 & 0.8 & 0 & 0.6 \\ 0.1 & 0 & 0.9 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0 \end{pmatrix}$ $\mathbf{Q}^n \to \begin{pmatrix} 0.0916 & 0.0916 & 0.0916 & 0.0916 \\ 0.6183 & 0.6183 & 0.6183 & 0.6183 \\ 0.1908 & 0.1908 & 0.1908 & 0.1908 \\ 0.0992 & 0.0992 & 0.0992 & 0.0992 \end{pmatrix}$ A: 0.0916 B: 0.6183 C: 0.1908 D: 0.0992	B1 M1 M1 A2	**5**	*Can be implied* Evaluating powers of **Q** *or Obtaining (at least) 3 equations from* $\mathbf{Qp} = \mathbf{p}$ Limiting matrix with equal columns *or Solving to obtain one equilib prob* *or M2 for other complete method* Give A1 for two correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001

(vii)	$\begin{pmatrix} 0 & 0.1 & a & 0.3 \\ 0.7 & 0.8 & b & 0.6 \\ 0.1 & 0 & c & 0.1 \\ 0.2 & 0.1 & d & 0 \end{pmatrix}\begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix}$	M1 A1	Transition matrix and $\begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix}$
	$0.075+0.04a+0.03=0.11$ $0.077+0.6+0.04b+0.06=0.75$ $0.011+0.04c+0.01=0.04$ $0.022+0.075+0.04d=0.1$	M1	Forming at least one equation *or* $a+b+c+d=1$
	$a=0.125,\ \ b=0.325,\ \ c=0.475,\ \ d=0.075$	A2 **5**	Give A1 for two correct

Post-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0 & 0.7 & 0.1 & 0.2\\ 0.1 & 0.8 & 0 & 0.1\\ 0 & 0 & 1 & 0\\ 0.3 & 0.6 & 0.1 & 0\end{pmatrix}$	B2	**2**	Give B1 for two rows correct
(ii)	$\begin{pmatrix}0.6 & 0.4 & 0 & 0\end{pmatrix}\mathbf{P}^{13}$ $=\begin{pmatrix}0.0810 & 0.5684 & 0.2760 & 0.0746\end{pmatrix}$	M1 A2	**3**	Using $\mathbf{P}^{13}$ (or $\mathbf{P}^{14}$) Give A1 for 2 probabilities correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001
(iii)	$0.5684\times 0.8+0.2760$ $=0.731$	M1M1 A1 ft	**3**	For 0.5684×0.8 and 0.2760 *Accept 0.73 to 0.7312*
(iv)	$\begin{pmatrix}0.6 & 0.4 & 0 & 0\end{pmatrix}\mathbf{P}^{30}=\begin{pmatrix}. & . & 0.4996 & .\end{pmatrix}$ $\begin{pmatrix}0.6 & 0.4 & 0 & 0\end{pmatrix}\mathbf{P}^{31}=\begin{pmatrix}. & . & 0.5103 & .\end{pmatrix}$ Level 32	M1 A1 A1	**3**	Finding P(C) for some powers of **P** For identifying $\mathbf{P}^{31}$
(v)	Expected number of levels including the next change of location is $\frac{1}{0.2}=5$ Expected number of further levels in B is 4	M1 A1 A1	**3**	For $1/(1-0.8)$ or $0.8/(1-0.8)$ For 5 or 4 For 4 as final answer
(vi)	$\mathbf{Q}=\begin{pmatrix}0 & 0.7 & 0.1 & 0.2\\ 0.1 & 0.8 & 0 & 0.1\\ 0 & 0 & 0.9 & 0.1\\ 0.3 & 0.6 & 0.1 & 0\end{pmatrix}$ $\mathbf{Q}^n\to\begin{pmatrix}0.0916 & 0.6183 & 0.1908 & 0.0992\\ 0.0916 & 0.6183 & 0.1908 & 0.0992\\ 0.0916 & 0.6183 & 0.1908 & 0.0992\\ 0.0916 & 0.6183 & 0.1908 & 0.0992\end{pmatrix}$ A: 0.0916 B: 0.6183 C: 0.1908 D: 0.0992	B1 M1 M1 A2	**5**	*Can be implied* Evaluating powers of **Q** *or Obtaining (at least) 3 equations from* $\mathbf{pQ}=\mathbf{p}$ Limiting matrix with equal rows *or Solving to obtain one equilib prob* *or M2 for other complete method* Give A1 for two correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001

(vii)	$(0.11 \quad 0.75 \quad 0.04 \quad 0.1)\begin{pmatrix} 0 & 0.7 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0 & 0.1 \\ a & b & c & d \\ 0.3 & 0.6 & 0.1 & 0 \end{pmatrix}$	M1	Transition matrix and $(0.11 \quad 0.75 \quad 0.04 \quad 0.1)$
	$= (0.11 \quad 0.75 \quad 0.04 \quad 0.1)$	A1	
	$0.075 + 0.04a + 0.03 = 0.11$ $0.077 + 0.6 + 0.04b + 0.06 = 0.75$ $0.011 + 0.04c + 0.01 = 0.04$ $0.022 + 0.075 + 0.04d = 0.1$	M1	Forming at least one equation *or* $a+b+c+d=1$
	$a = 0.125,\quad b = 0.325,\quad c = 0.475,\quad d = 0.075$	A2 **5**	Give A1 for two correct